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Marangoni Convection in a Fluid Saturated Porous Layer with a Prescribed Heat Flux at its Lower Boundary

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Abstract

The onset of Marangoni convection in a horizontal porous layer heated from below with a constant heat flux is investigated. The Brinkman model is used and the Darcy law is employed to describe the flow in the porous medium heated from below. We obtain for the first time the closed form analytical solution for the onset of steady Marangoni convection in a fluid saturated porous layer with a prescribed heat flux at its lower boundary. Besides, the asymptotic solution of the long-wavelength is also obtained using regular perturbation technique with wave number as a perturbation parameter. The Marangoni numbers are found to depend on the Darcy number and Biot number. Predictions for the onset of convection are studied in detail.

Keywords: Stability; Marangoni convection; Porous Media

1. Introduction

Convective flow driven by both buoyancy and surface tension play an important role in science, engineering and technology, especially in crystals growth and in materials processing (Pimutkar and Ostrach (1981) and Ostrach(1983)). One of the early attempts to study convective flows driven by surface tension, is known as Marangoni convection was made by Pearson (1958) under assumptions of infinitesimally small amplitude analysis with non-deformable free surface and no-slip at the bottom.

He showed that the variations in the surface tension at the free surface due to temperature gradients could induce motion within the fluid when the Marangoni number exceeds a critical value in the absence of buoyancy forces.

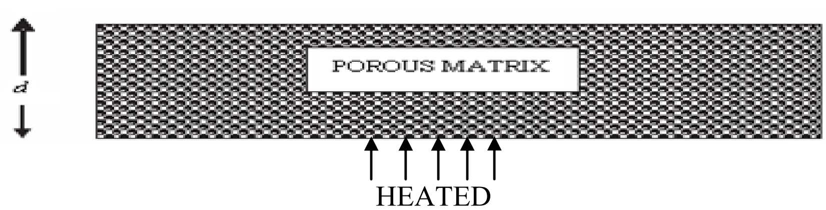
The problem of convective instability in a horizontal porous layer driven either by buoyancy effects or surface tension effects has been investigated extensively in the literature, using Darcy model (Lapwood (1948), Palm et. al(1972), Rudraiah et. al (1980), Nield (1977,1983)). The thermal stability for different system of superposed porous and fluid regions has also been considered by Pillatsis et. al (1987) and Taslim and Narusawa (1989). Hennenberg et.al.(1997) have considered a liquid saturated porous media in contact with air and subjected to an adverse gradient of temperature in the lower boundary is perfectly conducting. They have developed the model that can be described in terms of the Brinkman model. They solved the Brinkman approach over the whole saturated porous matrix and obtained a critical wave number which was highly dependent on the Darcy number. Recently, Shivakumara et al. (2006) studied the onset of Marangoni convection in a composite porous-layer system and the Beavers-Joseph slip condition is used at the interface and the Darcy law is employed to describe the flow in the porous medium. They showed that the linear stability curves for the onset of Marangoni convection depend on the parameter ζ, that is the ratio of the fluid layer depth to the porous layer depth. They interpreted their findings by showing that for ζ small, the instability was initiated in the porous medium, whereas for larger ζ, the instability was controlled by the fluid layer. They also suggested that the regular perturbation technique with small wave number a as a perturbation parameter can be conveniently used in solving convective instability problems in the case of insulating boundaries.

The purpose of the present work is to study the model developed by Hennenberg et al (1997) by considering the lower boundary is at the constant heat flux. We derived for the first time analytical expression for the onset of steady Marangoni convection for the case of constant-heat flux thermal condition at lower boundary. We also use regular perturbation technique to obtain the asymptotic solutions of the long-wavelength.

2. Mathematical Formulation

Consider a saturated isotropic porous matrix of thickness d and of infinite horizontal extent, heated from below. The physical configuration is shown in Fig. 1.

Figure 1: Physical Model



Its upper boundary is at a temperature T_0 and is in contact with a gaseous phase. The lower boundary is assumed to be a perfect insulator at a higher temperature $T_0 + \Delta T$. The free surface is assumed to be flat and undeformable. The saturated porous matrix is entirely described by the continuity, Brinkman momentum law and energy equation that are

$$\nabla \cdot \boldsymbol{V} = 0 \tag{1}$$

$$\rho_l c_a \frac{\partial \boldsymbol{V}}{\partial t} = -\nabla P - \frac{\mu}{K}\boldsymbol{V} + \mu_{\text{eff}} \nabla^2 \boldsymbol{V}, \tag{2}$$

$$\rho_m c_m \frac{\partial T}{\partial t} + \rho_l c_l \boldsymbol{V} \cdot \nabla T = k_m \nabla^2 T, \tag{3}$$

where $\boldsymbol{V}$ is the seepage velocity, ρ_m is the mean density, ρ_l is the clear liquid density, c_l is the specific solid heat capacity in the clear liquid, c_m is the specific solid heat capacity in the porous medium, k_m is the overall thermal conductivity of the porous medium, μ_{eff} is the effective saturated porous medium viscosity, c_a is the acceleration coefficient, P is the pressure, μ is the pure liquid viscosity and K the permeability of the porous matrix.

The variables are then nondimensionalized using d, $\varsigma d^2/\alpha_m$, α_m/d, ΔT, $\mu\alpha_m/K$ as the units of length, time, velocity, temperature and pressure respectively. Using the dimensionless variables, the Eqs. (1) – (3) are transform to the following dimensionless form:

$$\nabla \cdot \boldsymbol{V} = 0, \tag{4}$$

$$\gamma_{\boldsymbol{a}} \frac{\partial}{\partial \boldsymbol{t}} \nabla^2 \boldsymbol{V} = -\nabla^2 \boldsymbol{V} + \mathrm{Da}^{\mathrm{eff}} \nabla^4 \boldsymbol{V}, \tag{5}$$

$$\frac{\partial \theta}{\partial t} = W + \nabla^2 \theta, \tag{6}$$

where $\mathrm{Da}^{\mathrm{eff}} = \dfrac{\mu_{eff} K}{\mu d^2}$ and $\gamma_a = \dfrac{c_a \rho_f \alpha_m K}{\varsigma d^2 \mu}$ with α_m as the mean thermal diffusivity of the saturated porous medium. The boundary conditions at the bottom are for rigid boundary insulated to temperature perturbations:

$$W = \frac{\partial W}{\partial z} = \frac{\partial \theta}{\partial z} = 0, \tag{6}$$

evaluated at $z = 0$. The boundary conditions at the upper surface is based on rev averaged surface tension as detail in Hennenberg et. al (1997). The boundary conditions are

$$W = 0, \tag{7}$$

$$\frac{\partial^2 W}{\partial z^2} = \mathrm{Ma}_p^{\mathrm{eff}} \nabla_h^2 \theta, \tag{8}$$

$$-\frac{\partial \theta}{\partial z} = \mathrm{Bi}\, \theta, \tag{9}$$

evaluated at $z = 1$ and $\mathrm{Ma}_p^{\mathrm{eff}}$ is the equivalent of a Marangoni number for the upper surface, defined as

$$\mathrm{Ma}_p^{\mathrm{eff}} = -\left[\frac{\partial \sigma_m}{\partial T}\right] \frac{\Delta T d k_l}{\mu_{eff} \alpha_l k_m}$$

where $\mathrm{Ma}_p^{\mathrm{eff}}$ is the product of the pure liquid Marangoni number by a quantity which is a function of the porosity ϕ and of the thermal conductivity of the clear liquid and the solid (see detailed in Hennenberg et al.(1997)) and $\nabla_h^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator.

If f is a disturbance quantity, then following Hennenberg et. al (1997), and expressing this quantity as

$$f(x, y, z, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}k_x \mathrm{d}k_y f_k(z) \exp\left[i\left(k_x x + k_y y\right) + ist\right], \tag{10}$$

with $a = \sqrt{k_x^2 + k_y^2}$ is a wavenumber. Using (10), Eqs. (4) and (5) in dimensionless form become

$$\left((\gamma_a s + 1) - \mathrm{Da}^{\mathrm{eff}}\left(D^2 - a^2\right)\right)\left(D^2 - a^2\right) W(z) = 0, \tag{11}$$

$$\left(D^2 - \left(a^2 + s\right)\right)\theta(z) = -W(z), \tag{12}$$

where $W(z)$ is the vertical variation of the z-velocity and $D = d/dz$.

The dimensionless form boundary conditions (6) – (9) become

$$D = d/dz\,, \tag{13}$$

$$D\theta = 0, \tag{14}$$

$$DW = 0, \tag{15}$$

at $z = 0$ and

$$W = 0\,, \tag{16}$$

$$D\theta = -\mathrm{Bi}\,\theta\,, \tag{17}$$

$$D^2W + \mathrm{Ma}_p^{\mathrm{eff}} a^2\theta = 0\,, \tag{18}$$

at $z = 1$. The governing equations (11) and (12), subject to the boundary conditions (13) – (18), constitute an eigenvalue problem of order six can be solved exactly.

3. Method of Solutions

The resulting eigenvalue problem is solved exactly, in general, with $\mathrm{Ma}_p^{\mathrm{eff}}$ as an eigenvalue. Besides, an analytical expression for the critical Marangoni number is also obtained by regular perturbation method with wave number *a* as a perturbation parameter.

3.1. Exact Method

Since equation (11) is independent of θ and we want to look at the marginal non-oscillating case which we set $s = 0$, equation (11) can be directly solved to get the general solution in the form

$$W(z) = A_1 \sinh(az) + A_2 \cosh(az) + A_3 \sinh(az\alpha) + A_4 \cosh(az\alpha)\,, \tag{19}$$

where $A_1 - A_4$ are constants to be determined and $\alpha = \sqrt{\dfrac{1}{a^2\mathrm{Da}^{\mathrm{eff}}}}$. The parameter α plays a crucial role. When the permeability *K* and the Darcy number, $\mathrm{Da}^{\mathrm{eff}}$ becomes infinite, then the parameter α is equal to one. Using the boundary conditions (13), (15) and (16) to solve equation (11), we obtain

$$W(z) = A_1\left[\sinh(az) + \Delta_1 \cosh(az) + \Delta_2 \sinh(az\alpha) + \Delta_3 \cosh(az\alpha)\right] \tag{20}$$

where

$$\Delta_1 = -\frac{1}{\alpha}\left(\frac{\alpha\sinh(a) - \sinh(a\alpha)}{\cosh(a) - \cosh(a\alpha)}\right), \quad \Delta_2 = -\frac{1}{\alpha}, \text{ and } \Delta_3 = -\Delta_1. \tag{21}$$

The heat equation (12) has now to be solved defining their right-hand sides by the expressions given by equation (20). The solution obtained for θ using the boundary condition (14) is

$$\theta = A\left[\begin{aligned}&\left\{c^* + \frac{1 + 2a\alpha\beta(1)}{2a} z\right\}\cosh(az) - \left\{\frac{a\left(1 - a^2\mathrm{Da}^{\mathrm{eff}}\right)z + \alpha\left(1 - 3a^2\mathrm{Da}^{\mathrm{eff}}\right)\beta(1)}{2a^2\left(1 - a^2\mathrm{Da}^{\mathrm{eff}}\right)}\right\}\sinh(az)\\ &+\left\{\frac{\mathrm{Da}^{\mathrm{eff}}}{1 - a^2\mathrm{Da}^{\mathrm{eff}}}\right\}\cosh(a\alpha z) - \left\{\frac{\beta(1)\mathrm{Da}^{\mathrm{eff}}}{1 - a^2\mathrm{Da}^{\mathrm{eff}}}\right\}\sinh(a\alpha z)\end{aligned}\right] \tag{22}$$

where $\beta(1) = \dfrac{\cosh(a) - \cosh(\alpha a)}{\alpha\sinh(a) - \sinh(\alpha a)}$ and two unknown quantities *A* and *c** remain to be calculated. After using the last boundary conditions (17) and (18), we obtain the explicit value of $\mathrm{Ma}_p^{\mathrm{eff}}$ as a function of the wave number *a*, the porous Biot number, Bi and the Darcy number, $\mathrm{Da}^{\mathrm{eff}}$ that is given by

$$\mathrm{Ma}_p^{\mathrm{eff}} = \frac{2a\left(\alpha^2-1\right)^2\left[C\mathrm{Bi}+aS\right]\left(\alpha S\cosh(\alpha a)-C\sinh(\alpha a)\right)}{\left[\begin{array}{l}\left(\alpha^2-1\right)\left(a\alpha S-a\sinh(\alpha a)-\alpha\cosh(\alpha a)\right)-4\alpha C+\alpha\left(\alpha^2+3\right)C^2\cosh(\alpha a)\\ -\left(3\alpha^2+1\right)CS\sinh(\alpha a)\end{array}\right]} \tag{23}$$

where $C=\cosh(a), S=\sinh(a)$. From equation (23), it is seen that the Marangoni number whose explicit value is highly dependent on α and is thus a function of the permeability K.

3.2. Regular Perturbation Method

As the fluid is subjected to a uniform heat flux below and above ($\mathrm{Bi}=0$), the critical wave number is vanishing, $a_c \to 0$. When both boundaries are insulated to temperature perturbations, the long wavelength ($a_c \to 0$) approximation is usually invoked to find the solution for the eigenvalue problem in a closed form using regular perturbation technique with wave number a as a perturbation parameter. To study the validity of the small wave number analysis, the dependent variables in the porous layers are now expanded in powers of a^2 in the form

$$(W,\theta)=\sum_{i=0}^{N}\left(a^2\right)^i(W_i,\theta_i). \tag{24}$$

Substitute equation (24) into equations (11) – (18) yields a sequence of equations for the unknown functions $W_i(z)$ and $\theta_i(z)$.

At the zeroth order, equations (11) – (18) become, respectively,

$$D^2W_0-\mathrm{Da}^{\mathrm{eff}}\left(D^4W_0\right)=0, \tag{25}$$

$$D^2\theta_0+W_0=0, \tag{26}$$

at z = 0,

$$W_0=0, \tag{27}$$

$$D\theta_0=0, \tag{28}$$

$$DW_0=0, \tag{29}$$

at z = 1,

$$W_0=0, \tag{30}$$

$$D\theta_0=0, \tag{31}$$

$$D^2W_0=0. \tag{32}$$

The solution to the zeroth order for the equations (25) and (26) is given by

$$W_0=0 \text{ and } \theta_0=1. \tag{33}$$

The terms of order a2 are

$$D^2W_1-\mathrm{Da}^{\mathrm{eff}}\left(D^4W_1\right)=0, \tag{34}$$

$$D^2\theta_1+W_1=1, \tag{35}$$

at z = 0,

$$W_1=0, \tag{36}$$

$$DW_1=0, \tag{37}$$

$$D\theta_1=0, \tag{38}$$

at z = 1,

$$W_1=0, \tag{39}$$

$$D\theta_1=0, \tag{40}$$

$$D^2W_1+\mathrm{Ma}_p^{\mathrm{eff}}=0. \tag{41}$$

Using the symbolic algebra package MAPLE 11 to carry out much of the tedious algebraic manipulations, we obtained the critical Marangoni number, $\left(\mathrm{Ma}_p^{\mathrm{eff}}\right)_c$ as a function of Darcy number, $\mathrm{Da}^{\mathrm{eff}}$ is given by

$$\left(\mathrm{Ma}_p^{\mathrm{eff}}\right)_c = \frac{48B_1 + B_2 + 6\psi\mathrm{Da}^{\mathrm{eff}} + 8\psi^{-1}\left(\mathrm{Da}^{\mathrm{eff}}\right)^{3/2}}{B_3 B_4 \mathrm{Da}^{\mathrm{eff}}} \tag{42}$$

where

$$\psi = e^{\left(\frac{1}{\sqrt{\mathrm{Da}^{\mathrm{eff}}}}\right)},$$

$$B_1 = \left(\psi\mathrm{Da}^{\mathrm{eff}} + \psi\sqrt{\mathrm{Da}^{\mathrm{eff}}} + \psi^{-1}\sqrt{\mathrm{Da}^{\mathrm{eff}}}\right)\Big/24$$

$$B_2 = 4\left[\left(\mathrm{Da}^{\mathrm{eff}}\right)^2\left(\psi^2 - 1 - \psi + \psi^{-1}\right) - \left(\mathrm{Da}^{\mathrm{eff}}\right)^{3/2}\left(1+\psi^2\right)\right],$$

$$B_3 = -1 + 2\psi\mathrm{Da}^{\mathrm{eff}} - 2\sqrt{\mathrm{Da}^{\mathrm{eff}}} - 2\mathrm{Da}^{\mathrm{eff}},$$

$$B_4 = 4\mathrm{Da}^{\mathrm{eff}}\left(\psi - \psi^{-1}\right) - 4\left(\mathrm{Da}^{\mathrm{eff}}\right)^{3/2}\left(\psi + \psi^{-1} - 2\right) - \sqrt{\mathrm{Da}^{\mathrm{eff}}}\left(2 + \psi + \psi^{-1}\right).$$

4. Results and Discussion

The criterion for the onset of Marangoni convection in a one-layer system that is porous medium is investigated theoretically. The marginal curves in the (a, $\mathrm{Ma}_p^{\mathrm{eff}}$) plane are obtained by (23) where $\mathrm{Ma}_p^{\mathrm{eff}}$ is a function of the parameters a, Bi and $\mathrm{Da}^{\mathrm{eff}}$. For a given set of parameters, the critical Marangoni number for the onset of convection is defined as minimum of the global minima of marginal curve. Numerically calculated values of $\mathrm{Ma}_p^{\mathrm{eff}}$ and the corresponding of a are shown in Figure 2 and 3 below for a range of values of $\mathrm{Da}^{\mathrm{eff}}$ with Bi = 0 and Bi = 2 respectively.

Figure 2: Variation of $\mathrm{Ma}_p^{\mathrm{eff}}$ with a for different values of $\mathrm{Da}^{\mathrm{eff}}$ in the case of Bi = 0.

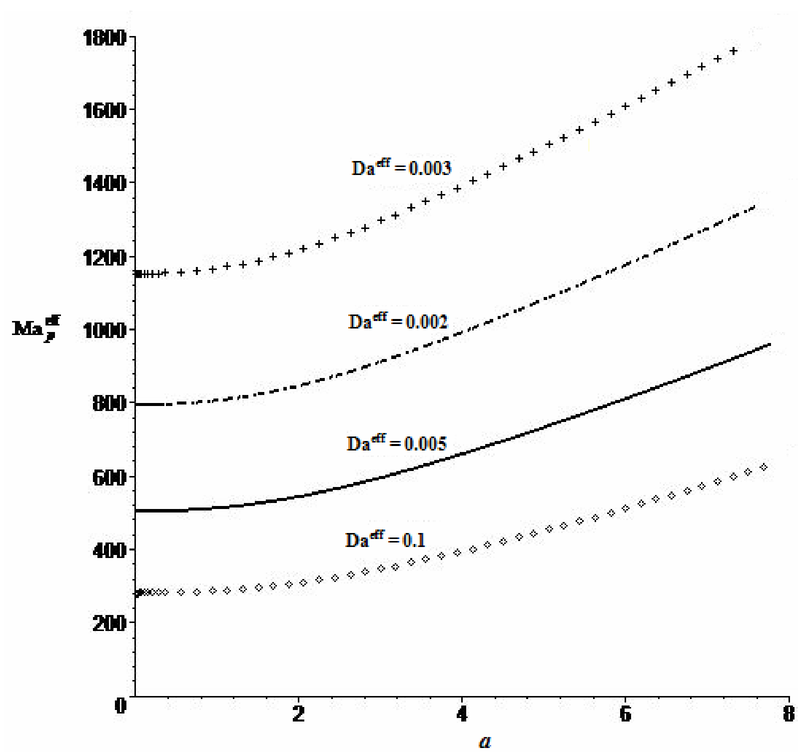
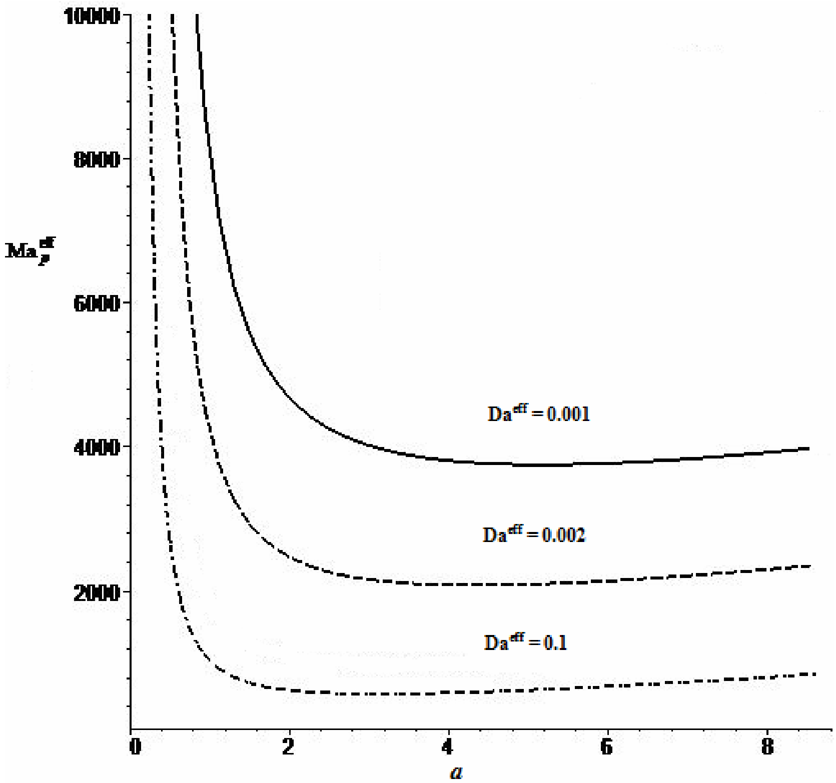


Figure 3: Variation of $\mathrm{Ma}_p^{\mathrm{eff}}$ with a for different values of $\mathrm{Da}^{\mathrm{eff}}$ in the case Bi = 2.



An inspection of the figures reveals that the critical Marangoni number increases as the Darcy number decreases and thus making the system more stable. This is due to the permeability decreases, one needs more driving forces to produce a given amount of flow.

At finite *a*, when the Darcy number, $\mathrm{Da}^{\mathrm{eff}} \rightarrow \infty$, a clear fluid ϕ tends towards 1 and *K* which is dependent upon the layer width *d*, becomes infinite, then the problem (11) – (18) reduce to the problem studied by Pearson (1958). When α equal to one, equation (23) will produce the explicit Marangoni function for an insulating rigid wall. By applying the l'Hospital rule, we derive the numerator and the denominator of equation (23) and we obtain

$$\lim_{\alpha \to 1} \mathrm{Ma}_p^{\mathrm{eff}} = \frac{8a\,[\,\mathrm{Bi}\,(a - CS) + a\,(aS - C^3 + C)}{a\,(a^2 + 2)\,S + (1 - a^2 - C^2)\,C} \tag{43}$$

The compatibility condition (43) produces for Darcy numbers, $\mathrm{Da}^{\mathrm{eff}}$ much larger than one, exactly the results derived by Pearson (1958). To verify our results, test computations have been performed for large value of Darcy number (i.e $\mathrm{Da}^{\mathrm{eff}}$ = 50 and $\mathrm{Da}^{\mathrm{eff}}$ = 100). The marginal stability curves obtained by (43) are plotted in Figure 4 and 5 for a range of values of Bi. As expected, the classical curve (1958) is reproduced and the critical Marangoni number attains the minimum, $\mathrm{Ma}_p^{\mathrm{eff}}$ = 48 at $a = 0$ in the case Bi = 0.

Figure 4: Variation of Ma_p^{eff} with a for different values of Bi in the case $Da^{eff} = 50$

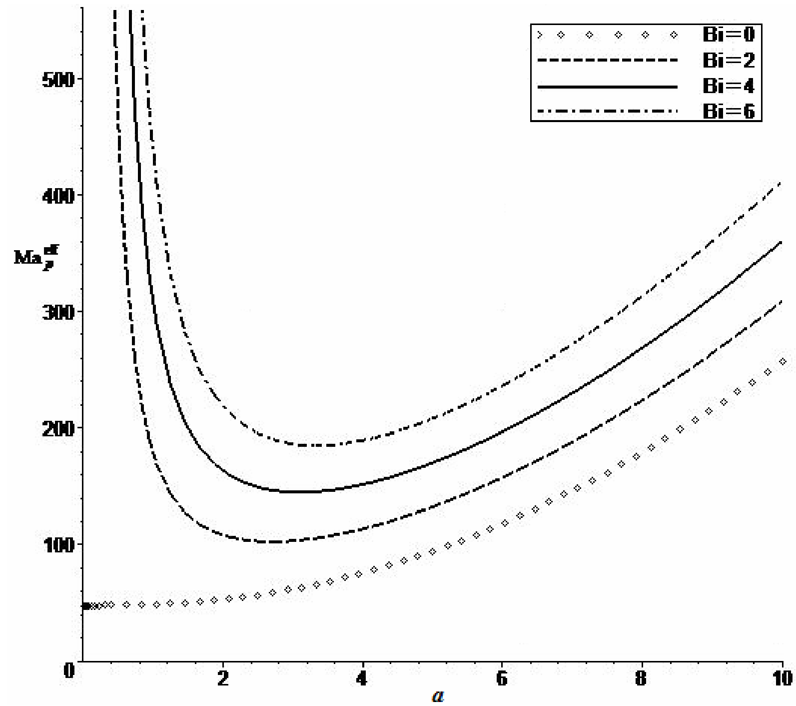
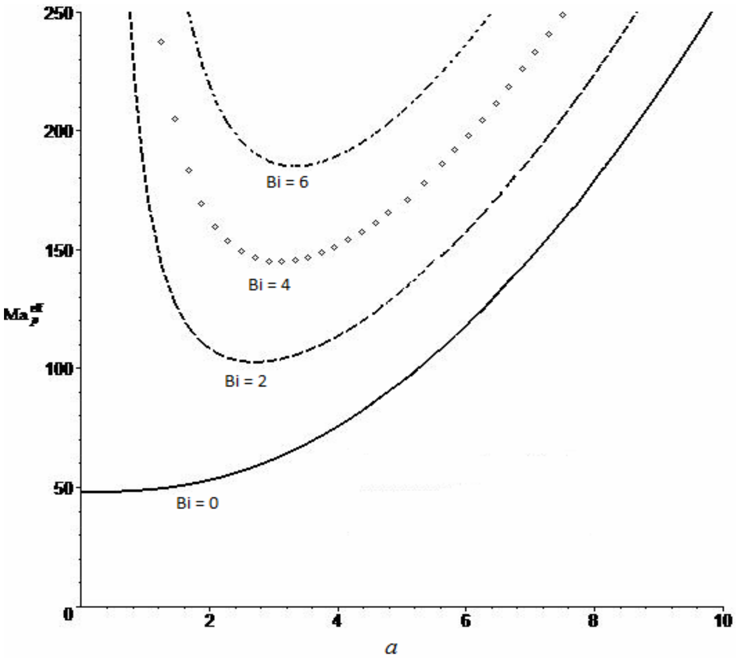
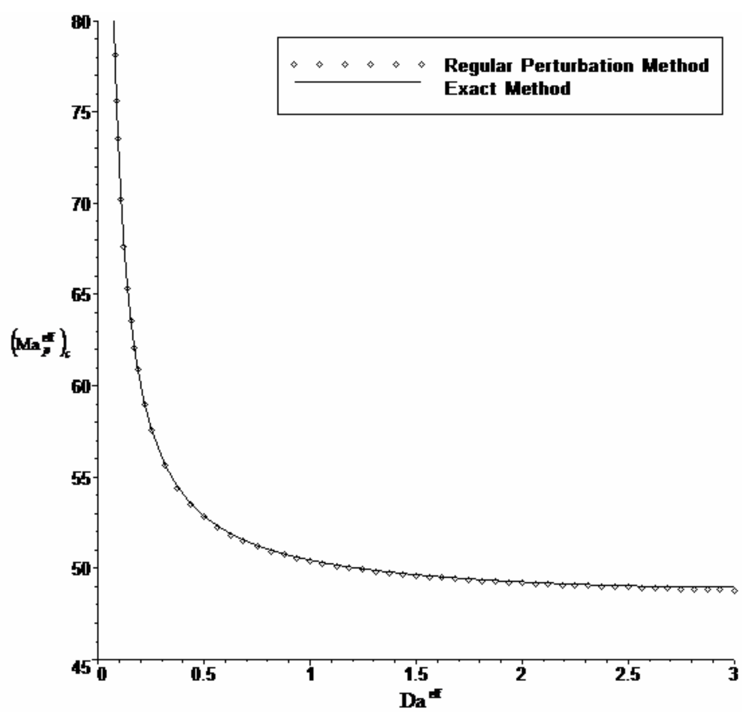


Figure 5: Variation of $\mathrm{Ma}_p^{\mathrm{eff}}$ with a for different values of Bi in the case $\mathrm{Da}^{\mathrm{eff}}=100$.



The critical Marangoni numbers obtained by the regular perturbation method, that is Eqn. (42) are shown in Fig. 6 by the symbol lines. It can be seen that there is an excellent agreement between the results of the exact method in the case of Bi = 0 and the regular perturbation method. This also proved that the regular perturbation method with wave number a as a perturbation parameter can conveniently be used in solving this convective instability problems.

Figure 6: Variation of $\left(\mathrm{Ma}_p^{\mathrm{eff}}\right)_c$ with $\mathrm{Da}^{\mathrm{eff}}$ for different values of Bi



5. Conclusions

The stability of Marangoni convection in a liquid saturated porous media in contact with air and subjected to an adverse gradient of temperature by considering the lower boundary is at the constant heat flux has been investigated. We derived for the first time analytical expression for the onset of steady Marangoni convection for the case of constant-heat-flux thermal condition at the lower boundary. We also use regular perturbation technique to obtain the asymptotic solutions of the long-wavelength. Results are presented for the Marangoni number, Darcy number and Biot number on the convective heat transfer.

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