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NUMERICAL SOLUTION OF INTEGRALS AND NONLINEAR INTEGRAL EQUATIONS BY WAVELETS

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By

MOHAMMAD HASAN BIN ABDUL SATHAR

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2016

DEDICATIONS

I would like to dedicate this thesis to my loving family
To my parents, Mum and Dad for their support and guidance.
To my LOVELY Wife, Aznira
For her Patience, Love and Support.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chair: Anvarjon Ahmedov, PhD
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In recent years, wavelets have found their way into many different fields of science and engineering. This is because wavelets possess several important properties, such as orthogonality, compact support, exact representation of polynomials at certain degree and the ability to represent functions on different levels of resolution. In this thesis, new methods based on wavelet expansion are considered to solve problems arising in approximation of functions, integrals and integral equations. Mainly we deal with the numerical approximations by Haar wavelets, linear Legendre multi-wavelets and Chebyshev wavelets.

Numerous work has been done to solve numerical integration in terms of quadrature rule. Regardless of the simplicity of quadrature rule, there exist some drawbacks. In order to overcome these existing drawbacks, new methods based on Haar wavelets and linear Legendre multi-wavelets are proposed to obtain numerical solutions of double, triple and N dimensional integrals. Main advantages of these methods are its efficiency and simple applicability. Furthermore, the error analysis for double and triple integral where functions belong in the class of $C^2(R)$ and $C^3(R)$ is worked out to show the efficiency of the methods.

The second part of the thesis focus on obtaining error estimations for the approximation by Haar and Chebyshev wavelets and linear Legendre multi wavelets. Error estimations are established for functions from Holder $H^s[0,1]$ and Holder Zygmund $C^{m.\alpha}[0,1]$ classes. Therefore functions can be consider in a wider class compared to the previous work. The smoothness of functions from Holder and Holder Zygmund classes is reflected in the error estimation.

Finally, new numerical techniques to solve nonlinear Fredholm and Volterra integral equation of the second kind by Haar and Chebyshev wavelets are developed. These methods reduce the nonlinear integral equation to a linear algebraic system of equation. Newton Kantorovich method is implemented to reduce the nonlinear integral equations into linear integral equations. This allows us to establish approximation solutions for nonlinear integrals. The comparison of error and accuracy between other methods are shown.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENYELESAIAN BERANGKA BAGI MASALAH KAMIRAN DAN PERSAMAAN KAMIRAN TAK LINAR DENGAN MENGGUNAKAN KAEDAH WAVELETS

Oleh

MOHAMMAD HASAN BIN ABDUL SATHAR

April 2016

Pengerusi: Anvarjon Ahmedov, PhD
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Sejak kebelakangan ini, wavelets mendapat sambutan dalam pelbagai bidang sains dan kejuruteraan. Ini kerana wavelets mempunyai beberapa ciri-ciri penting, seperti ortogonal, sokongan padat, keupayaan mewakili polinomial pada darjah tertentu dan keupayaan untuk mewakili fungsi pada tahap resolusi yang berbeza. Dalam tesis ini, kaedah baru berasaskan pengembangan wavelet dipertimbangkan untuk menyelesaikan masalah yang wujud dalam penghampiran fungsi, kamiran dan persamaan kamiran. Kami memberi lebih penekanan terhadap penghampiran berangka oleh wavelet Haar, wavelet linear Legendre pelbagai dan wavelet Chebyshev.

Berbagai kajian yang telah dilakukan sebelum ini untuk menyelesaikan masalah kamiran berangka dalam sebutan petua kuadratur. Walaupun dengan segala kemudahan yang ada dengan menggunakan kaedah kuadratur, wujud beberapa kelemahan. Bagi mengatasi kelemahan yang sedia ada ini, kaedah baru berdasarkan wavelet Haar dan linear Legendre dicadangkan untuk mendapatkan penyelesaian kamiran berangka berdimensi dua, tiga dan N. Kelebihan utama kaedah ini adalah kecekapan dan kebolehgunaan yang mudah. Tambahan pula, analisis ralat untuk kamiran berdimensi dua dan tiga di mana fungsi dalam golongan kelas C^2 (R) dan $C^3(R)$ diselesaikan untuk menunjukkan kecekapan kaedah tersebut.

Bahagian kedua tesis ini tetumpu untuk mendapatkan anggaran ralat bagi penghampiran oleh wavelet Haar dan wavelet Chebyshev dan wavelet linear Legendre pelbagai. Anggaran ralat yang diperolehi adalah untuk fungsi dari kelas Holder $H^s[0,1]$ dan Holder Zygmund $C^{m.\ alpha}[0,1]$. Oleh sebab itu, fungsi yang digunakan boleh diambill dari kelas fungsi yang lebih luas berbanding dengan kerja sebelumnya. Kelicinan fungsi dari kelas Holder dan Holder Zygmund terserlah di dalam ralat anggaran.

Akhir sekali, teknik berangka baru untuk menyelesaikan masalah persamaan kamiran tak linear Fredholm dan Volterra dari jenis yang kedua oleh Haar dan Chebyshev wavelet dibentuk. Kaedah-kaedah ini menurunkan persamaan kamiran tak linear kepada sistem persamaan linear algebra. Kaedah Newton Kantorovich dilaksanakan

untuk menurunkan persamaan kamiran tak linear ke dalam persamaan kamiran linear. Ini membolehkan kami untuk membentuk penyelesaian hampiran untuk kamiran tak linear. Perbandingan ralat dan ketepatan antara kaedah lain akan ditunjukkan.

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My deepest gratitude and love to my parents and wife for their support, patience and prayers for my success.

Finally I will like to thank to all of my friend for their help and moral support during my study in Universiti Putra Malaysia.

I certify that a Thesis Examination Committee has met on 6 April 2016 to conduct the final examination of Mohammad Hasan bin Abdul Sathar on his thesis entitled "Numerical Solution of Integrals and Nonlinear Integral Equations by Wavelets" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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LIST OF ABBREVIATIONS

MRA	Multiresolution Analysis
$C[a,b]$	Continuous space in the interval $a \leq x \leq b$
$C^{m,\alpha}[0,1]$	Holder-Zygmund class in the interval $0 \leq x \leq 1$
ONB	Orthonormal Basis
$H^s[0,1]$	Holder class in the interval $0 \leq x \leq 1$
HW	Haar Wavelets
DHW	Definite Haar Wavelets
LLMW	Linear Legendre Multi Wavelets
NKHW	Newton-Kantrovich Haar Wavelets
NKCW	Newton-Kantrovich Chebyshev Wavelets

CHAPTER 1

INTRODUCTION

The research in this thesis introduce new methods based on wavelet expansion to solve problem arising in approximation of functions, integrals and integral equations. The significant part of the research depends on the new technique of the study of special wavelets and their multiresolution analysis (MRA). This technique can be applied to many areas of science and engineering. In recent years numerical approximation by wavelets becoming one of the most popular, powerful and reliable tool in this area.

1.1 Motivation

The wavelets are designed to approximate the functions and have an application in many fields of science, such as to store fingerprint electronically, signal processing, compress data and many more. Many statistical phenomena have wavelets structure and the theory of wavelets help to better understand the reason for this phenomena. Wavelets expansion theory is different from Fourier analysis and spectral theory since it is based on the local frequency representation of the function being expanded. The main idea behind the approximation for the solutions of nonlinear integral equations is by using the theory of derivatives of nonlinear operators in Banach space. Classical approximation of nonlinear integral equations is based on the construction of successive sequence of functions containing derivative of the nonlinear operator. The challenge faced when solving linear and nonlinear integral equations is the existence of an unknown function appearing under the integral. The motivation comes when we want to overcome this obstacle and for this reason the approximation by wavelets of unknown functions is represented. The latter approximation method expands the unknown function which allows the integral equation to be reduced to a system of algebraic equation. This then can be solved by any conventional methods, such as polynomial, spline, pulse basic function and etc.

1.2 Approximation Theory

The theory of approximation of functions is the branch of mathematical analysis which started with the work of Chebyshev and a well known theorem by Weiestrass on the approximation of continuous functions by polynomials.

Theorem 1.2.1 (Weiestrass) *(Christensen (2004)). For any continuous function $f(x), x \in [a,b]$ there exists a sequence of polynomials which converges uniformly to $f(x)$ on $[a,b]$.*

Theorem 1.2.1 is of great importance in the development of the whole mathematical analysis. The proof of Weierstrass theorem (refer to Christensen (2004)) is based on the construction of the sequence of polynomials

$$B_n(x) = (b-a)^{-1} \sum_{k=0}^{n} f\left(a + \frac{k}{n}(b-a)\right)(x-a)^k (b-x)^{n-k}$$

, for each function $f(x)$ in $[a,b]$.

Research related to mathematical approximation has attracted the interest of many mathematicians since Archimedes's approximation of π which was two centuries ago. In this thesis the branch of mathematical approximation that we will focus on is approximation theory. Approximation theory can be divided into computational methods and theoretical estimations. When referring to the computational methods, its main aspect is the estimation of errors. Where as, the theoretical estimation deals with existence and uniqueness problems. In the classical approximation theory, polynomials are essential approximation tools. As an example, when solving problems involving periodic functions the most efficient methods used are trigonometric polynomials. To better understand the nature of these polynomials, many mathematicians have studied their properties such as orthogonality, compact support and etc.

1.3 Functional Analysis

To understand the real picture of modern approximation theory, we introduce some concepts related to this field.

1.3.1 Metric spaces.

Let consider the set X and a function $d(x,y)$ defined on X with the following properties:

1. $d(x,y) \geq 0, \quad \forall x,y \in X.$
2. $d(x,y) = 0$ if and only if $x = y$.
3. $d(x,y) = d(y,x), \quad \forall x,y \in X.$
4. $d(x,z) \leq d(x,y) + d(y,z). \quad \forall x,y \in X$ (triangle inequality).

Such set X with $d(x,y)$ is called metric space and will be denoted by (X,d).

As an example, let the set X be all real valued functions defined and continuous on $[a,b]$ be denoted as $C[a,b]$. The following two functions:

$$d_1(x,y) = \max_{t \in [a,b]} |x(t) - y(t)|,$$

or

$$d_2(x,y) = \int_a^b |x(t) - y(t)|\,dt.$$

are metric in $C[a,b]$.

Definition 1.3.1 (Cauchy sequence) *A sequence $\{x_n\}$ in a metric space (X,d) is said to be Cauchy sequence if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that*

$$d(x_m, y_n) < \varepsilon \quad for\ every\ n,m \geq N.$$

Every convergent sequence in a metric space (X,d) is a Cauchy sequence, but the converse not necessarily be true. The following example shows this statement.

Example 1.1:
Let $X = (0,1]$ and the metric function $d(x,y) = |x-y|$. Consider the sequence $x_n = \frac{1}{n}$ in the metric space (X,d). It is a Cauchy sequence, but is not a convergent sequence because the sequence $\{x_n\}$ converge to a limit point 0 and its not the element in the metric space.

Definition 1.3.2 (Complete) *The metric space (X,d) is said to be complete if every Cauchy sequence in X converges to a limit within the metric space.*

The above Example 1.1 shows the metric space (X,d) is not complete.

Definition 1.3.3 (Density) *A set M in a metric space (X,d) is dence in X if $\overline{M} = X$.*

This means that, every $x \in X$ is limit of elements in M.

Definition 1.3.4 (Contraction operator) *Let (X,d) be a metric space. A mapping $T : X \to X$ is a contraction mapping if there exists a constant λ, with $0 \leq \lambda < 1$, such that*

$$d(Tx, Ty) \leq \lambda d(x,y), \ \forall x,y \in X.$$

Theorem 1.3.5 *(Shestopalov (2002)) Let (X,d) be a complete metric space. If $T : X \to X$ is a contraction mapping, then there is exists a unique $x_0 \in X$ such that $Tx_0 = x_0$.*

The proof of this Theorem 1.3.5 is very easy consequence of the definition of the contraction operator. The point of x_0 which satisfy in the Theorem 1.3.5 is called fixed point of operator T and denoted by $Fix(T) = x_0$.

1.3.2 Normed Spaces.

Normed space X is a vector space with a norm defined on it. The norm on a real $(\mathbb{R})$ vector space V is a real-value function on X which satisfies:

1. $\|x\| \geq 0, \ \forall x \in X.$
2. $\|x\| = 0$ if and only if $x = \mathbf{0}$.
3. $\|\alpha x\| = |\alpha| \, \|x\|, \ \forall x \in X$ and $\forall \alpha \in \mathbb{R}$.
4. $\|x + y\| \leq \|x\| + \|y\|$, (triangle inequality) $\forall x, y \in X$.

The metric function $d(x,y)$ on a normed space X is defined as

$$d(x,y) = \|x - y\|,$$

and is called the metric induced by the norm.

As an example, let consider the class $L^p(\mathbb{R}^n)$, $p \geq 1$, which consists of all functions $f(x)$, such that $|f|^p$ is integrable on $(\mathbb{R}^n)$:

$$L^p(\mathbb{R}^n) := \left\{ f : \Big| \int_{\mathbb{R}^n} |f(x)|^p \, dx < \infty \right\}.$$

The norm in $L^p(\mathbb{R}^n)$ is defined as follows:

$$\|f\|_p = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

A bounded linear operator T from a normed space X_1 to a normed space X_2 is an operator which satisfies:

1. $T(\alpha x + \beta y) = \alpha T x + \beta T y$, for all $x, y \in X$ and $\alpha, \beta \in \mathbb{R}$.
2. $\|Tx\|_2 \leq c \, \|x\|_1$ for some real number $c \geq 0$.

Consider an integral operator $T : C[a,b] \to C[a,b]$ by

$$Tx(t) = \int_a^b k(t,s) x(s) ds,$$

where k is the kernel function. We assume that k is continuous on the $[a,b] \times [a,b]$ ts-plane which implies $|k(t,s)| \leq K$.

1.

$$
\begin{aligned}
T(\alpha x+\beta y) &= \int_a^b k(t,s)\,(\alpha x(s)+\beta y(s))\,ds,\\
&= \alpha \int_a^b k(t,s)x(s)ds + \beta \int_a^b k(t,s)y(s)ds,\\
&= \alpha Tx+\beta Ty.
\end{aligned}
$$

2.

$$
\begin{aligned}
\|Tx\| &= \max_{t\in[a,b]} \left| \int_0^1 k(t,s)x(s)ds \right|,\\
&\leq \max_{t\in[a,b} \int_a^b |k(t,s)||x(s)|ds,\\
&\leq K\|x\|,\ \|x\| = \max_{J\in[01]} |x(s)|.
\end{aligned}
$$

that T is bounded linear operator with $c = K$.

1.3.3 Hilbert Spaces.

Let X be a (complex) vector space. An inner product on X is a mapping $\langle\cdot,\cdot\rangle : X\times X \to \mathbb{C}$ for which

1. $\langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle\ \ \forall x,y,z\in X, \alpha,\beta\in\mathbb{C}$.

2. $\langle x,y\rangle = \overline{\langle y,x\rangle}\ \ \forall x,y\in X$.

3. $\langle x,x\rangle \geq 0,\ \ \forall x\in X$, and $\langle x,x\rangle = 0 \Leftrightarrow x=0$.

The function $\langle\cdot,\cdot\rangle$ is called an inner product. A vector space with an inner product that satisfies all the following conditions is called an inner product space.

If x and y are vectors in an inner product space then,

$$|\langle x,y\rangle| \leq \|x\|\,\|y\|,$$

this inequality is called Cauchy-Schwarz inequality.

Every inner product space X is a normed space with the norm defined as $\|x\| = \sqrt{\langle x,x\rangle}$. Then the metric function $d(x,y)$ can be denote as $d(x,y) = \sqrt{\langle x-y,x-y\rangle}$.

A complete inner product space is called Hilbert space $\mathbb{H}$. The $L^p(\mathbb{R}^n)$ space is not a Hilbert space except when $p=2$ that is $L^2(\mathbb{R}^n)$ space.

The space $L^2(\mathbb{R}^n)$ is the set of all square-integrable function defined on

$$L^2(\mathbb{R}^n) = \left\{ f : \left| \int_{(\mathbb{R}^n)} |f(t)|^2 dt < \infty \right. \right\},$$

and can be equipped with an inner product as:

$$\langle f, g \rangle = \int_{(\mathbb{R}^n)} f(x)\overline{g(x)} dx, \ \ \forall f, g \in L^2(\mathbb{R}^n).$$

Consider some classes of linear operator on $L^2(R)$:

1. $\forall a \in \mathbb{R}$, the translation operator $T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as

$$T_a f(x) = f(x-a), \ a \in \mathbb{R}.$$

2. $\forall c > 0$, the dilation operator $D_c : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined as

$$D_c f(x) = c^{\frac{1}{2}} f(cx).$$

Note that if $c > 1$ the graph of $D_c f(x)$ is a compressed version of the graph $f(x)$, while if $0 < c < 1$ is a spread out version of $f(x)$. For the translation operator if $a > 0$ the graph $T_a f(x)$ will appear to the right of the graph $f(x)$ by a units and verse vise if $a < 0$.

The collection of vectors $\{e_k\}_{k=1}^{\infty} \in \mathbb{H}$ form an orthonormal system if

$$\langle e_k, e_j \rangle = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad \forall k, j \in \mathbb{N}.$$

An orthonormal system $\{e_k\}_{k=1}^{\infty}$ in Hilbert space $\mathbb{H}$ forms an orthonormal basis (ONB) for $\mathbb{H}$, if each $f \in \mathbb{H}$ there exist unique scalar coefficient $\{c_k\}_{k=1}^{\infty}$ such that

$$f = \sum_{k=1}^{\infty} c_k v_k,$$

where $c_k = \langle f, c_k \rangle$.

1.4 Newton-Kantorovich Method

The important of Newtons method come into consideration when we face with the problem of finding the root of nonlinear equation. Many scholars, contribute to the development of the method, such Cauchy, Fourier, Thomas Simpson and many more. Later an important contribution were established by Kantorovich (1948) where he considers nonlinear operator equation and expand the work of Newton. Kantorovich construct a new way for finding an approximate solution of the nonlinear operator equations which become a powerful tool in numerical analysis. The method is called Newton-Kantorovich's. To investigate the nonlinear operator we need to understand how to differentiate an operator. Consider the first derivative of an operator P as follow:

Let consider an operator P is a mapping from an open set $\Omega \subset X$ of Banach space X into a set Δ of another Banach space Y

$$P : \Omega \to \Delta.$$

Fix $x_o \in \Omega$. If there exist a continuous linear operator $U : X \to Y$ such that, $\forall x \in X$,

$$\lim_{h \to 0} \frac{P(x_0 + hx) - P(x_0)}{h} = U(x),$$

then the linear operator U is called a derivative of P at point x_0 and denoted as follows:

$$U = P'(x_0).$$

Let us consider the following cases to illustrate this statement.

Case 1: First derivative $P'(x_0)$

Let operator P defined in $X = C[a,b]$ as

$$P(x) = \int_a^b (t+s)e^{x(s)}\,ds.$$

We will show that, $U = P'(x_0)$ is a continuous linear operator.

$$\begin{aligned}
U &= \lim_{h \to 0} \frac{P(x_0 + hx) - P(x_0)}{h}, \\
&= \lim_{h \to 0} \frac{1}{h} \int_a^b (t+s)e^{x_0(s)+hx(s)}\,ds - \int_a^b (t+s)e^{x_0(s)}\,ds, \\
&= \lim_{h \to 0} \int_a^b (t+s)e^{x_0(s)} \left(\frac{e^{hx(s)} - 1}{hx(s)} \right) x(s)\,ds,
\end{aligned}$$

since $\lim_{h \to 0} \frac{e^{hx(s)} - 1}{hx(s)} = 1$, then

$$\begin{aligned}
U &= \lim_{h \to 0} \int_a^b (t+s)e^{x_0(s)} \left(\frac{e^{hx(s)} - 1}{hx(s)} \right) x(s)\,ds, \\
&= \int_a^b (t+s)e^{x_0(s)} x(s)\,ds = \int_a^b K(s,t)x(s)\,ds,
\end{aligned}$$

where $K(s,t) = (t+s)e^{x_0(s)} \in C[a,b]$.

Now assume that operator P has first derivative in $\Omega \subset X$. Let $P'(x)$ be an operator mapping from an open set $\Omega \subset X$ of Banach space X into a Banach space Y. Hence, if the derivative of this operator exists at a given point $x_0 \in \Omega$ then, is called the second derivative of the given operator denoted by $U' = P''(x_0)(x,x')$.

$$P''(x_0)(x,x') = \lim_{h\to 0} \frac{P'(x_0+hx')x - P'(x_0)x}{h} = U'(x).$$

The second case will explain the second derivative of the operator.

Case 2: Second derivative $P''(x_0)(x,x')$

Let choose

$$P'(x_0)x = \int_a^b (t+s)e^{x_0(s)}\, x(s)\, ds,$$

and

$$P'(x_0+hx')x = \int_a^b (t+s)e^{x_0(s)+hx'(s)}\, x(s)\, ds,$$

be a continuous operator in $C[a,b] \in X$.

Thus,

$$\begin{aligned} P''(x_0)(x,x') &= \lim_{h\to 0} \frac{P'(x_0+hx')x - P'(x_0)x}{h}, \\ &= \lim_{h\to 0} \frac{1}{h} \int_a^b (t+s)e^{x_0(s)+hx'(s)}\, x(s)\, ds - \int_a^b (t+s)e^{x_0(s)}\, x(s)\, ds, \\ &= \lim_{h\to 0} \int_a^b (t+s)e^{x_0(s)} x(s) \left(\frac{e^{hx'(s)}-1}{hx'(s)} \right) x'(s)\, ds, \\ &= \int_a^b (t+s)e^{x_0(s)} x(s)\, x'(s)\, ds = \int_a^b K(s,t)\, x(s)\, x'(s)\, ds. \end{aligned}$$

1.5 Approximation of function from Holder classes

The non-differentiable functions are important in various application of signal analysis. The importance of studying continuous but nowhere differentiable functions was emphasized a long time ago. It is possible for a continuous function to be sufficiently irregular so that its graph is a fractal. This observation points out to a connection between the lack of differentiability of such a function and the dimension of its graph. These functions are known as Irregular functions.

Irregular functions arise naturally in various branches of physics. It is well known that the graphs of projections of Brownian paths are nowhere differentiable and have dimension 3/2. A generalization of Brownian motion called fractional Brownian motion gives rise to graphs having dimension between 1 and 2. Typical Feynmann paths, like the Brownian paths are continuous but nowhere differentiable. Also, passive scalars adverted by a turbulent fluid can have is scalar surfaces which are highly irregular, in the limit of the diffusion constant going to zero. Attractors of some dynamical systems have been shown to be continuous but nowhere differentiable. All these irregular functions are characterized at every point by a local Holder exponent typically lying between 0 and 1. Let proceed with the definition of the Holder classes of order $s \in (0,1)$(Kolwankar and Gangal (1996), pp 2)

Definition 1.5.1 (Holder space) *The set of all continuous functions on* $[0,1]$*, which satisfies the inequality:*

$$|f(x)-f(y)| \leqslant L|x-y|^s, \forall x,y \in [0,1], L>0,$$

is called a Holder space of order s and denoted by $H^s[0,1]$ *whereas, the norm is given by*

$$\|f\|_{H^s[0,1]} = \|f\|_{C[0,1]} + \sup_{x \neq y} \frac{|f(x)-f(y)|}{|x-y|^s},$$

for all $x,y \in [0,1]$.

The Holder spaces are nested as follows :

$$H^\alpha \subset H^\beta, \ 0<\alpha<\beta<1.$$

Holder spaces are medium spaces between $C[0,1]$ and $C^1[0,1]$ such that:

$$C^1[0,1] \subset H^\alpha[0,1] \subset C[0,1], \ 0<\alpha<1.$$

1.6 Integral Equation

Integral equation is an equation where the unknown variable, we are attempting to solve appears under one or more integral sign. Consider the following integral equations

$$x(t)-\lambda \int_D K(t,s)x(t)ds = f(t), \quad \lambda \neq 0, \tag{1.1}$$

where $D=\{R[0,1]^m, m \geq 1\}$ and

$$\int_D K(t,s)x(s)ds = f(t). \tag{1.2}$$

The functions can be complex-value functions of the variable t and s.There are two primary types of integral equations, known as the Fredholm equation and the Volterra equation. Integral equations (1.1) and (1.2) are known as the Fredholm integral equation of the second kind and first kind. Here $x(s)$ are unknown function while $K(s,t)$, $f(s)$ are known functions and λ is parameter.

The properties of $K(s,t)$ will determine many things about the integral, how we can solve the integral equation, what are the possible solutions and etc. $K(s,t)$ is known as the kernel function of two variable defined in the square

$$\Omega = \{(s,t) : a \le s \le b, a \le t \le b\},$$

and must satisfy one of the following conditions:

1. $K(s,t)$ is continuous function defined in Ω.
2. If $K(s,t)$ is discontinuous, then the square integrable function must be finite

$$\int_a^b \int_a^b \|K(s,t)\|^2 ds dt < \infty.$$

The Volterra integral of the second kind and the first kind are precisely the same as the Fredholm integral equations (1.1) and (1.2) except that the upper limit of integration is variable. Model of the system in the physical, engineering such measured the current flow in an electric circuit at certain times, image analysis, the displacement moved by the spring by some pressured gives rise to the integral and differential equation. The integral equation is actually self sufficient means that the integral equation doesnt need to specify the boundary and initial condition separately, whereas for the differential equation need to specify the boundary condition so that we can solve the problem. Differentiable equations and integral equations are connected to each other in the way that the differentiable equation can be transformed to the integral equation and similarly it is possible to transformed the integral to differentiable equation. Detail of this transformation can be found in Kanwal (1997)

1.7 Wavelets

Theory of wavelets is important in constructing an orthonormal bases in $L^2(\mathbb{R})$. The analysis shall take place in $L^2(\mathbb{R})$ is because of the fact that many applications do exist in this area. A wavelets is a function $\psi(x)$ such that the collections $\psi_{j,k}(x)$ form an orthonormal basis (ONB) in $L^2(\mathbb{R})$ as follows:

$$\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}} = \left\{2^{\frac{j}{2}}\psi(2^j x - k)\right\}_{j,k\in\mathbb{Z}} = \{D_{2^j} T_k \psi(x)\}_{j,k\in\mathbb{Z}},$$

where T_k and D_{2^j} is the translation and dilation operator. The ONB of wavelet functions $\psi_{j,k}(x)$ that are form have very special structure such that all of them will be scaled and translate version of a fixed function $\psi(x) \in L^2(\mathbb{R})$. The structure of all functions $f(x) \in L^2(\mathbb{R})$ can be represent as

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \psi_{j,k}(x), \\ &= \sum_{j,k\in\mathbb{Z}} c_{j,k} \psi_{j,k}(x). \end{aligned} \tag{1.3}$$

is also important in application where $c_{j,k}$ are the coefficients.

Mallat (1987) introduced Multiresolution Analysis (MRA) a framework to construct an orthonormal wavelet bases. The MRA consists two main ingredient that is the collection of subspace $V_j \in L^2(\mathbb{R})$ with certain conditions and the function $\phi(x) \in L^2(\mathbb{Z})$.

A MRA for $L^2(R)$ consists of a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(R)$ and a function $\phi \in V_0$, such that the following conditions hold:

1. $\cdots V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$. (nested)

2. $\overline{\underset{j\in\mathbb{Z}}{\cup} V_j} = L^2(R)$ (density) and $\cap_{j\in\mathbb{Z}} V_j = \{0\}$.(*uniqueness*)

3. For all $j \in \mathbb{Z}$, $V_{j+1} = D(V_j)$. (scaling)

4. $f \in V_0 \to T_k f \in V_0,\ \forall k \in \mathbb{Z}$.

5. $\{T_k\phi\}_k \in \mathbb{Z}$ is an ONB for V_0.

For the second condition $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ means that $\cup_{j\in\mathbb{R}} V_j$ is dense in $L^2(\mathbb{R})$, shows that for any $f(x) \in L^2(\mathbb{R})$ and $\forall \varepsilon > 0$ there exists a function $g(x) \in \cup_{j\in\mathbb{R}} V_j$ such that $\|f - g\| \leq \varepsilon$.

The condition of the MRA above show that the choice of function $\phi(x)$ actually determined all the spaces V_j uniquely with $j \in \mathbb{Z}$. Assume that 3 and 4 satified the MRA conditons. Let V_0 be the space of all functions of the form as

$$\sum_{k\in\mathbb{Z}} c_k \phi(x-k) = \overline{span}\{T_k\phi(x)\}, \quad c_k \in \mathbb{R}$$

Then for any $j \geq 0$

$$\begin{aligned} V_j =& DV_{j-1} = D^2 V_{j-2} = \cdots = D^j V_0, \\ =& D^j \{\overline{span}\{T_k\phi(x)\}\}, \\ =& \overline{span}\{D^j T_k \phi(x)\}. \end{aligned}$$

Thus, the function $\phi(x)$ determine the space V_j uniquely.

In order to construct an orthonormal basis $\psi_{j,k}(x)$ for $L^2(\mathbb{R})$ we need to consider a class of vector spaces associated with $\{V_j\}\ j \in \mathbb{Z}$. Assume that the property 1 in MRA is satisfied, then there exist a subspace W_j which are orthogonal complement of V_j in V_{j+1} that is

$$V_{j+1} = V_j \oplus W_j, \qquad j \in \mathbb{Z}.$$

and for any $j, j' \in \mathbb{Z}$. with $j \neq j'$ the space $W_j \perp W_{j'}$. Furthermore the space V_j can be successively decomposed as an orthogonal direct sum,

$$\begin{aligned} V_j &= W_{j-1} \oplus V_{j-1}, \\ &= W_{j-1} \oplus W_{j-2} \oplus V_{j-2}, \\ &\ \ \dots \\ &= W_{j-1} \oplus W_{j-2} \oplus \cdots \oplus W_0 \oplus V_0. \end{aligned}$$

Here the decomposition $\{V\}_{j\in\mathbb{Z}}$ plays an important part in filtering out noise problem in application. Since by removing the spikes that belong in the V_j does not affect members of V_{j-1}.

Let $\psi(x) \in W_0$ such that $\{\psi(x-k)\}_{k\in\mathbb{Z}}$ form an orthonormal basis for the space W_0, then the set of all function

$$\left\{\psi_{j,k}(x)\right\}_{j,k\in\mathbb{Z}} = \left\{2^{\frac{j}{2}}\psi(2^j x - k)\right\}_{j,k\in\mathbb{Z}} = \{D_{2^j} T_k \psi(x)\}_{j,k\in\mathbb{Z}},$$

is an orthonormal basis of W_j. Therefore any function $f(x) \in L^2(\mathbb{R})$ can be represent as equation (1.3).

1.7.1 Haar wavelets

As defined in Walnut (2013), the Haar scaling and wavelets function can be represented as follows:

$$\phi(x) = \begin{cases} 1, & \text{for} \quad x \in [0,1) \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$\psi(x) = \begin{cases} 1, & \text{for} \quad x \in [0,\frac{1}{2}) \\ -1, & \text{for} \quad x \in [\frac{1}{2},1) \\ 0, & \text{elsewhere}. \end{cases}$$

To decompose V_j to W_l for $l < j$ the following relation between $\phi(x)$ and $\psi(x)$ are needed. The two-scale relation for Haar scaling and wavelets function are

$$\phi(x) = \phi(2x) + \phi(2x-1),$$

and

$$\psi(x) = \phi(2x) - \phi(2x-1).$$

Then the equation can be rewrite as

$$\phi(2^j x) = \frac{\left(\psi(2^{j-1}x) + \phi(2^{j-1}x)\right)}{2}, \tag{1.4}$$

and

$$\phi(2^j x - 1) = \frac{\left(\phi(2^{j-1}x) + \psi(2^{j-1}x)\right)}{2}, \tag{1.5}$$

where x is replaced by $2^{j-1}x$.

Suppose the function $f(x)$ is

$$f(x) = \phi(4x) + \phi(4x-1) + 2\phi(4x-2) - 2\phi(4x-3),$$

and the graph of $f(x)$ is given in Figure 1.1. Using equation (1.4) and (1.5) each of functions V_2 can be computed as below

$$\phi(4x) = \frac{(\psi(2x) + \phi(2x))}{2},$$

$$\phi(4x-1) = \frac{(\phi(2x) - \psi(2x))}{2},$$

$$\phi(4x-2) = \psi\left(4\left(x-\frac{1}{2}\right)\right) = \frac{\psi\left(2\left(x-\frac{1}{2}\right)\right) + \phi\left(2\left(x-\frac{1}{2}\right)\right)}{2},$$

$$\phi(4x-3) = \psi\left(4\left(x-\frac{1}{2}\right)-1\right) = \frac{\phi\left(2\left(x-\frac{1}{2}\right)\right)\psi\left(2\left(x-\frac{1}{2}\right)\right)}{2}.$$

Substitute all the functions in $f(x)$ to obtain

$$f(x) = 2\psi(2x-1) + \phi(2x) = 2\psi(2x-1) + \psi(x) + \phi(x).$$

The results show that the function can be decomposed as W_1, W_0 and V_0 that is an important tool in analyzing problem occurring in engineering and science.

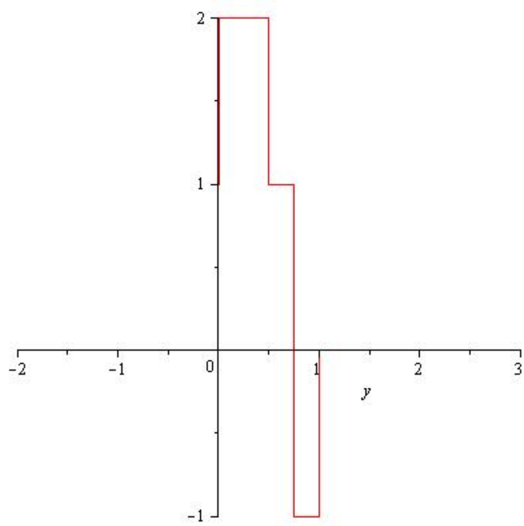

Figure 1.1: The graph of function $f(x)$

1.7.2 Chebyshev wavelets

The Chebyshev wavelets are defined on the interval $[0,1)$ by

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} \tilde{T}_m(2^k t - 2n + 1), & \frac{n-1}{2^{k-1}} \leq t < \frac{n}{2^{k-1}}, \\ 0, & otherwise, \end{cases} \tag{1.6}$$

where

$$\tilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}}, & m = 0, \\ \sqrt{\frac{2}{\pi}} T_m(t), & m > 0, \end{cases}$$

and $m = 0, 1, ..., M-1, n = 1, 2, ..., 2^{k-1}$. Here $T_m(t), m = 0, 1, ...,$ are Chebyshev polynomials of first kind of degree m

$$T_m(t) = cos(m cos^{-1} t), \ m \geq 0.$$

Also, it can derived from the following recursive formula:

$$T_0(t) = 1$$
$$T_1(t) = t$$
$$T_{m+1}(t) = 2t T_m(t) - T_{m-1}(t), \ n = 1, 2, 3, ...$$

Chebyshev polynomials are orthogonal with respect to the weight function $w(t) = \frac{1}{\sqrt{1-t^2}}$ defined on the interval [-1,1]. We should note that in dealing with Chbyshev wavelets the weight function $w(t)$ have to be dialated and translated as

$$w_k(t) = w(2^k t - 2n + 1),$$

which the Chebyshev wavelets will form an orthonormal set with the given weight function:

$$\langle \psi_{nm}, \psi_{n'm'} \rangle = \int_{2^{-(k+1)}(n-1)}^{2^{-(k+1)}n} \psi_{nm}(t) \psi_{NM}(t) w_k(t) dt = \begin{cases} 1, & n = n', m = m', \\ 0, & elsewhere. \end{cases}$$

Any function $x(t)$ defined over the interval [0,1) may be expanded as:

$$x(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \tag{1.7}$$

with $c_{nm} = \langle x, \psi_{nm} \rangle$.

1.7.3 Linear Legendre multi-wavelets (LLMW)

For constructing the linear Legendre multi-wavelets, first we need to describe scaling functions $\phi_0(x)$ and $\phi_1(x)$ as follows:

$$\phi_0(x) = 1, \;\; \phi_1 = \sqrt{3}(2x-1), \;\; 0 \le x < 1.$$

By the condition of MRA the scaling functions can be describe as the linear span of $2^{\frac{j}{2}}\phi_0(2^j x - n), 2^{\frac{j}{2}}\phi_1(2^j x - n)$. By applying suitable condition Khellat and Yousefi (2006) on the corresponding mother wavelets $\psi^0(x)$ and $\psi^1(x)$, then explicit formula for the linear Legendre mother wavelets are obtained as

$$\psi^0_{00}(x) = \begin{cases} -\sqrt{3}(4x-1), & 0 \le x < \frac{1}{2}, \\ \sqrt{3}(4x-3), & \frac{1}{2} \le x < 1, \end{cases}$$

$$\psi^1_{00}(x) = \begin{cases} 6x-1, & 0 \le x < \frac{1}{2}, \\ 6x-5, & \frac{1}{2} \le x < 1, \end{cases}$$

and the family

$$\psi^j_{kn}(x) = \begin{cases} 2^{\frac{k}{2}}\psi^j(2^k x - n), n2^{-k} \le x < (n+1)2^{-k}, \\ 0, \quad\quad otherwise, \end{cases}$$

$n = 0, 1, \ldots, 2^{k-1}$, $k, \in \mathbb{Z}+$, and $j = 0, 1$, form an orthonormal basis for $L^2(\mathbb{R})$.

Any function $f(x) \in L^2(\mathbb{R})$ in the interval [0,1) can be expanded as

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + \sum_{k=0}^{\infty}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1} c^j_{kn}\psi^j_{kn}(x), \tag{1.8}$$

where, $c^j_{kn} = \langle f, \psi^j_{kn} \rangle$.

The next four functions are defined below

$$\psi_{10}^{0}(x)=\begin{cases}-\sqrt{6}(8x-1), & 0\le x<\dfrac{1}{4},\\ \sqrt{6}(8x-3), & \dfrac{1}{4}\le x<\dfrac{1}{2},\\ 0, & \dfrac{1}{2}\le x<1,\end{cases}$$

$$\psi_{11}^{0}(x)=\begin{cases}0, & 0\le x<\dfrac{1}{2},\\ -\sqrt{6}(8x-5), & \dfrac{1}{2}\le x<\dfrac{3}{4},\\ \sqrt{6}(8x-7), & \dfrac{3}{4}\le x<1,\end{cases}$$

$$\psi_{10}^{1}(x)=\begin{cases}\sqrt{2}(12x-1), & 0\le x<\dfrac{1}{4},\\ \sqrt{2}(12x-5), & \dfrac{1}{4}\le x<\dfrac{1}{2},\\ 0, & \dfrac{1}{2}\le x<1,\end{cases}$$

$$\psi_{11}^{1}(x)=\begin{cases}0, & 0\le x<\dfrac{1}{2},\\ \sqrt{2}(12x-7), & \dfrac{1}{2}\le x<\dfrac{3}{4},\\ \sqrt{2}(12x-11), & \dfrac{3}{4}\le x<1,\end{cases}$$

and the graph of these eight functions is described in Figure 1.2-1.9

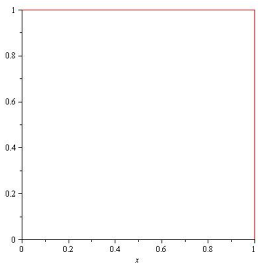

Figure 1.2: LLMW scaling $\phi_0(x)$

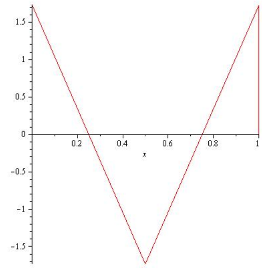

Figure 1.4: LLMW wavelet $\psi_{00}^{0}(x)$

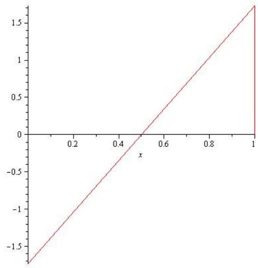

Figure 1.3: LLMW scaling $\phi_1(x)$

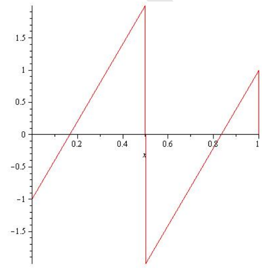

Figure 1.5: LLMW wavelet $\psi_{00}^{1}(x)$

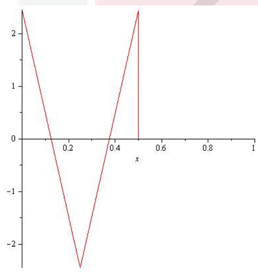

Figure 1.6: LLMW wavelet $\psi_{10}^{0}(x)$

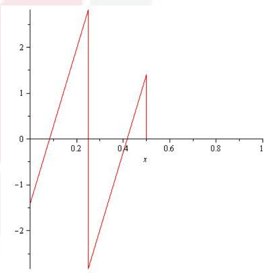

Figure 1.8: LLMW wavelet $\psi_{10}^{1}(x)$

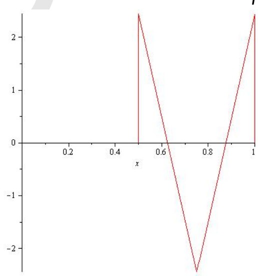

Figure 1.7: LLMW wavelet $\psi_{11}^{0}(x)$

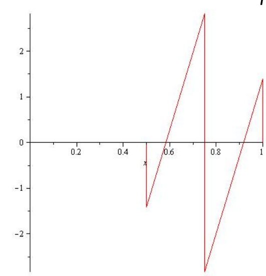

Figure 1.9: LLMW wavelet $\psi_{11}^{1}(x)$

1.8 Problem statement

Numerical integrations by quadrature rule are widely used in the application of science and engineering. A lot of work devoted to quadrature rule such as Trapezoid, Gauss quadrature and Newton-Cotes quadrature are used to find numerical solution of definite integrals. Unfortunately, quadrature rule bears some drawbacks. For instance, the gaussian quadrature is based on polynomial interpolation. To derive this method the unknown nodes as well as its weights are determined by solving $2n$ nonlinear equation. This procedure is quite difficult because the nonlinear equations might have multiple solution. Due to the large number of node points needed to achieve high accuracy, the quadrature rule may require dealing with a high degree of polynomial interpolation. In the case of Newton-cotes quadrature, the use of large number of equally spaced nodes may cause erratic behavior with high degree polynomial interpolation. Therefore to overcome this problem, new methods based on wavelets (Haar and Daubechies) were attempts by researchers (see section 2.1 of the thesis) with the objective to find algorithms to solve definite integrals without finding the optimum weight and the nodes.

When developing a new numerical method, establishing its error analysis is important. The efficiency of the error analysis directly relates to the accuracy of the new method. Therefore the error estimation by extension is a validation of the reliability of the method. Previous attempts on new methods only established error estimations by assuming functions belonging to the $C^1(R)$ or $C^2(R)$. While others only provide the new method without considering the error estimation. In order to overcome these error estimation issues, further theoretical analysis in the Holder classes need to be investigate.

The popular technique uses wavelets to reduce non linear integral equations to nonlinear algebraic systems, then solve these nonlinear algebraic systems using some other numerical methods (Gaussian integration, trapezoidal formula, etc) which compromises the entire purpose of wavelet method. In this thesis, we propose a new method to reduce the nonlinear integral equation to linear algebraic systems and then solved by wavelets. By doing so, we are able to preserve the advantage of the wavelet method.

1.9 Research Objectives

In this section, each research objective are elaborate to describe the method used towards achieving the state objective as follows.

1. To obtain generalized solution for double, triple and N-dimensional definite integrals by Haar wavelets.

   In previous works, Haar wavelets are used to solve double and triple integral with variable limits of integration. Algorithm based on Haar wavelets have been obtain for single integral and assumed true for double and triple integral.

Here not only are we able to validate those assumptions but also establish a general algorithm for any N-dimensional integrals. Our next aim is to solve these double and triple integral by changing the variable limit of integration to definite integral by Haar wavelets.

2. To establish error estimation in differentiable functions.

   The error estimation for single integral has been done for Haar wavelets where as for double and triple integral remains open. We attempt to establish the error estimation for double and triple integral in $C^2(R)$ and $C^3(R)$ respectively for Haar wavelets. It is important to see how many bases are needed to solve the integrals and why must we consider functions from $C^2(R)$ and $C^3(R)$.

3. To obtain generalized solution for double and triple definite integrals by LLMV wavelets.

   The LLMW is a type of wavelet which consists of 2 scaling functions and two mother wavelet. LLMW displays properties similar to Haar wavelets. Due to these similarity, the LLWM should be able to solve the multi dimensional integral easily. To validate this, numerical results are compared to the Haar wavelets.

4. To find error estimation for the Haar, LLMW and Chebychev wavelets of the function from Holder classes.

   The Holder class is a medium between the continuous function and a differentiable $C^1(R)$ class. In this research, the error estimation is expanded to a wider group of functions, which are functions from Holder class. By obtaining this estimation, we are able extend the validity of the wavelet method to a larger group of integral equations.

5. To approximate the solution of nonlinear Fredholm and Volterra integrals equations of the second kind by Haar and Chebychev wavelets respectively.

   The main aim is to construct new methods that is simple applicable and more efficient to solve the nonlinear Fredholm and Volterra integral equation of the second kind. Combination of Newton-Kantorovich and wavelets methods are used to solve nonlinear integral equations of the second kind.

1.10 Thesis Outline

In this thesis, we have organized the chapter as following.

Chapter 1 provide background tools to guide the reader to the motivation of this research. This chapter also highlight the problem statement and the objective of this research.

In chapter 2, we present the literature and the main figure in developing of wavelets. Moreover we review the research done by previous studies that dealt with issues related to our topic.

Chapter 3, we describe the method used in solving the double, triple and N-dimensional definite integrals by Haar wavelets. We also developed a new technique to solve the double and triple integrals with variable limit of integrations. This is followed by providing the error estimation and numerical results for double and triple integrals.

In Chapter 4 we look into the error estimation for the Haar and LLMW in Holder class. Types of function that satisfy the Holder class are also discuss and proven. Next, a new algorithm by LLMW was established to approximate 1,2,3-dimensional definite integrals. Finally the error estimation and numerical validation for Haar and LLMW in Holder class are given.

In Chapter 5, new approach for solving nonlinear Fredholm and Volterra integral equations of the second kind by using Haar and Chebyshev wavelets are constructed. Numerical examples are given in order to determined the accuracy of the new method with the previous results.

In the last chapter of the thesis we will present the outcome of this thesis and its contribution to the field of wavelets. We also discuss the possibility for future research and give suggestions on open problems for future researchers of the subject.

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