UNIVERSITI PUTRA MALAYSIA

A HIERARCHICAL MATRIX ADAPTATION ON A FAMILY OF ITERATIVE METHOD FOR SOLVING POISSON EQUATION

NIK AMIR SYAFIQ NIK MAZLAN

IPM 2016 11

A HIERARCHICAL MATRIX ADAPTATION ON A FAMILY OF ITERATIVE METHOD FOR SOLVING POISSON EQUATION

By

NIK AMIR SYAFIQ NIK MAZLAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

June 2016

DEDICATIONS

To my family.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

A HIERARCHICAL MATRIX ADAPTATION ON A FAMILY OF ITERATIVE METHOD FOR SOLVING POISSON EQUATION

By

NIK AMIR SYAFIQ NIK MAZLAN

June 2016

Chairman : **Professor Mohamed Othman, PhD**
Faculty : **Institute for Mathematical Research**

This thesis deals with an adaptation of hierarchical matrix ($\mathscr{H}$-matrix) techniques in iterative methods for solving the Poisson equation, which is a representative of partial differential equations. The research examines different iterative techniques and ordering strategies in Gauss-Seidel method which are easy to implement on a computer. The $\mathscr{H}$-matrix techniques allows an efficient treatment of a dense matrix. This treatment will lead to less memory utilizations.

Three types of finite-difference approximations in the form of the full-sweep (FS), half-sweep (HS) and quarter-sweep (QS) approaches are considered in this research. An extension of this approach where a faster convergence rate can be achieved is by grouping the iteration points into a single iteration unit. Implemented with the finite-difference schemes mentioned above, this approach produces Explicit Group (EG), Explicit Decoupled Group (EDG) and Modified Explicit Group (MEG) methods. All of these iterative methods are yet to be implemented with $\mathscr{H}$-matrix.

The construction of an $\mathscr{H}$-matrix relies on a hierarchical partitioning of the dense matrix. To set up this partitioning, a so-called admissibility condition must be satisfied. Two types of admissibility conditions namely the *weak* admissibility and *standard* admissibility will be considered in this research. This will produce two different $\mathscr{H}$-matrix structures, $\mathscr{H}_W$- and $\mathscr{H}_S$-matrices, which consists of different memory utilizations.

The main objective of this thesis is to develop an adaptation of the $\mathscr{H}$-matrix structures with the iterative method. Both of these structures will be compared with each other. The $\mathscr{H}_W$-matrix should produce a more accurate solution with a faster execution time and utilizes less memory when compared to the $\mathscr{H}_S$-matrix.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk Master Sains

ADAPTASI MATRIKS BERHIERARKI PADA FAMILI KAEDAH LELARAN UNTUK MENYELESAIKAN PERSAMAAN POISSON

Oleh

NIK AMIR SYAFIQ NIK MAZLAN

Jun 2016

Pengerusi : **Profesor Mohamed Othman, PhD**
Fakulti : **Institut Penyelidikan Matematik**

Tesis ini berkaitan dengan adaptasi teknik matriks hierarki ($\mathscr{H}$-matriks) dalam kaedah lelaran untuk menyelesaikan persamaan Poisson, yang mewakili persamaan pembezaan separa. Kajian ini mengkaji teknik lelaran yang berbeza dan strategi susunan dalam kaedah Gauss-Seidel yang mudah untuk dilaksanakan pada komputer. Teknik-teknik $\mathscr{H}$-matriks membolehkan pergolakkan yang cekap kepada matriks padat. Pergolakkan ini akan membawa kepada penggunaan memori yang kurang.

Tiga jenis anggaran perbezaan terhingga dalam bentuk pendekatan sapuan penuh (FS), separuh sapuan (HS) dan suku sapuan (QS) yang dipertimbangkan dalam kajian ini. Lanjutan daripada pendekatan ini adalah di mana kadar penumpuan yang lebih cepat boleh dicapai dengan mengumpulkan titik lelaran ke dalam satu unit lelaran tunggal. Apabila dilaksanakan dengan skema perbezaan terhingga yang dinyatakan di atas, pendekatan ini menghasilkan kaedah Kumpulan Tak Tersirat (EG), Kumpulan Tak Tersirat Nyah Pasangan (EDG) dan Kumpulan Tak Tersirat Terubahsuai (MEG). Semua kaedah ini belum pernah dilaksanakan dengan $\mathscr{H}$-matriks.

Pembinaan sebuah $\mathscr{H}$-matriks bergantung kepada pembahagian hierarki matriks padat. Untuk menyediakan pembahagian ini, satu keadaan yang dipanggil syarat kebolehterimaan mesti berpuas hati. Terdapat dua jenis syarat kebolehterimaan iaitu kebolehterimaan *lemah* dan kebolehterimaan *biasa* akan dipertimbangkan dalam kajian ini. Ini akan menghasilkan dua struktur $\mathscr{H}$-matriks yang berbeza, matriks-matriks $\mathscr{H}_W$- dan $\mathscr{H}_S$-, yang terdiri daripada

penggunaan memori yang berbeza.

Objektif utama tesis ini adalah untuk membangunkan adaptasi daripada struktur $\mathscr{H}$-matriks dengan kaedah lelaran. Kedua-dua struktur akan dibandingkan antara satu sama lain. $\mathscr{H}_W$-matriks akan menghasilkan penyelesaian yang lebih tepat dengan masa pelaksanaan yang lebih cepat dan kurang menggunakan memori apabila dibandingkan dengan $\mathscr{H}_S$-matriks.

ACKNOWLEDGEMENTS

Praised be to the All Mighty Allah for of giving me health, strength, faith, endurance, guidance, to mentioned but nothing to be able to complete this thesis. May his peace and blessing be upon the Prophet Muhammad (SAW), who guides us to the right path we follow.

I would like to express my utmost gratitude to my parents Nik Mazlan Mamat and Sriviowarti Noerdin, and my brother Nik Amir Hasyim on whose upbringing and, continuous support and prayers I am rising up to this level.

My sincere gratitude goes to my supervisor Prof. Dr. Mohamed Othman and co-supervisor Assoc. Prof. Dr Norazak Senu for their tireless support, encouragement and motivation as well as their immense knowledge which translate to what I am concluding today. I am really grateful to both of them.

I am grateful to all staff and students of Institute for Mathematical Research, INSPEM, my lecturers and particularly the postgraduate unit. This goes along with the entire management of UPM for providing an amazing environment and limitless resources.

Finally I wish to extend my gratitude to all my colleagues that have, assisted me throughout this study either directly or indirectly.

I certify that a Thesis Examination Committee has met on 30 June 2016 to conduct the final examination of NIK AMIR SYAFIQ NIK MAZLAN on his thesis entitled "A HIERARCHICAL MATRIX ADAPTATION ON A FAMILY OF ITERATIVE METHOD FOR SOLVING POISSON EQUATION" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

Members of the Thesis Examination Committee were as follows:

Zanariah Abdul Majid, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Fudziah Ismail, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Jumat Sulaiman, PhD
Associate Professor
School of Science and Technology
Universiti Malaysia Sabah
(External Examiner)

ZULKARNAIN ZAINAL, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 23 August 2016

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science.

The members of the Supervisory Committee were as follows:

Mohamed Othman, PhD
Professor
Faculty of Computer Science and Information Technology
Universiti Putra Malaysia
(Chairperson)

Norazak Senu, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

____

BUJANG KIM HUAT, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

Declaration by graduate student

I hereby confirm that:

- this thesis is my original work;
- quotations, illustrations and citations have been duly referenced;
- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
- intellectual property from the thesis and copyright of thesis are fully-owned by Universiti Putra Malaysia, as according to the Universiti Putra Malaysia (Research) Rules 2012;
- written permission must be obtained from supervisor and the office of Deputy Vice-Chancellor (Research and Innovation) before thesis is published (in the form of written, printed or in electronic form) including books, journals, modules, proceedings, popular writings, seminar papers, manuscripts, posters, reports, lecture notes, learning modules or any other materials as stated in the Universiti Putra Malaysia (Research) Rules 2012;
- there is no plagiarism or data falsification/fabrication in the thesis, and scholarly integrity is upheld as according to the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) and the Universiti Putra Malaysia (Research) Rules 2012. The thesis has undergone plagiarism detection software.

Signature:________________________ Date:________________________

Name and Matric No: Nik Amir Syafiq Nik Mazlan, GS39791

Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________________
Name of Chairman of Supervisory Committee
Professor Mohamed Othman, PhD

Signature: ______________________________
Name of Member of Supervisory Committee
Associate Professor Norazak Senu, PhD

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

AOR	Accelerated Over-Relaxation
EDG	Explicit Decoupled Group
EG	Explicit Group
FS	Full Sweep
GS	Gauss-Seidel
HS	Half Sweep
HZL	Horizontal-Zebra-Line
IADE	Iterative Alternating Decomposition Explicit
MEDG	Modified EDG
MEG	Modified Explicit Group
MSOR	Modified SOR
NA	Natural
OMEG	Octo MEG
QS	Quarter Sweep
RB	Red-Black
Rtd51h	Rotated Five Point
SOR	Successive Over-Relaxation
Std51h	Standard Five Point 1h
Std52h	Standard Five Point 2h

CHAPTER 1

INTRODUCTION

1.1 Overview

In mathematical modeling of many physical phenomena, a greater number of processes can be represented through Partial Differential Equations (PDEs), particularly the Poisson equation, which involve functions of several variables. Generally, the Poisson equation is very complicated to solve analytically, hence numerical methods are applied. The operational speed of modern computers makes it possible to obtain a fast approximate solutions, taking into account a satisfactory computational error. For such equations, the theory of numerical methods has become one of the most sought-after research areas of modern science.

In 1998, Hackbush and his colleagues introduced a hierarchical ($\mathscr{H}$-)matrix technique. This technique acts as an efficient treatment of dense matrices as it stores them in a special data-sparse way in the sense that these matrices are described by only few data. This treatment will eventually reduce the memory utilization of the matrices. In the up coming chapter we will discuss more on the fundamentals of $\mathscr{H}$-matrix and how it is constructed.

1.2 Problem Statement

Many scientific problems, involving the Poisson equation, occur in fast and real time applications. Fast solutions with minimal absolute error require big sizes of discretization which in return requires a large-capacity computational memory. One of the earlier methods when solving the PDEs, in particular the Poisson equation, is using the finite-difference approximation where the full-sweep (FS) point iterative method was considered as a foundation for newer iterative methods. Recent research then discovered the half-sweep (HS) point, quarter-sweep (QS) point, Explicit Group (EG), Explicit Decoupled Group (EDG), and Modified Explicit Group (MEG) iterative methods. All of these methods were also studied with the basic Gauss-Seidel (GS) scheme. However, none of these methods have been applied with $\mathscr{H}$-matrix. This shows that there is a high demand in improving the performance of the methods for solving the Poisson equation but not many researches are concern about the memory utilization.

1.3 Objectives

The primary objective of this thesis is to propose a more data-sparse algorithms, which will outstrip the existing methods by memory utilizations. The objective is accomplished through research on the $\mathscr{H}$-matrix technique. This technique will be adapted to all of the point and group iterative methods mentioned above. The performance of the new algorithms comprises such characteristics as accuracy, computational time and memory utilization.

1.4 Research Scope

The thesis is focusing on adapting the $\mathscr{H}$-matrix technique onto the iterative method. The solution domain will be discretize using the finite difference approximations and solved using the FS point, HS point, QS point, EG, EDG and MEG iterative methods based on GS scheme. This thesis will only consider domains with group cases which is why the discretized domain for EG and EDG iterative methods will be of the same size. This will differ from the discretized domain for MEG iterative method. The discretized domain for point iterative methods will follow their respective group iterative methods according to their sweeping approach. This will be explained further in the upcoming chapters. In the following literature, it will show that there are two types of $\mathscr{H}$-matrix structures that can be constructed. Both of these structures will be adapted on all of the mentioned points and group iterative methods.

1.5 Research Methodology

In order to develop a new adaptive $\mathscr{H}$-matrix on iterative methods for solving the Poisson equation, there are several research methodology steps to follow, which are

1. Literature review on:
   - the Poisson equation and general theory of PDEs;
   - point and group iterative methods;
   - the fundamentals of $\mathscr{H}$-matrix.
2. Implementation of the recent methods on the solution of Poisson equation and on $\mathscr{H}$-matrix for benchmarking, such as:
   - FS, HS and QS point iterative methods;
   - EG, EDG and MEG iterative methods;

- different ordering strategies for all of the methods mentioned above;
- an $\mathscr{H}$-matrix structure proposed by Börm et al. (2003) ($\mathscr{H}_S$-).

3. Constructing the $\mathscr{H}$-matrix structures with respect to:
   - standard admissibility conditions which will result in an $\mathscr{H}_S$-matrix;
   - weak admissibility conditions which will result in an $\mathscr{H}_W$-matrix.
4. Proposed a new algorithm for adapting an $\mathscr{H}_W$-matrix structure with:
   - FS point and EG iterative methods;
   - HS point and EDG iterative methods;
   - QS point and MEG iterative methods.
5. Implementation of different ordering strategies with the new algorithm.
6. Experiments to benchmark the new method with the iterative methods and $\mathscr{H}_S$-matrix studied in Step 2 with different grid sizes.

The ($\mathscr{H}_S$-) and ($\mathscr{H}_W$-) notation will be futher explained in the next chapter. The computer hardware, software and initial values that will be used to conduct the experiments is shown in Table 1.1.

Figure 1.1 shows the research methodology framework of this thesis.

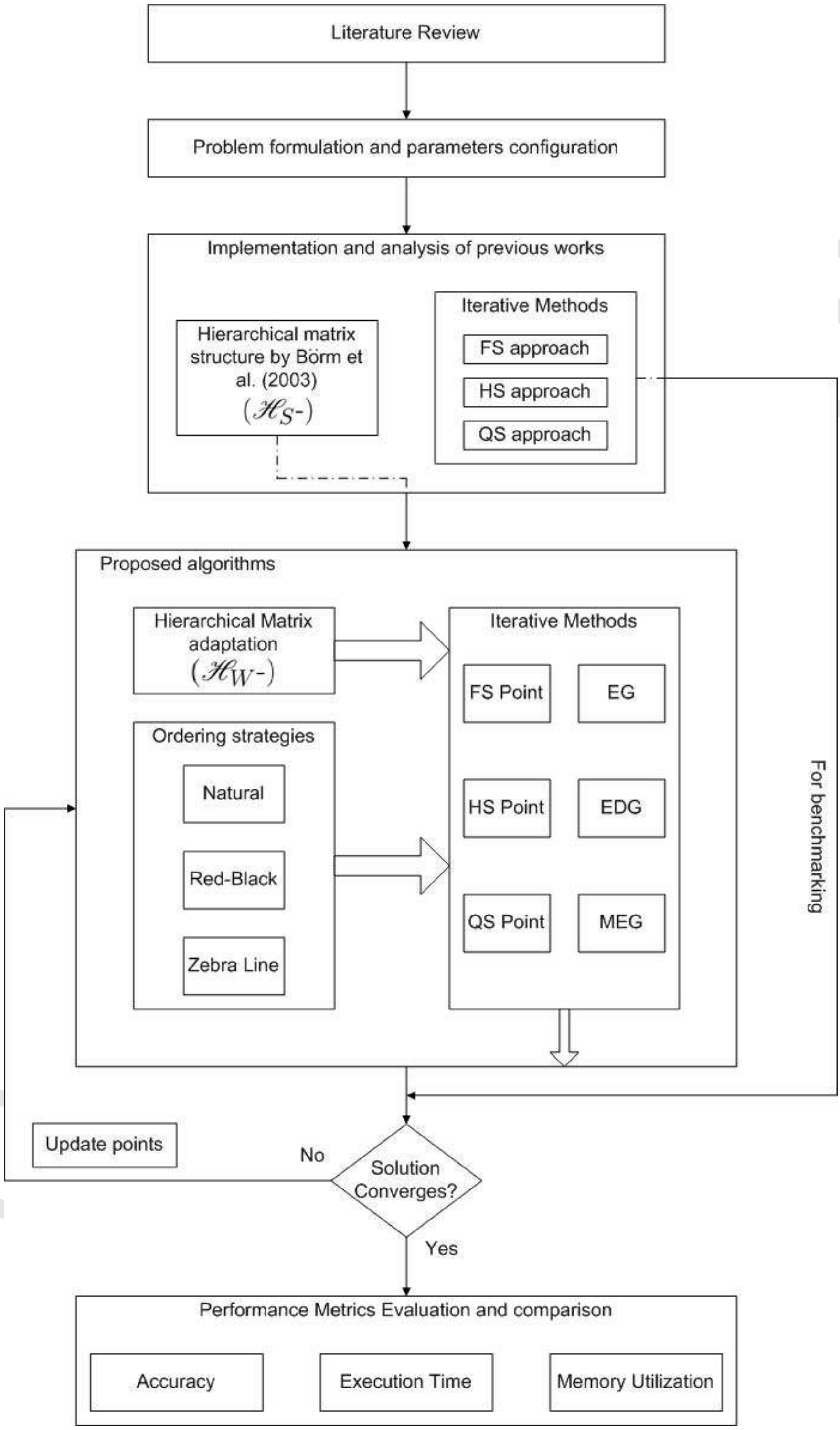

Figure 1.1: The research methodology framework.

Table 1.1: The computer hardware, software and initial values used to conduct the experiments.

Test Problem	The Poisson equation: $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = (x^2 + y^2)e^{xy}, \quad (x, y) \in \Omega = [0, 1] \times [0, 1]$, subject to the Dirichlet boundary conditions and satisfying the exact solution $u(x, y) = e^{xy}$.
Hardware	Intel(R) Core(TM)2 Duo, 3.16 GHz CPU, 4.00 GB(RAM), 32-bit OS, Windows 7.
Software	Matlab R2011a, 32-bit, for Windows.
Matrix Sizes	$n = 16, 32, 64, 128$ for FS point, HS point, EG and EDG iterative methods. $n = 17, 33, 65, 129$ for QS point and MEG iterative methods.
Performance metrics	Total memory cost: Measured at computer's task manager Execution time: Time taken for new method to converge, measured in MATLAB software. Accuary: Average absolute error $= \dfrac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\lvert u_{i,j} - e_{i,j}\rvert$, where $e_{i,j}$ is the exact solution.
Error tolerance	$\varepsilon = 10^{-10}$ which is similar to researches in Othman et al. (2000) and Othman and Abdullah (2000b).

1.6 Thesis Structure

The remaining chapters of this thesis are organized as follows. Chapter 2 presents the Poisson equation and the approximate solution of it. An introduction to point and group iterative methods to solve the discrete Poisson equation is given. Moreover, the fundamentals of $\mathscr{H}$-matrix technique are also considered. Recent research related to these subjects of this thesis is reviewed and analyzed at the end of the chapter.

Chapter 3 is devoted to the derivation and formulation of the FS GS and EG GS iterative methods and their $\mathscr{H}$-matrix adaptations. Several ordering strategies are applied with the new method. To benchmark the new method, an experimental comparison with the mentioned point iterative methods is given. An experiment to determine the optimal ordering strategy is conducted and discussed.

The derivation of the HS GS and EDG GS iterative methods with their $\mathscr{H}$-matrix adaptations is given in Chapter 4. The chapter is then continued with description and comparison of different ordering strategies. In the experimental part of this chapter, the new method is compared with the mentioned group iterative methods. The results are presented and the performance of the method is evaluated.

Chapter 5 is focused on the derivation of the QS GS and MEG GS iterative method with their $\mathscr{H}$-matrix adaptations. The chapter is then continued with description and comparison of different ordering strategies. An experiment to determine the optimal ordering strategy is conducted and discussed.

Finally, conclusions and recommendations for future research are given in Chapter 6.

REFERENCES

Akhir, M. K. M., Othman, M., Majid, Z. A., Suleiman, M., and Sulaiman, J. (2012a). Four Point Explicit Decoupled Group Iterative Method Applied to Two-Dimensional Helmholtz Equation. *International Journal of Mathematical Analysis*, 6(20):963–974.

Akhir, M. K. M., Othman, M., Sulaiman, J., Majid, Z. A., and Suleiman, M. (2011a). The Four Point-EDGMSOR Iterative Method for Solution of 2D Helmholtz Equations. In Manaf, A. A., Sahibuddin, S., Ahmad, R., Daud, S. M., and Qawasmeh, E. E., editors, *Informatics Engineering and Information Science*, volume 253, pages 218–227. Springer Berlin Heidelberg.

Akhir, M. K. M., Othman, M., Sulaiman, J., Majid, Z. A., and Suleiman, M. (2011b). Half-Sweep Modified Successive OverRelaxation for Solving Two-Dimensional Helmholtz Equations. *Australian Journal of Basic and Applied Sciences*, 5(12):3033–3039.

Akhir, M. K. M., Othman, M., Sulaiman, J., Majid, Z. A., and Suleiman, M. (2012b). The Solution of 2D Helmholtz Equations by Modified Explicit Group Iterative Method. In *International Conference on Advances in Computing, Control, and Telecommunication Technologies*, pages 70–77. Elsevier.

Akhir, M. K. M., Othman, M., Suleiman, M., Majid, Z. A., Sulaiman, J., Muthuvalu, M. S., and Aruchuan, E. (2014). QSMSOR Iterative Method for the Solution of 2D Homogeneous Helmholtz Equations. *International Journal of Mathematical Analysis*, 8(51):2493–2505.

Akhir, M. K. M., Othman, M., Suleiman, M., and Sulaiman, J. (2012c). A New Octo Modified Explicit Group Iterative Method for the Solution of 2D Elliptic PDEs. *Applied Mathematical Sciences*, 6(31):1505–1524.

Akhir, M. K. M., Sulaiman, J., Othman, M., Majid, Z. A., Muthuvalu, M. S., and Aruchuan, E. (2015). Application of Four-Point MEGMSOR Method for the Solution of 2D Helmholtz Equations. *International Journal of Mathematical Analysis*, 9(38):1847–1862.

Ali, N. H. M. and Fu, N. K. (2007). Modified Explicit Decoupled Group Method in the Solution of 2D Elliptic PDEs. In Demiralp, M., Udriste, C., Bognar, G., Soni, R., and Nassar, H., editors, *MATH'07 Proceedings of the 12th WSEAS International Conference on Applied Mathematics*, pages 162–167. World Scientific and Engineering Academy and Society (WSEAS).

Banjai, L. and Hackbusch, W. (2008). Hierarchical Matrix Techniques for Low- and High-Frequency Helmholtz Problems. *IMA Journal of Numerical Analysis*, 28:46–79.

Baur, U. and Benner, P. (2006). Factorized Solution of Lyapunov Equations Based on Hierarchical Matrix Arithmetic. *Computing*, 78:211–234.

Bebendorf, M. and Krämer, F. (2012). Hierarchical Matrix Preconditioners for Low-Frequency-Full-Maxwell Simulations. In *Proceedings of the IEEE*, volume 101 of *2*, pages 423–433. IEEE.

Benedetti, I., Aliabadi, M. H., and Davì, G. (2008). A Fast 3D Dual Boundary Element Method Based on Hierarchical Matrices. *International Journal of Solids and Structures*, 45:2355–2376.

Benner, P. and Mach, T. (2010). On the QR Decomposition of $\mathscr{H}$-Matrices. *Computing*, 88:111–129.

Benner, P. and Mach, T. (2013). The LR Cholesky Algorithm for Symmetric Hierarchical Matrices. *Linear Algebra and its Applications*, 439:1150–1166.

Börm, S. (2010). Approximation of Solution Operators of Elliptic Partial Differential Equations by $\mathscr{H}$- and $\mathscr{H}^2$-Matrices. *Numerische Mathematik*, 115:165–193.

Börm, S., Grasedyck, L., and Hackbusch, W. (2003). Introduction to Hierarchical Matrices with Applications. *Engineering Analysis with Boundary Elements*, 27:405–422.

Börm, S., Grasedyck, L., and Hackbusch, W. (2006). Lecture notes in Hierarchical Matrices.

Borne, S. L. (2005). Hierarchical Matrices for Convection-Dominated Problems. In Barth, T. J., Griebel, M., Keyes, D. E., Nieminen, R. M., Roose, D., Schlick, T., Kornhuber, R., Hoppe, R., Périaux, J., Pironneau, O., Widlund, O., and Xu, J., editors, *Domain Decomposition Methods in Science and Engineering*, volume 40, pages 631–638. Springer Berlin Heidelberg.

Borne, S. L. (2008). Hierarchical Matrix Preconditioners for the Oseen Equations. *Computing and Visualization in Science*, 11:147–157.

Chen, H. B., Li, Y. C., Tan, S. X. D., Huang, X., Wang, H., and Wong, N. (2015). $\mathscr{H}$-Matrix-Based Finite-Element-Based Thermal Analysis for 3D ICs. *ACM Transactions on Design Automation of Electronic Systems*, 20(4):47:1–25.

Engquist, B. and Ying, L. (2011). Sweeping Preconditioner for the Helmholtz Equation: Hierarchical Matrix Representation. *Communications on Pure and Applied Mathematics*, 64:697–735.

Fauzi, N. I. M. and Sulaiman, J. (2012). Half-Sweep Modified Successive Over Relaxation Method for Solving Second Order Two-Point Boundary Value Problems Using Cubic Spline. *International Journal of Contemporary Mathematical Sciences*, 7(32):1579–1589.

Grasedyck, L. and Hackbusch, W. (2003). Construction and Arithmetics of $\mathscr{H}$-Matrices. *Computing*, 70:295–334.

Grasedyck, L., Hackbusch, W., and Kriemann, R. (2008). Performance of $\mathscr{H}$-LU Preconditioning for Sparse Matrices. *Computational Methods in Applied Mathematics*, 8(4):336–349.

Grytsenko, T. and Galybin, A. N. (2010). Numerical Analysis of Multi-Crack Large-Scale Plane Problems with Adaptive Cross Approximation and Hierarchical Matrices. *Engineering Analysis with Boundary Elements*, 34:501510.

Hackbusch, W. (1999). A Sparse Matrix Arithmetic Based on $\mathscr{H}$-Matrices. Part I: Introduction to $\mathscr{H}$-Matrices. *Computing*, 62:89–108.

Hackbusch, W., Grasedyck, L., and Börm, S. (2002). An Introduction to Hierarchical Matrices. *Mathematica Bohemica*, 127(2):229–241.

Hackbusch, W. and Khoromskij, B. N. (2000). A Sparse $\mathscr{H}$-Matrix Arithmetic. Part II: Application to Multi-Dimensional Problems. *Computing*, 64:21–47.

Hackbusch, W., Khoromskij, B. N., and Kriemann, R. (2004). Hierarchical Matrices Based on a Weak Admissibility Criterion. *Computing*, 73:207–243.

Hasan, M. K., Sulaiman, J., and Othman, M. (2008). Implementation of Red Black Strategy to Quarter Sweep Iteration for Solving First Order Hyperbolic Equations. In *Information Technology, 2008. ITSim 2008. International Symposium*, volume 3, pages 1–5. IEEE.

Hoe, N. Y. and Hasan, M. K. (2013). Investigation of Steady State Problems via Quarter Sweep Schemes. *Sains Malaysiana*, 42(6):837–844.

Izadi, M. (2012). *Hierarchical Matrix Techniques on Massively Parallel Computers*. PhD thesis, Fakultät für Mathematik und Informatik, Universität Leipzig.

Kazempour, M. and Gürel, L. (2013). Solution of Low-Frequency Electromagnetics Problems Using Hierarchical Matrices. In *Computational Electromagnetics Workshop (CEM), 2013*, pages 9–10. IEEE.

Krämer, F., Bebendorf, M., Ostrowski, J., and Hiptmair, R. (2009). Hierarchical Matrix Preconditioners for Eddy Current Problems. In *Electromagnetics in Advanced Applications, 2009. ICEAA '09. International Conference*, volume 1, pages 724–727. IEEE.

Kriemann, R. (2005). Parallel $\mathscr{H}$-matrix Arithmetics on Shared Memory Systems. *Computing*, 74:273–297.

Lin, L., Lu, J., and Ying, L. (2011). Fast Construction of Hierarchical Matrix Representation from Matrix-Vector Multiplication. *Journal of Computational Physics*, 230:4071–4087.

Muthuvalu, M. S. and Sulaiman, J. (2010). Quarter-Sweep Iteration for First Kind Linear Fredholm Integral Equations. *International Journal of Open Problems in Computer Science*, 3(3):357–367.

Nusi, N. M. and Othman, M. (2009). Half- and Quarter-Sweeps Implementation of Finite-Difference Time-Domain Method. *Malaysian Journal of Mathematical Sciences*, 3(1):45–53.

Ohtania, M., Hirahara, K., Takahashi, Y., Hori, T., Hyodo, M., Nakashima, H., and Iwashita, T. (2011). Fast Computation of Quasi-Dynamic Earthquake Cycle Simulation with Hierarchical Matrices. *Procedia Computer Science*, 4:1456–1465.

Othman, M. and Abdullah, A. R. (2000a). An Efficient Four Points Modified Explicit Group Poisson Solver. *International Journal of Computer Mathematics*, 72(2):203–217.

Othman, M. and Abdullah, A. R. (2000b). An Efficient Parallel Quarter-sweep Point Iterative Algorithm for Solving Poisson Equation on SMP Parallel Computer. *Pertanika Journal of Science & Technology*, 8(2):161–174.

Othman, M. and Abdullah, A. R. (2004). A Parallel New High Order Iterative Algorithm on Shared Memory Multiprocessors Parallel Computer. *Jurnal Teknologi Maklumat & Multimedia*, 1:53–61.

Othman, M., Abdullah, A. R., and Evans, D. J. (2006). A Parallel Four Points Modified Explicit Group Algorithm on Shared Memory Multiprocessors. *International Journal of Parallel, Emergent and Distributed Systems (formerly known as Parallel Algorithms and Applications)*, 19(1):1–9.

Othman, M., Sulaiman, J., and Abdullah, A. R. (2000). A Parallel Halfsweep Multigrid Algorithm on the Shared Memory Multiprocessors. *Malaysian Journal of Computer Science*, 13(2):1–26.

Ovall, J. S. (2007). Hierarchical Matrix Techniques for a Domain Decomposition Algorithm. *Computing*, 80:287–297.

Pan, C., Zhang, M., Liu, L., and Bo, Y. (2014). An Adaptive Cross Approximation Recompressed Degenerate Kernel Algorithm with Radial Bases for Scattering from PEC Objects. In *Antennas and Propagation (APCAP), 2014 3rd Asia-Pacific Conference*, pages 1006–1008. IEEE.

Rakhimov, S. (2010). New Quarter Sweep Based Accelerated Over-Relaxation Iterative Algorithms and their Parallel Implementations in Solving the 2D Poisson Equation. Master's thesis, Universiti Putra Malaysia.

Schmitz, P. G. (2010). *Fast Direct Algorithms for Elliptic Equations via Hierarchical Matrix Compression.* PhD thesis, University of Texas, Austin.

Schmitz, P. G. and Ying, L. (2014). A Fast Nested Dissection Solver for Cartesian 3D Elliptic Problems using Hierarchical Matrices. *Journal of Computational Physics*, 258:227–245.

Sulaiman, J., Hasan, M. K., and Othman, M. (2004a). The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) Method for Diffusion Equation. In Zhang, J., He, J. H., and Fu, Y., editors, *Computational and Information Science*, volume 3314, pages 57–63. Springer Berlin Heidelberg.

Sulaiman, J., Hasan, M. K., Othman, M., and Karim, S. A. A. (2012). MEG-SOR Iterative Method for the Triangle Element Solution of 2D Poisson Equations. In *Procedia Computer Science*, volume 1 of *1*, pages 377–385. Elsevier.

Sulaiman, J., Hasan, M. K., Othman, M., and Yaacob, Z. (2009a). Quarter-Sweep Arithmetic Mean Iterative Algorithm to Solve Fourth-Order Parabolic Equations. In *Electrical Engineering and Informatics, 2009. ICEEI '09. International Conference*, volume 1, pages 194–198. IEEE.

Sulaiman, J., Othman, M., and Hasan, M. K. (2004b). Quarter-Sweep Iterative Alternating Decomposition Explicit Algorithm Applied to Diffusion Equations. *International Journal of Computer Mathematics*, 81(12):1559–1565.

Sulaiman, J., Othman, M., and Hasan, M. K. (2007a). Red-Black EDG-SOR Iterative Method Using Triangle Finite Element Approximation for 2D Poisson Equations. In Gervasi, O. and Gavrilova, M. L., editors, *Computational Science and Its Applications - ICCSA 2007*, volume 4707, pages 298–308. Springer Berlin Heidelberg.

Sulaiman, J., Othman, M., and Hasan, M. K. (2007b). Red-Black Half-Sweep Iterative Method Using Triangle Finite Element Approximation for 2D Poisson Equations. In Shi, Y., Albada, G. D. V., Dongarra, J., and Sloot, P. M. A., editors, *Computational Science - ICCS 2007*, volume 4487, pages 326–333. Springer Berlin Heidelberg.

Sulaiman, J., Othman, M., and Hasan, M. K. (2008a). Half-Sweep Algebraic Multigrid (HSAMG) Method Applied to Diffusion Equations. In Bock, H. G., Kostina, E., Phu, H. X., and Rannacher, R., editors, *Modeling, Simulation and Optimization of Complex Processes*, pages 547–556. Springer Berlin Heidelberg.

Sulaiman, J., Othman, M., and Hasan, M. K. (2009b). A New Quarter Sweep Arithmetic Mean (QSAM) Method to Solve Diffusion Equations. *Chamchuri Journal of Mathematics*, 1(2):93–103.

Sulaiman, J., Othman, M., and Hasan, M. K. (2009c). Nine Point-EDGSOR Iterative Method for the Finite Element Solution of 2D Poisson Equations. In Gervasi, O., Taniar, D., Murgante, B., Laganà, A., Mun, Y., and Gavrilova, M. L., editors, *Computational Science and Its Applications - ICCSA 2009*, volume 5592, pages 764–774. Springer Berlin Heidelberg.

Sulaiman, J., Othman, M., and Hasan, M. K. (2010). MEGSOR Iterative Scheme for the Solution of 2D Elliptic PDEs. *International Journal of Mathematical, Computational, Physical, Electrical and Computer Engineering*, 4(2):264–270.

Sulaiman, J., Saudi, A., Abdullah, M. H., Hasan, M. K., and Othman, M. (2008b). Quarter-Sweep Arithmetic Mean Algorithm for Water Quality

Model. In *Information Technology, 2008. ITSim 2008. International Symposium*, volume 3, pages 1–5. IEEE.

Wan, T., Du, L., and Hu, Y. (2015). Fast Simulation of Microwave Devices via a Data-sparse and Explicit CN-FETD Method. In *Advanced Materials and Processes for RF and THz Applications (IMWS-AMP), 2015 IEEE MTT-S International Microwave Workshop Series*, pages 1–3. IEEE.

Wan, T., Hu, X. Q., and Chen, R. S. (2011). Hierarchical LU Decomposition Based Direct Method with Improved Solution for 3D Scattering Problems in FEM. *Microwave and Optical Technology Letters*, 53(8):1687–1694.

Yaacob, Z. and Hasan, M. K. (2014). Family of Gauss-Seidel Method for Solving 2D Reynolds Equation in Hydrodynamic Lubrication Problem. In *International Conference of Recent Trends in Information and Communication Technologies*, pages 361–369.

Yang, F. (2008). *Construction and Application of Hierarchical Matrix Preconditioners*. PhD thesis, University of Iowa.

Yousif, W. S. and Martins, M. M. (2008). Explicit De-couple Group AOR Method for Solving Elliptic Partial Differential Equations. *Neural, Parallel and Scientific Computations*, 16(4):531–542.