UNIVERSITI PUTRA MALAYSIA

DIAGONALLY IMPLICIT TWO POINT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING STIFF ORDINARY DIFFERENTIAL EQUATIONS AND FUZZY DIFFERENTIAL EQUATIONS

ISKANDAR SHAH BIN MOHD ZAWAWI

FS 2014 49

DIAGONALLY IMPLICIT TWO POINT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING STIFF ORDINARY DIFFERENTIAL EQUATIONS AND FUZZY DIFFERENTIAL EQUATIONS

By

ISKANDAR SHAH BIN MOHD ZAWAWI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

January 2014

COPYRIGHT

DEDICATIONS

To

Mohd Zawawi Abdullah

Rozita Arifin

Anis Syazween

Noreen Natasha

Muzzaffar Shah

Alauddin Shah

and

Assoc. Prof. Dr. Zarina Bibi Ibrahim

for their support and patience.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

DIAGONALLY IMPLICIT TWO POINT BLOCK BACKWARD DIFFERENTIATION FORMULAS FOR SOLVING STIFF ORDINARY DIFFERENTIAL EQUATIONS AND FUZZY DIFFERENTIAL EQUATIONS

By

ISKANDAR SHAH BIN MOHD ZAWAWI

January 2014

Chair: Zarina Bibi Binti Ibrahim, PhD
Faculty: Science

Our main contribution in the thesis is the development of a new block method which is called diagonally implicit two point block backward differentiation formulas of order two (DI2BBDF(2)), order three (DI2BBDF(3)) and order four (DI2BBDF(4)) for solving stiff ordinary differential equations (ODEs) and fuzzy differential equations (FDEs). This method is constructed to compute multiple approximations concurrently in a block using various back values. The performance of the method is compared with existing methods. Furthermore, the convergence and stability properties of the method are investigated. The strategy of choosing suitable step size is also discussed. This thesis also explored the numerical solution of first order FDEs. The fully implicit two point block backward differentiation formulas of order three (FI2BBDF(3)) is reviewed and modified in fuzzy version for solving fuzzy initial value problems (FIVPs) under a new interpretation of Hukuhara Differentiability Theorem (HDT). Based on HDT, the exact and approximated solutions for two cases are compared to investigate the accuracy of the method. Finally, the derived method is modified in fuzzy version for solving FIVPs under HDT. The efficiency of the method is compared with several existing methods. In conclusion, the proposed method can be an alternative method for solving stiff ODEs and FDEs.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra sebagai memenuhi keperluan untuk ijazah Master Sains

FORMULASI DUA TITIK BLOK PEMBEZAAN KE BELAKANG PEPENJURU TERSIRAT BAGI MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA KAKU DAN PERSAMAAN PEMBEZAAN KABUR

Oleh

ISKANDAR SHAH BIN MOHD ZAWAWI

Januari 2014

Pengerusi: Zarina Bibi Binti Ibrahim, PhD
Fakulti: Sains

Sumbangan utama kami dalam tesis ini ialah pembangunan kaedah blok baru yang dipanggil formula dua titik blok pembezaan ke belakang pepenjuru tersirat peringkat kedua (F2BPBPT(2)), peringkat ketiga (F2BPBPT(3)) dan peringkat keempat (F2BPBPT(4)) bagi penyelesaian persamaan pembezaan biasa (PPB) kaku dan persamaan pembezaan kabur (PPK). Kaedah ini dibina untuk menghitung pelbagai penyelesaian anggaran secara serentak dalam satu blok menggunakan pelbagai nilai belakang. Prestasi kaedah dibandingkan dengan kaedah sedia ada. Tambahan lagi, sifat-sifat kestabilan dan penumpuan kaedah diselidik. Strategi pemilihan saiz langkah yang sesuai turut dibincangkan. Tesis ini juga meneroka penyelesaian berangka bagi persamaan pembezaan biasa (PPB) kabur peringkat pertama. Formula dua titik blok pembezaan ke belakang tersirat penuh peringkat ketiga (F2BPBTP(3)) dikaji semula dan diubahsuai dalam versi kabur untuk menyelesaikan Masalah Nilai Awal (MNA). Oleh itu, pentafsiran kabur baru di bawah Teorem Pembezaan Hukuhara (TPH) ditunjukkan. Berdasarkan TPH, penyelesaian tepat dan anggaran bagi dua kes dibandingkan untuk menyiasat ketepatan kaedah. Akhirnya, kaedah yang dibina telah diubahsuai dalam versi kabur bagi menyelesaikan PPK di bawah TPH. Ketepatan kaedah ditunjukkan dan dibandingkan dengan beberapa kaedah sedia ada. Kesimpulannya, kaedah yang dicadangkan boleh menjadi salah satu kaedah alternatif bagi menyelesaikan PPB dan PPK.

ACKNOWLEDGEMENTS

In the name of Allah, the Most Compassionate and the Most Merciful

First and foremost, I would like to express my sincere and deepest gratitude to the chairman of the supervisory committee, Associate Professor Dr. Zarina Bibi Ibrahim for her moral support, guidance, constructive criticism and encouragement. I am also thankful to the member of the supervisory committee, Professor Dr. Fudziah Ismail and Associate Professor Dr. Zanariah Abdul Majid for their assistance and ideas to improve my research.

A special thanks to all my friends, Nurzeehan Ismail, Siti Ainor Mohd Yatim, Noraini Zainuddin, Noraini Jamaludin, Atiyah Wan Mohd Sham, Norshakila Abd Rasid, Latifah Md Arifin, Mohd Mughti Hasni, Ahmad Shah Abdullah, Radzi Aridi, Wan Mohd Zulqhairi and Faidzan Hasmunira Hasbollah. Special acknowledgement is extended to Phang Pei See, Hoo Yann Seong and Nor Ain Azeany Mohd Nasir for spending exceptional amount of time in discussing and exchanging opinion. I am also indebted to the Ministry of High Education for the MYBRAIN scholarship which enables me to pursue this research.

Finally, I appreciate the continuous support, understanding and patience of my parents, Mohd Zawawi Abdullah and Rozita Arifin, and my siblings, Anis Syazween, Noreen Natasha, Muzzaffar Shah and Alauddin Shah for their encouragement.

I certify that a Thesis Examination Committee has met on 20 December 2013 to conduct the final examination of Iskandar Shah Bin Mohd Zawawi on his thesis entitled “Diagonally Implicit Two Point Block Backward Differentiation Formulas for solving Stiff Ordinary Differential Equations and Fuzzy Differential Equations” in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A)106] 15 March 1998. The Committee recommends that the student be awarded the degree of Master of Science.

Members of the Thesis Examination Committee were as follows:

Mohd Rizam bin Abu Bakar, PhD
Associate Professor
Department of Mathematics
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Mansor Bin Monsi, PhD
Department of Mathematics
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Leong Wah June, PhD
Associate Professor
Department of Mathematics
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Arsmah Binti Ibrahim, PhD
Professor
Faculty of Computer and Mathematical Sciences
Universiti Teknologi MARA
Malaysia
(External Examiner)

NORITAH OMAR, PhD
Assoc. Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 21 January 2014

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

Zarina Bibi Binti Ibrahim, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Fudziah Binti Ismail, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Zanariah Binti Abdul Majid, PhD
Associate Professor
Faculty of Sciences
Universiti Putra Malaysia
(Member)

BUJANG BIN KIM HUAT, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

DECLARATION

Declaration by graduate student

I hereby confirm that:

- this thesis is my original work;
- quotations, illustrations and citations have been duly referenced;
- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
- intellectual property from the thesis and copyright of thesis are fully-owned by Universiti Putra Malaysia as according to the Universiti Putra Malaysia (Research) Rules 2012;
- written permission must be obtained from supervisor and the office of Deputy Vice-Chancellor (Research and Innovation) before thesis is published (in the form of written, printed or in electronic form) including books, journals, modules, proceedings, popular writings, seminar papers, manuscripts, posters, reports, lecture notes, learning modules or any other materials as stated in the Universiti Putra Malaysia (Research) Rules 2012;
- there is no plagiarism or data falsification/fabrication in the thesis, and scholarly integrity is upheld as according to the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) and the Universiti Putra Malaysia (Research) Rules 2012. The thesis has undergone plagiarism detection software.

Signature: ______________________ Date: 20 December 2013

Name and Matric No: Iskandar Shah Bin Mohd Zawawi (GS31077)

Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________
Name of Chairman of Supervisory Committee: ______________________

Signature: ______________________
Name of Member of Supervisory Committee: ______________________

Signature: ______________________
Name of Member of Supervisory Committee: ______________________

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

IVPs	Initial value problems
LMM	Linear multistep method
ODEs	Ordinary differential equations
FDEs	Fuzzy differential equations
FIVPs	Fuzzy initial value problems
HDT	Hukuhara Differentiability Theorem
BBDF	Block backward differentiation formulas
FI2BBDF(3)	Fully implicit two point block backward differentiation formulas of order three
FI3BBDF(3)	Fully implicit three point block backward differentiation formulas of order three
FI2BEBDF(3)	Fully implicit two point block extended backward differentiation formulas order three
FI3BEBDF(3)	Fully implicit three point block extended backward differentiation formulas of order three
DI2BBDF(2)	Diagonally implicit two point block backward differentiation formulas of second order
DI2BBDF(3)	Diagonally implicit two point block backward differentiation formulas of third order
DI2BBDF(4)	Diagonally implicit two point block backward differentiation formulas of fourth order
FFI2BBDF(3)	Fuzzy fully implicit two point block backward differentiation formulas of order three
FDI2BBDF(2)	Fuzzy diagonally implicit two point block backward differentiation formulas of order two
FDI2BBDF(3)	Fuzzy diagonally implicit two point block backward differentiation formulas of order three
FDI2BBDF(4)	Fuzzy diagonally implicit two point block backward differentiation formulas of order four
DI2BBDF	Diagonally implicit two point block backward differentiation formula
AB2SM	Adam Bashforth 2 −step method
PC3SM	Predictor-corrector 3 −step method
EULER	Euler method
ERK4	Extended Runge-Kutta method of order four
RK3	Runge-Kutta of order three
RK4	Runge-Kutta of order four

CHAPTER 1

INTRODUCTION

1.1 Introduction

Differential equations serve as mathematical models for many exciting problems, not only in science and technology but also in such diverse fields such as economics, psychology, defense, and demography. The general form of differential equation is given by Butcher (2008) as follows:

$$y'(x) = f(x, y(x)), \qquad y' = dy/dx. \tag{1.1}$$

Rapid growth in the theory of differential equations and in its applications to almost every branch of knowledge has resulted in a continued study by researchers in many disciplines. However, ordinary differential equations (ODEs) are the most popular differential equations in mathematics curricula all over the world and it is now being taught at various levels in almost every institution of higher learning. Lambert (1991) has presented the general form of first order system of ODEs as

$$\left.\begin{aligned} y'_1(x) &= f_1(x, y_1, y_2, \dots, y_m), \\ y'_2(x) &= f_2(x, y_1, y_2, \dots, y_m), \\ &\ \vdots \\ y'_m(x) &= f_m(x, y_1, y_2, \dots, y_m). \end{aligned}\right\} \tag{1.2}$$

Currently, the study of differential equations with uncertainty plays an important role in many disciplines and real world phenomena. This type of differential equations is called fuzzy differential equations (FDEs). Developing an accurate numerical method is one of the important parts in studying ODEs and FDEs. The numerical method can be classified as single step methods and multistep methods. The single step methods is used to calculate approximated solution using one previous point while for multistep methods, the approximated solution is evaluated using several previous points. The examples of multistep methods are Adams method and backward differentiation formulas (BDF).

1.2 Stiff initial value problems

The following definition is given by Lambert (1991) to define stiff ODEs.

Definition 1.1
The system of (1.2) is said to be stiff if $Re(\lambda_t) < 0, t = 1, 2, \ldots, m$ and $max_t|Re(\lambda_t)| \gg min_t|Re(\lambda_t)|$ where λ_t are the eigenvalues of the Jacobian matrix, $J = \left(\frac{\partial f}{\partial y}\right)$. Stiff problems often have $Re(\lambda_t)$ of greatly varying magnitude, which adds to the difficulty of their solution.

1.3 Linear multistep method

The theory of linear multistep method (LMM) is developed in large scale by Dahlquist (1956) and has become widely known through the exposition by Henrichi (1962, 1963). In this section, we briefly present some definitions of LMM which are introduced by Lambert (1991).

Definition 1.2
The LMM can be represented in standard form by an equation:

$$\sum_{j=0}^{k} a_j y_{n+j} = h \sum_{j=0}^{k} b_j f_{n+j}, \qquad (1.3)$$

where $y_{n+j} \approx y(x_{n+j})$ and $f_{n+j} \equiv f\left(x_{n+j}, y_{n+j}\right)$, a_j and b_j are real constants and k is defined as the order of the particular method applied. The formula (1.3) is explicit if $b_k = 0$, and it is implicit if $b_k \neq 0$.

Definition 1.3
The LMM is said to be of order p if $C_0 = C_1 = \cdots = C_p = 0$, $C_{p+1} \neq 0$. The general form of constant C_q is defined as:

$$C_0 = \sum_{j=0}^{k} j\alpha_j \,,$$

$$C_q = \sum_{j=0}^{k} \left(\frac{1}{q!} j^q \alpha_j - \frac{1}{(q-1)!} j^{q-1} \beta_j\right) \,, q = 1,2, \ldots, k. \qquad (1.4)$$

The general form of block method is given by Ibrahim *et al.* (2008) as follows:

Definition 1.4
The k −block r −point method is a matrix finite difference equation of the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}, \qquad (1.5)$$

where α_j and β_j are properly chosen $r \times r$ matrix coefficients.

Majid and Suleiman (2006) stated that the block method is defined to be diagonally implicit if the coefficients of the upper-diagonal entries are zero.

Definition 1.5
We consider a_{11}, a_{12}, a_{21} and a_{22} are coefficients of y_{n+1} and y_{n+2} in the matrix form below.

$$\begin{bmatrix} a_{11} & a_{12} \\ & \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{n+1} \\ \\ y_{n+2} \end{bmatrix}. \tag{1.6}$$

The equation (1.5) is defined to be diagonally implicit if a_{12} is zero whereas a_{11} and a_{22} are equal.

1.4 Convergence

Convergence refers to the ability of a method to approximate the exact solution to a differential equation to any required accuracy. Butcher (2008) mentioned that the LMM is convergent if and only if it is consistent and stable.

Definition 1.6
The equation (1.5) proved to be consistent if and only if the following conditions are satisfied:

$$\sum_{j=0}^{k} \alpha_j = 0, \tag{1.7}$$

$$\sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j, \tag{1.8}$$

Definition 1.7
LMM is said to be zero-stable if the roots $R_j, j = 1(1)k$ of the first characteristic polynomial, $\rho(R) = det\,[\sum_{i=0}^{k} A_i R^{k-i}] = 0,\ A_0 = -I$ satisfies $|R_j| \leq 1$. If one of the roots is $+1$, we call this root the principal root of $\rho(R)$.

Definition 1.8
The LMM is said to be zero-stable if no root of the first characteristic polynomial, $\rho(t)$ has modulus greater than one, and if every root with modulus one is simple.

Theorem 1.1
The necessary and sufficient conditions for the LMM to be convergent are that it is consistent and zero-stable.

1.5 Stability theory

The stability properties of LMM are generally considered as the most important for the effectiveness solution of some problems. In fact, Shampine and Watts (1969) emphasized that the stability problem would appear to be the most serious limitation of block methods. The following definitions demonstrate the absolute stable and A-stable of LMM.

Definition 1.9
The LMM is said to be absolute stable in a region $\Re$ (real part) of the complex plane if, for all $\hat{h} \in \Re$, all roots of the stability polynomial $\pi(t, \hat{h})$ associated with the method, satisfy $|t_s| < 1, s = 1,2,\cdots,k$.

Definition 1.10
The LMM is A-stable if its region of absolute stability contains the whole of the left-hand half-plane, $Re(h\lambda) < 0$.

1.6 Fuzzy theory

Here, we present some definitions of fuzzy number, triangular fuzzy number, trapezoidal fuzzy number and fuzzy initial value problems which are described by Nguyen and Walker (2000).

Definition 1.11
A fuzzy number satisfies the following conditions.
1) $A(t) = 1$ for at least one t.
2) The support $\{t: A(t) > 0\}$ of A is bounded.
3) The $\alpha -$ cuts of A are closed intervals.

Definition 1.12
A fuzzy number, $\mu(t)$ can be determined by any pair $\mu(t) = \left(\underline{\mu}(t), \overline{\mu}(t)\right)$, where $0 \le \alpha \le 1,$ which satisfies the three conditions:
1) $\underline{\mu}(t)$ is a bounded left continuous increasing function $\alpha \in (0,1]$.
2) $\overline{\mu}(t)$ is a bounded left continuous decreasing function $\alpha \in (0,1]$.
3) $\underline{\mu}(t) \le \overline{\mu}(t)$, $0 \le \alpha \le 1$.

Definition 1.13
A triangular fuzzy number is determined by a triplet (a, b, c) of crisp number with $a < b < c$ where its membership function is given by

$$\mu(t) = \begin{cases} \dfrac{t-a}{b-a}, & if\ a \le t \le b \\ \dfrac{t-c}{b-c}, & if\ b < t \le c \\ 0, & \text{otherwise} \end{cases}$$

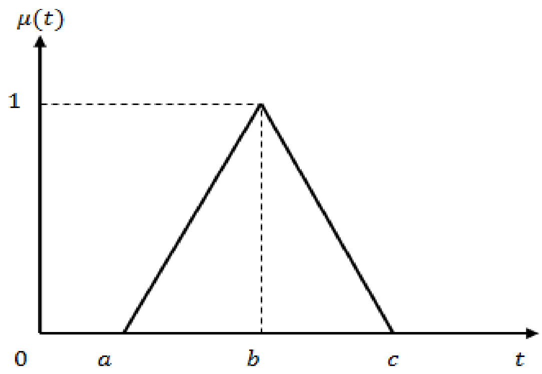

Figure 2.1: Triangular fuzzy on $[a, c]$

Definition 1.14
A trapezoidal fuzzy number is determined by the quadruplet (a, b, c, d) of crisp number with $a < b < c < d$ where its membership function is given by

$$\mu(t) = \begin{cases} \dfrac{t-a}{b-a}, & if\ a \le t \le b \\ 1, & if\ b < t < c \\ \dfrac{t-c}{d-c}, & if\ c \le t \le d \\ 0\,, & \text{otherwise} \end{cases}$$

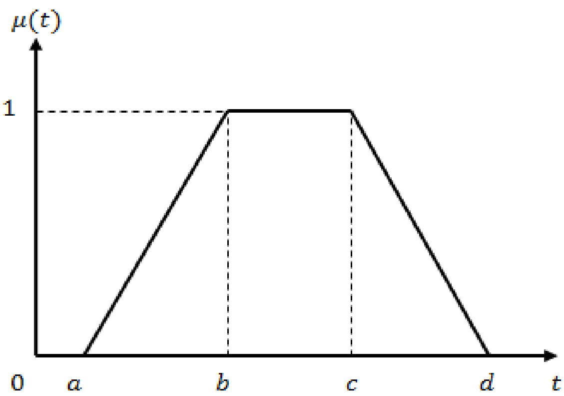

Figure 2.2: Trapezoidal fuzzy on $[a, d]$

1.7 Fuzzy initial value problems

The general form of fuzzy initial value problem (FIVP) is first introduced by Seikkala (1987) as follows:

$$x'(t) = f\big(t, x(t)\big)\,, \quad x(0) = x_0, \tag{1.9}$$

where x_0 is a fuzzy number with α –level intervals $[x_0]_\alpha = [x^\alpha{}_{01}, x^\alpha{}_{02}]$ and $0 < \alpha \le 1$.

The general form of FIVP which is given by Shokri (2007) in following form:

$$y'(t;\alpha) = f(t; y(t;\alpha);\alpha), \qquad t \in [t_0, T],$$
$$y(t_0;\alpha) = y_0, \qquad \alpha \in [0,1] \tag{1.10}$$

where $y(t;\alpha)$ is a fuzzy function of t, $f(t; y(t;\alpha);\alpha)$ is a fuzzy function of variable t and the fuzzy variable $y(t;\alpha)$, $y'(t;\alpha)$ is the fuzzy derivative of $y(t;\alpha)$ and $y(t_0;\alpha)$ is a trapezoidal shaped fuzzy number.

The definition and theorem of Hukuhara differentiability is given by Stefanini and Bede (2009) as follows:

Definition 1.15
Let $f : T \to E(\mathbb{R})$ and $t_0 \in (a,b)$ where f is differentiable at t_0. Then we consider two cases:

(I) For all $h > 0$ sufficiently close to 0, the Hukuhara differences $f(t_0 + h) \ominus f(t_0)$ and $f(t_0) \ominus f(t_0 - h)$ exist (in metric D) such that

$$\lim_{h \to o^+} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \to o^+} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0), \tag{1.11}$$

(II) For all $h > 0$ sufficiently close to 0, the Hukuhara differences $f(t_0) \ominus f(t_0 + h)$and $f(t_0 - h) \ominus f(t_0)$ exist (in metric D) such that

$$\lim_{h \to o^-} \frac{f(t_0) \ominus f(t_0 + h)}{h} = \lim_{h \to o^-} \frac{f(t_0 - h) \ominus f(t_0)}{h} = f'(t_0), \tag{1.12}$$

Theorem 1.2
Let $f : T \to y(\mathbb{R})$ where $t \in (t_0, T)$ and y is a fuzzy function and denote $[f(t;\alpha)] = \left[\underline{y}(t;\alpha), \bar{y}(t;\alpha)\right]$ for each $\alpha \in [0,1]$. Then two cases will be considered.

Case 1: If $f(t;\alpha)$ is Hukuhara differentiable in the first form (1.11), then $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ are differentiable functions in the following form:

$$f'(t;\alpha) = \left[\underline{y}'(t;\alpha), \bar{y}'(t;\alpha)\right]. \tag{1.13}$$

Case 2: If $f(t;\alpha)$ is Hukuhara differentiable in the second form (1.12), then $\underline{y}(t;\alpha)$ and $\bar{y}(t;\alpha)$ are differentiable functions in the following form:

$$f'(t;\alpha) = \left[\bar{y}'(t;\alpha), \underline{y}'(t;\alpha)\right]. \tag{1.14}$$

1.8 Objective of the thesis

The objectives of this thesis are as follows:

1) To derive the diagonally implicit two point block backward differentiation formulas of order two, three and four for solving ODEs and FDEs.
2) To study the convergence and stability properties of the derived methods.
3) To compare the efficiency of the derived methods in terms of accuracy and computational time when applied to stiff ODEs.
4) To investigate the accuracy of the proposed method when applied to FDEs.

1.9 Scope of the thesis

This thesis comprises the formulation of a new block backward differentiation formulas (BBDF) which is called diagonally implicit two point block backward differentiation formulas (DI2BBDF) of order two (DI2BBDF(2)), order three (DI2BBDF(3)) and order four (DI2BBDF(4)). Meanwhile, the scope of this thesis is limited to the numerical solution of stiff initial value problems (IVPs) and first order fuzzy initial value problems (FIVPs). For a fair comparison, the numerical results obtained from the existing methods are collected and compared with the proposed method.

1.10 Outline of the thesis

This thesis covers the following:

Chapter 1 provides the interest of problems and some relevant definitions when solving stiff ODEs and FDEs.

In Chapter 2, the evolution of block method, BBDF, diagonally implicit method and FDEs are reviewed.

Chapter 3 contains the derivation of second order, third order and fourth order diagonally implicit two point BBDF. The order of the method is verified. This chapter focuses on solving stiff ODEs under implementation of Newton iteration. In the last section of this chapter, the performance of the derived method is compared with the existing methods in terms of accuracy and computational time.

In Chapter 4, the consistency and zero stability of the derived method are discussed for the purpose of convergence properties. The stability region of the methods are illustrated and discussed. The restriction of the step size is calculated to determine the suitable step size.

The formulation of fully implicit two point BBDF is reviewed in Chapter 5. This method is modified in fuzzy version to solve FIVPs. A new interpretation of FIVPs is presented based on Case 1 and Case 2 of HDT. The performance of the method is observed based on comparison of approximate solutions and exact solutions.

In Chapter 6, the diagonally implicit two point BBDF of order two, order three and order four are modified in fuzzy version to solve FIVPs. The accuracy of the numerical results is compared with several existing methods.

Finally, the summary of the thesis and recommendation for future research are discussed in Chapter 7.

REFERENCES

Ababneh, O. Y., Ahmad, R. and Ismail, E. S. 2009. Design of new diagonally implicit Runge–Kutta methods for stiff problems. *Applied Mathematical Sciences* 3 (45): 2241 – 2253.

Abbasbandi, S. and Viranloo, T. A. 2002. Numerical solutions of fuzzy differential equations by Taylor method. *Comp. Method in Applied Mathematics* 2: 113-124.

Alexander, R. 1977. Diagonally implicit runge-kutta for stiff ordinary differential equations. *SIAM J. NUMER. ANAL.* 14 (6).

Allahviranloo, T., Ahmady, N. and Ahmady, E. 2007. Numerical solution of fuzzy differential equations by predictor-corrector method. *Information Sciences* 177: 1633-1647.

Balooch Shahryari, M. R., and Salahshour, S. 2012. Improved predictor corrector method for solving fuzzy differential equations under generalized differentiability. *Journal of Fuzzy Set Valued Analysis*, Volume 2012. ISPACS: 16 pages.

Bede, B. 2008. Note on numerical solutions of fuzzy differential equations by predictor-corrector method. *Information Sciences* 178: 1917-1922.

Bede, B. and Gal, S. G. 2005. Generalizations of the differentiability of fuzzy-number-valued functions with applications to fuzzy differential equations. *Fuzzy Sets and Systems* 151: 581-599.

Burden, R. and Faires, J. 2001. *Numerical analysis*. Brooks-Cole.

Butcher, J. C. 2008. *Numerical methods for ordinary differential equations*. John Wiley and Sons, England.

Calvo, M., Gonzalez-Pinto, S. and Montijano, J. I. 2010. Extending convergence results of Runge–Kutta methods for stiff semi linear initial value problems, *Monograf´ias de la Real Academia de Ciencias de Zaragoza* 33: 141–154.

Cash, J. R. 1983. The integration of stiff initial value problems in ordinary differential equations using modified extended backward differentiation formulae. *Comput. Math. Appl* 9: 645-660.

Chang, S. L. and Zadeh, L. A. 1972. On fuzzy mappings and control. *IEEE Trans. Systems, Man Cybernet.* 2: 30-34.

Chu, M.T. and Hamiltoon, H. 1987. Parallel solution of ODEs by multi-block methods, *SIAM J. Sci. Stat. Comp.* 8(3): 342-353.

Curtiss, C. and Hirschfelder, J. O. 1952. Integration of stiff equations. *Proceeding of the National Academy of Sciences of the United States of America* 38: 235-243.

Dahaghin, M. S. and Moghadam, M. M. 2010. Analysis of a two step method for numerical solution of fuzzy ordinary differential equations. *Italian Journal of Pure and Applied Mathematics.* 27: 333-340.

Dahlquist, G. 1956. Convergence and stability in the numerical integration of ordinary differential equations. *Math. Scand.* 4: 33-53.

Darvishi, M., Khani, F. and Soliman, A. 2007. The numerical simulation for stiff systems of ordinary differential equations. *Computers and Mathematics with Applications* 27: 1055-1063.

Duraisamy, C. and Usha, B. 2011. Numerical Solution of Fuzzy Differential Equations By Runge-Kutta Method of order three. *International journal of Scientific Research* 1 (1): ISSN 2249-9954.

Duraisamy, C. and Usha, B. 2012. Numerical Solution of fuzzy differential equations by Runge-Kutta method of order four. *European journal of Advanced Scientific and Technical Research* 1: 324-337.

Ehigie, J. O., Okunuga, S. A. and Sofoluwe, A. B. 2011. 3-Point block methods for direct integration of general second-order ordinary differential equations. *Advances in Numerical Analysis,* volume 2011, Article ID 513148: 14 pages.

Fatunla, S. O. 1990. Block methods for second order ODEs. *Intern. J. Comp. Math.* 41: 55-63.

Ghanbari, M. 2009. Numerical solutions of fuzzy initial value problems under generalized differentiability by HPM. *International Journal Industrial Mathematics* 1 (1): 19-39.

Ghazanfari, B. and Shakerami, A. 2012. Numerical solutions of fuzzy differential equations by extended Runge-Kutta-like formulae of order four. *Fuzzy Sets and Systems* 189 (1): 74-91.

Henrici, P. 1962. *Discrete variable methods in ordinary differential equations*. John Wiley and Sons, New York.
Henrici, P. 1963. *Error propagation for difference methods*. John Wiley and Sons, New York.

Hukuhara, M. 1967. *Intégration des applications mesurables dont la valeur est un compact convex.* Funkcial. Ekvac. 10: 205-229.

Ibrahim, Z. B. 2006. *Block method for multistep formulas for solving ordinary differential equations.* PhD Thesis, Universiti Putra Malaysia.

Ibrahim, Z. B., Suleiman, M. B. and Othman, K. I. 2007. Implicit r −point block backward differentiation formula for solving first order stiff ordinary differential equations. *Applied Mathematics and computation* 186: 558-565.

Ibrahim, Z. B., Suleiman, M. B. and Othman, K. I. 2008. Fixed coefficients block backward differentiation formulas for the numerical solution of stiff ordinary differential equations. *European Journal of Scientific Research* 21(3): 508-520.

Ibrahim, Z. B., Suleiman, M. B., Nasir, N. A. A. M. 2011. Convergence of the 2-Point Block Backward Differentiation Formulas. *Applied Mathematical Sciences* 5(70): 3473 – 3480.

Ismail, F., Al-Khasawneh, R. A., Suleiman, M. and Hassan, M. A. 2010. Embedded pair of diagonally implicit Runge-Kutta method for solving ordinary differential equations. *Sains Malaysiana* 39 (6): 1049–1054.

Kandel, A. and Byatt, W. J. 1978. Fuzzy differential equations. *Proceedings of International Conference Cybernetics and Society, Tokyo*: 1213-1216.

Kandel, A. and Byatt, W. J. 1980. Fuzzy processes. *Fuzzy Sets and Systems* 4: 117-152.

Lambert, J. D. 1991. *Numercial methods for ordinary differential systems: The initial value problem.* John Wiley and Sons, New York.

Majid, Z. A. and Suleiman, M. B. 2006. Performance of four point diagonally implicit block method for solving first order stiff ordinary differential equations. *Department of Mathematics UTM* 22 (2): 137-146.

Majid, Z. A., Suleiman, M. B., Ismail, F. I. and Othman, K. I. 2004. 2-point 1 block diagonally and 2-point 1 block fully implicit method for solving first order ordinary differential equations. In *Proceeding of the 12th National Symposium on Mathematical Science.*

Mehrkanoon, S., Suleiman, M. and Majid, Z. A. 2009. Block method for numerical solution of fuzzy differential equations. *International Mathematical Forum* 4: 2269-2280.

Milne, W. E. 1953. *Numerical Solution of Differential Equations*. Wiley, New York.

Musa, H., Suleiman, M. B. and Senu, N. 2012. A-stable 2-Point block extended backward differentiation formulas for stiff ordinary differential equations. *The 5th International Conference on Research and education in Mathematics*. AIP Conf. Proc. 1450: 254-258.

Musa, H., Suleiman, M. B. and Senu, N. 2012. Fully Implicit 3-Point block extended backward differentiation formula for stiff initial value problems. *Applied Mathematical Sciences* 6 (85): 4211 – 4228.

Nasir, N. A. A. M., Ibrahim, Z. B., Othman, K. I. and Suleiman, M. B. 2011. Fifth order two-point block backward differentiation formulas for solving ordinary differential equations. *Applied Mathematical Sciences* 5 (71): 3505 - 3518.

Nguyen, H. T. and Walker, E. A. 2000. *A first course in fuzzy logic*. Chapman & Hall/CRC, USA.

Pearson, D. W. 1997. A property of linear fuzzy differential equations, *Applied Mathematics Letters*. 10 (3): 99-103.

Sedaghatfar, O., Darabi, P. and Moloudzadeh, S. 2013. A method for solving first order fuzzy differential equations. *International Journal Industrial Mathematics* 5 (3): 251-257.

Seikkala, S. 1987. On the fuzzy initial value problem. *Fuzzy Sets and Systems* 24: 319-330.

Shampine, L. F. and Watts, H. A. 1969. Block implicit one step methods. *Math. Comp.* 23: 731-740.

Shokri, J. 2007. Numerical solution of fuzzy differential equations. *Applied Mathematical Sciences* 1: 2231-2246.

Stefanini, L. and Bede, B. 2009. Generalized Hukuhara differentiability of interval-valued functions and interval differential equations. *Nonlinear Analysis* 71: 1311-1328.