UNIVERSITI PUTRA MALAYSIA

TWO-POINT DIAGONALLY IMPLICIT MULTISTEP BLOCK METHOD FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATION OF SECOND KIND

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By

NUR AUNI BINTI BAHARUM

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

January 2018

DEDICATIONS

To
My Beloved Parent:
Dad: Baharum Bin Sulaiman,
Mum: Mek Yah Binti Ya,
...
My Amazing Siblings:
Asrul Aiman Bin Baharum,
Hafidzul Aiman Bin Baharum,
Nur Alya Binti Baharum,

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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NUR AUNI BINTI BAHARUM

January 2018

Chairman : **Zanariah Abdul Majid, PhD**
Faculty : **Institute for Mathematical Research**

The first part of the thesis focuses on solving Volterra integro-differential equation (VIDE) of the second kind with the multistep block method. The two points diagonally implicit multistep block (2PDIB) method is formulated for the numerical solution of the second kind of VIDE. The derivation of the 2PDIB method can be obtained using Lagrange interpolating polynomial. The numerical solution of the second kind of VIDE computed at two points simultaneously in block form using the proposed method using constant step size. These numerical solutions are executed in the predictor-corrector mode.

Since an integral part of VIDE cannot be solved explicitly and analytically, the idea to approximate the solution of the integral part is discussed and the appropriate order of numerical integration formulae is chosen to approximate the solution of the integral part of VIDE which include trapezoidal rule, Simpson's rule and Boole's rule. Regarding the general form of VIDE, there are two cases of the kernel which are $K(x,s) = 1$ and $K(x,s) \neq 1$. Two different procedures are developed to obtain the solution for these cases. The stability region is discussed based on the stability polynomial of the 2PDIB method paired with the appropriate quadrature rule.

Linear and nonlinear problems of VIDE have been solved numerically using the 2PDIB method. Six tested problems are presented in order to study the

performance and efficiency of the 2PDIB method in terms of maximum error, total function calls, total steps taken and the execution time taken. Numerical results showed that the efficiency of 2PDIB method when solving VIDE compared to the existing methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KAEDAH BLOK MULTILANGKAH DUA TITIK TERSIRAT PEPENJURU DIGUNAKAN UNTUK MENYELESAIKAN JENIS KEDUA BAGI PERSAMAAN PEMBEZAAN-KAMIRAN VOLTERRA

Oleh

NUR AUNI BINTI BAHARUM

Januari 2018

Pengerusi : **Zanariah Abdul Majid, PhD**
Fakulti : **Institut Penyelidikan Matematik**

Pada bahagian pertama tesis ini adalah untuk memfokuskan penyelesaian terhadap persamaan pembezaan kamilan Volterra (PPKV) dengan menggunakan kaedah blok multilangkah. Kaedah blok multilangkah dua titik tersirat pepenjuru (BM2TTP) telah dirumuskan untuk penyelesaian berangka terhadap jenis kedua PPKV. Terbitan kaedah BM2TTP boleh diperolehi dengan menggunakan polinomial interpolasi Lagrange. Penyelesaian berangka untuk PPKV dihitung pada dua titik secara serentak dalam bentuk blok dengan menggunakan kaedah yang telah dicadangkan pada ukuran langkah yang tetap. Penyelesaian berangka ini dilaksanakan dengan cara peramal-pembetul.

Disebabkan bahagian kamilan PPKV tidak dapat diselesaikan secara jelas dan analitikal, satu idea untuk menyelesaikan bahagian kamilan PPKV telah dibincangkan dan formula kamilan berangka yang mempunyai urutan yang sesuai telah dipilih untuk mencari penyelesaian untuk bahagian kamilan PPKV yang merangkumi petua trapezium, petua Simpson dan petua Boole. Daripada bentuk PPKV yang umum, terdapat dua jenis kes inti yang merangkumi $K(x,s) = 1$ dan $K(x,s) \neq 1$. Dua jenis prosedur telah dibentuk untuk menyelesaikan kes-kes tersebut. Rantau kestabilan telah dibincangkan berdasarkan kestabilan polinomial untuk kaedah BM2TTP yang telah dipasangkan dengan petua kuadratur yang sesuai.

Masalah linear dan tak linear bagi PPKV telah diselesaikan secara berangka menggunakan kaedah BM2TTP. Enam masalah telah diuji untuk mengkaji prestasi dan kecekapan kaedah BM2TTP dari segi ralat maksimum, jumlah panggilan fungsi, jumlah langkah yang diambil dan masa pelaksanaan yang diambil. Hasil kajian menunjukkan kecekapan kaedah BM2TTP semasa menyelesaikan masalah PPKV berbanding kaedah sedia ada.

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I certify that a Thesis Examination Committee has met on 4 January 2018 to conduct the final examination of Nur Auni binti Baharum on her thesis entitled "Two-Point Diagonally Implicit Multistep Block Method for Solving Volterra Integro- Differential Equation of Second Kind" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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Declaration by graduate student

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This is to confirm that:

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Signature: ____________________________
Name of Member of Supervisory Committee
Norazak Senu

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LIST OF ABBREVIATIONS

ODE	Ordinary Differential Equations
RK3	Runge-Kutta method of order three
RK4	Runge-Kutta Method of order four
ABM3	Third-order Adams-Bashforth-Moulton method
ABM4	Fourth-order Adams-Bashforth-Moulton method
ABM5	Fifth-order Adams-Bashforth-Moulton method
VIDE	Volterra Integro-Differential Equation
2PBM3	Two-point multistep block method of third order
2PBM4	Two-point multistep block method of fourth order
2PBM5	Two-point multistep block method of fifth order
2PDIB	Two-point diagonally implicit multistep block method
2PDIBM3	Two-point diagonally implicit multistep block method of order three
2PDIBM4	Two-point diagonally implicit multistep block method of order four
2PDIBM5	Two-point diagonally implicit multistep block method of order five

CHAPTER 1

INTRODUCTION

1.1 Introduction

Integro-differential equation plays major roles in mathematical modeling of real life phenomena with in such as biology, physics, engineering and natural sciences. Integro-differential equation happen when any equation consists of both integrals and derivatives of the unknown function $y(x)$. The general form of integro-differential equation is given as

$$y^{(n)}(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,s)y(s)\,ds, \quad y^{(n)}(x) = \frac{d^n y}{dx^n} \tag{1.1}$$

where the limits of integration are $g(x)$ and $h(x)$, λ is the variable parameter, and $K(x,s)$ is known function which is called as the kernel or the nucleus of an integral equation. Kernel consists of two variables x and s, (Wazwaz, 2015).

Integro-differential equation occur in several forms. Two distinct ways that depend on the limit of integration are used to characterize integro equations. Fredholm integro-differential equation consist fixed of limits of integration in the form,

$$y^{(n)}(x) = f(x) + \lambda \int_{a}^{b} K(x,s)y(s)\,ds, \tag{1.2}$$

where a and b are constants. While, Volterra integro-differential equation (VIDE) has at least one variable in the integral limit in the form,

$$y^{(n)}(x) = f(x) + \lambda \int_{a}^{x} K(x,s)y(s)\,ds, \tag{1.3}$$

where $y^{(n)}(x)$ indicates the nth derivative of $y(x)$. The derivatives of the unknown function $y(x)$ may appear in any order of functions depending on the problem studied. There are two types of integro-differential equation and can be referred to as the first kind and second kind.

However, Volterra integro-differential equation of the second kind will be investigated in this study because the Volterra integro-differential equation of first kind is more complicated to solve compared to the Volterra integro-differential equation of second kind. In most real life situations, the numerical technique was chosen to solve the integral equations since the problem is complicated and cannot be solved analytically. Several numerical methods are required to acquire the accurate approximate solution. The multistep block method will be applied in order to determine the solution for the Volterra integro-differential equation of the second kind.

1.2 Volterra Integro-Differential Equation

Since 1844, Volterra has started the research on integral equation and to take the study on integral equation seriously in 1896, (Wazwaz, 2011). Volterra developed a new type of equations when Volterra conducted a survey of population growth and focused on the hereditary influences. According to his research work, the general form of Volterra integro-differential equation was developed and given in the form,

$$y'(x) = F(x, y(x), z(x)), \tag{1.4}$$

where

$$z(x) = \int_0^x K(x, s, y(s))\, ds. \tag{1.5}$$

VIDE can be classified into two types, they are the first kind and the second kind. VIDE of the second kind were involved with two cases of the kernel.

1. VIDE of the first kind

$$\int_0^x K_1(x,s)y(s)\, ds + \int_0^x K_2(x,s)y'(s)\, ds = f(x), \quad K_2(x,t) \neq 0. \tag{1.6}$$

2. VIDE of the second kind
    - Case I: $K(x,t) = 1$

$$y'(x) = f(x) + \int_0^x y(s)\, ds. \tag{1.7}$$

- Case II: $K(x,t) \neq 1$

$$y'(x) = f(x) + \int_0^x K(x,s)y(s)\, ds. \tag{1.8}$$

VIDE fall into two other types of classifications according to homogeneity and linearity concepts. Homogeneity and linearity concepts employ major role in the structures of the solutions. If the function $f(x)$ in VIDE of the second kind is identically zero, the equations are called homogeneous or otherwise it is called as inhomogeneous.

VIDE is classified as linear when the power of $y(s)$ inside the integral part is one, but a nonlinear function of VIDE occurs when the power of unknown function $y(s)$ in the integral part is more than one or it consists nonlinear function of $y(s)$ such as $e^{(y)}$, $\sin(y)$, $\cosh(y)$ and $\ln(1+y)$, (Wazwaz, 2015).

1.3 Research Problem

Many of Volterra integro-differential equation cannot be solve analytically, then the numerical method are proposed to solve Volterra integro-differential equation. Most numerical methods for solving Volterra integro-differential equation produce only one new approximation value at each step such as Runge-Kutta method, Adam-Bashforth-Moulton method and Simpson's rule. There are only a small number of researcher who solve Volterra integro-differential equation using block method such as Mohamed and Majid (2015) and Mohamed and Majid (2016).

1.4 Motivation

The motivation of this research is to improve the block method and to enhance the efficiency of the block method in solving Volterra integro-differential equation of second kind in terms of total function calls and the execution times taken.

1.5 Objectives of the thesis

The main objective of this thesis is to develop the two-point diagonally implicit multistep block (2PDIB) method and the method will be implemented for solving Volterra integro-differential equation of the second kind. The objectives can be determined by

(i) Deriving the two points diagonally implicit multistep block method for solving VIDE of the second kind.

(ii) Determining the order of the proposed diagonally implicit multistep block methods for solving VIDE of the second kind.

(iii) Investigating the stability analysis of diagonally implicit multistep block method combined with the quadrature rule for solving VIDE of the second kind.

(iv) Developing the algorithms of diagonally implicit multistep block method combined with a quadrature rule using constant step size for solving VIDE of the second kind.

1.6 Scope of the Study

The scope of the research will be focused on solving the linear and nonlinear problem of the Volterra integro-differential equation of the second kind. The two-point diagonally implicit multistep block method of third, fourth and fifth order are proposed to evaluate the approximate solution of the VIDE using constant step size.

Since the integral part in VIDE cannot be solved explicitly, the Newton-Cotes rule will be adapted for solving the integral part. There are two cases of the kernel in VIDE which are $K(x,t) = 1$ and $K(x,t) \neq 1$. The quadrature rule is applied to solve the kernel function of VIDE. Simpson's rule and composite Simpson's rule are applied for solving the kernel function of VIDE for the 2PDIB method of order three and four. While Boole's rule and composite Boole's rule are adapted to the 2PDIB method of order five for solving the integral part of VIDE.

1.7 Outline of the thesis

This thesis is divided into six chapters. In Chapter 1, a brief introduction of Volterra integro-differential equation is given. This chapter covers the objectives of the thesis, the scope of the study and the outline of the thesis.

Chapter 2 presents the review of previous works which is related to VIDE. Moreover, the basis definitions and properties of Lagrange interpolation polynomial, and linear multistep method are discussed. Chapter 2 also included the relevant mathematical concepts of VIDE.

The derivation of two points diagonally implicit multistep block method of the third order (2PDIBM3) is discussed in Chapter 3 and their stability regions are plotted. The error constant and zero stable of the methods are determined to show that the two-point of diagonally implicit multistep block method is stable. Hence, the implementation of the method with two approaches is given. Some examples of linear and nonlinear VIDE problem are tested and the numerical results are determined.

Chapter 4 deals with the derivation of fourth order of two points diagonally implicit multistep block method (2PDIBM4). The quadrature rule approach is used for the implementation of the proposed methods for obtaining the accurate approximate solution. The stability region of the proposed method is determined. Numerical results are presented and comparisons of the performance of the methods with the existing block methods are made.

Chapter 5 focused on the derivation of the fifth order method of two points diagonally implicit multistep block method (2PDIBM5) for solving VIDE of the second kind. The algorithms which would solve the VIDE of the second kind are developed. The error constant and zero stable of the method are determined. Chapter 5 end with the presentation and discussion of the numerical results. Lastly, the conclusion of the thesis was presented in Chapter 6 and suggestion for potential future research is also provided in this chapter.

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