UNIVERSITI PUTRA MALAYSIA

GENERAL LINEAR METHODS FOR SOLVING ORDINARY AND FUZZY DIFFERENTIAL EQUATIONS

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GENERAL LINEAR METHODS FOR SOLVING ORDINARY AND FUZZY DIFFERENTIAL EQUATIONS

By

FATIN NADIAH BINTI ABD HAMID

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

December 2017

DEDICATIONS

TO MY FAMILY:
Dad: Abd Hamid Bin Daud,
Mom: Nurjannah Khoo Binti Abdullah,
. . .
Abang, Adik, Hafiy, Atuk

Gadis-Gadis Inspem
Friends

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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FATIN NADIAH BINTI ABD HAMID

December 2017

Chairman : **Faranak Rabiei, PhD**
Faculty : **Institute for Mathematical Research**

In this thesis, a third order General Linear Method (GLM) is proposed for solving ordinary and fuzzy differential equations as well as second kind of fuzzy Volterra integro-differential equations (FVIDEs). Also, the fuzzy version of Improved Runge-Kutta (IRK) method is adapted to solve the second kind of FVIDEs.

Third order General Linear Method is derived using the technique of rooted trees and B-series. The algebraic order conditions of the method are obtained up to trees of order four. Using the order conditions, we obtained the different sets of coefficients for GLM of order three. Stability region is discussed and numerical results of GLMs for solving first order ordinary differential equations (ODEs) are compared with the existing method.

Subsequently, numerical solutions of first order fuzzy differential equations (FDEs) are proposed using fuzzy version of obtained GLM. The approach of generalized Hukuhara differentiability is used to define the FDEs. Based on this approach, the characterization theorem which converted the FDEs into systems of ODEs is explored. Then, a fuzzy version of third order GLM for solving FDEs using the generalized Hukuhara differentiability is developed. The convergence of the method is given and numerical results compared with different existing methods are presented.

The study of FDEs is then extended to the first order fuzzy Volterra integro-differential equations. Unlike FDEs, a differential and integral operators appear simultaneously in FVIDEs. Therefore, suitable numerical quadrature rules which are the combination of composite Simpson's rule together with Lagrange interpolation polynomial and Trapezoidal rule are used to solve the integral part whereas the third order GLM is considered for the differential part. Again, the generalized Hukuhara differentiability is applied to develop the GLM combined with given numerical quadrature rules for solving FVIDEs. Numerical results are tabulated to illustrate the performance of the proposed method.

Finally, Improved Runge-Kutta method of order four with three stages is proposed to obtain the numerical solutions of FVIDEs. A similar strategy is used to develop the Improved Runge-Kutta method by adapting the same numerical quadrature rules as used for General Linear Method and is based on generalized Hukuhara differentiability. The performance of Improved Runge-Kutta method is demonstrated by comparing the numerical results with the existing method of same order.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KAEDAH LINEAR UMUM UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA DAN KABUR

Oleh

FATIN NADIAH BINTI ABD HAMID

Disember 2017

Pengerusi : **Faranak Rabiei, PhD**
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Dalam tesis ini, Kaedah Linear Umum (KLU) peringkat ketiga dicadangkan untuk menyelesaikan persamaan pembezaan biasa dan kabur serta jenis kedua persamaaan pembezaan-kamiran kabur Volterra (PPKKV). Juga, kaedah Runge-Kutta (RKP) versi kabur disesuaikan untuk menyelesaikan jenis kedua PPKKV.

Kaedah Linear Umum peringkat ketiga diperoleh dengan menggunakan teknik pokok berakar dan B-siri. Syarat-syarat peringkat bagi kaedah ini diperolehi sehingga pokok peringkat keempat. Menggunakan syarat-syarat peringkat, set-set pekali yang berlainan untuk KLU peringkat ketiga diperoleh. Rantau kestabilan dibincangkan dan hasil berangka KLU untuk menyelesaikan persamaan pembezaan biasa (PBB) bagi peringkat pertama dibandingkan dengan kaedah sedia ada.

Seterusnya, penyelesaian berangka persamaan pembezaan kabur (PPK) peringkat pertama dicadangkan menggunakan versi kabur KLU yang diperolehi. Pendekatan kebolehbezaan umum Hukuhara digunakan untuk menakrifkan PPK. Berdasarkan pendekatan ini, teorem pencirian yang mengubah PPK menjadi sistem PBB dikaji. Kemudian, KLU peingkat ketiga untuk menyelesaikan PPK menggunakan kebolehbezaan umum Hukuhara dibangunkan. Penumpuan KLU versi kabur diberikan dan hasil berangka yang dibandingkan dengan kaedah yang sedia ada dibentangkan.

Kajian PBK kemudian dilanjutkan kepada peringkat pertama PPKKV. Tidak seperti PPK, operasi pembezaan dan kamiran muncul serentak dalam PPKKV. Oleh itu, aturan kuadratur berangka yang sesuai dimana gabungan aturan komposit Simpson bersama dengan interpolasi polinomial Lagrange dan aturan Trapezoidal digunakan untuk menyelesaikan bahagian kamiran manakala KLU peringkat ketiga dipertimbangkan untuk bahagian pembezaan. Sekali lagi, kebolehbezaan umum Hukuhara digunakan untuk membangunkan KLU yang digabungkan dengan aturan kuadratur berangka bagi menyelesaikan PPKKV. Hasil berangka dijadualkan untuk membentang prestasi kaedah yang dicadangkan.

Akhir sekali, kaedah Runge-Kutta penambahbaikan peringkat keempat dengan tahap tiga dicadang untuk mendapatkan penyelesaian PPKKV. Strategi yang sama digunakan untuk membangunkan kaedah Runge-Kutta penambahbaikan dengan menyesuaikan aturan kuadratur berangka yang sama seperti dalam KLU dan adalah berdasarkan kebolehbezaan umum Hukuhara. Prestasi kaedah Runge-Kutta penambahbaikan ditunjukkan dengan membandingkan dengan kaedah sedia ada yang berperingkat sama.

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THESIS EXAMINATION COMMITTEE

I certify that a Thesis Examination Committee has met on 20 December 2017 to conduct the final examination of FATIN NADIAH BINTI ABD HAMID on her thesis entitled ”GENERAL LINEAR METHODS FOR SOLVING ORDINARY AND FUZZY DIFFERENTIAL EQUATIONS” in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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LIST OF ABBREVIATIONS

IVP	Initial Value Problem
ODEs	Ordinary Differential Equations
FDEs	Fuzzy Differential Equations
FVIDEs	Fuzzy Volterra Integro-Differential Equations
GLM	General Linear Method
GLM1	General Linear Method with coefficients Set 1
GLM2	General Linear Method with coefficients Set 2
GLM3	General Linear Method with coefficients Set 3
RK(3)	Third order Runge-Kutta Method
GLM(3)	General Linear Method with coefficients Set 1
ABM(3)	Third order Adams-Bashorth-Moulton predictor-corrector method
ANN	Artificial Neural Network method
IRK(4)	Fourth order, three stages Improved Runge-Kutta method
RK(4)	Fourth order Runge-Kutta Method

CHAPTER 1

INTRODUCTION

In today's world, knowledge in mathematics is fundamental in real world applications. Many situations or problems are displayed as mathematical models which are then actively studied by scientists and mathematicians. One of the prominent mathematical models is known as the differential equation. The appearance of differential equations can be seen in major fields such as engineering, medicine, physics and other sciences. For example, describing the population growth of living species, exponential decay and growth, designing optimum business strategies and finding the flow of electricity are some of the uses of differential equations.

In some situations, the parameters required as initial values for differential equations are uncertain. Therefore, it is necessary that the problems with uncertainty are resolved by employing the fuzzy differential equations (FDEs). The FDEs emerged in many applications such as in hydraulic systems (Bencsik et al., 2006), modeling of earthquake engineering vibration problems (Marano et al., 2010), medicine (Torres and Nieto, 2006) and other branches of sciences and engineering.

Mathematical problems are typically solved using exact formulae. However, many problems are complicated to deal with just by using the analytic method and hence numerical methods are suggested.

1.1 Motivation

The search of numerical methods to find the best solutions of differential equations are continuously studied. A particular numerical method had our attention which is the General Linear Method (GLM). The GLM introduced by Butcher (2006) is a natural generalization of the linear multistep method and Runge-Kutta method. Adapting the behavior of both linear multistep method and Runge-Kutta method potentially leads to a more accurate method. Since the existing derivation of GLM can be complicated to understand, it is interesting to offer a simpler strategy of deriving the GLM using the rooted trees and B-series. Additionally, it is compelling to solve the first order ODEs and FDEs using the proposed method. Hopefully, we could obtain improve results by employing the proposed method.

1.2 Objective of the Thesis

The objectives of this thesis are given as following:

(a) to derive new sets of coefficients for a third order GLM using the rooted trees and theory of B-series,
(b) to obtain the stability polynomials and the stability regions of GLM,
(c) to develop and implement the third order GLM with a new set of coefficient for solving first order ODEs and FDEs using the generalized Hukuhara differentiability,
(d) to prove the convergence of third order GLM for solving FDEs,
(e) to construct the third order GLM for solving first order FVIDEs using the generalized Hukuhara differentiability,
(f) to construct a fourth order with three stages Improved Runge-Kutta (IRK) method for solving first order FVIDEs using the generalized Hukuhara differentiability.

1.3 Scope of the Study

In this research, in depth details on derivation of proposed method using the rooted trees and B-series are studied. There will be two types of problems to be focused on solving which are the ODEs and FDEs. Fuzzy Volterra integro-differential equations appear as a part of FDEs are included to be solved as well. In FDEs, the approach of generalized Hukuhara differentiability is highlighted. The third order general linear method is implemented to solve the ODEs, FDEs and FVIDEs. Also, IRK is implemented to solve the FVIDEs.

1.4 Outline of the Thesis

In Chapter 1, the scope of study and objectives of this thesis are given.

In Chapter 2, the introduction to ordinary differential equations, fuzzy differential equations, Volterra integro-differential equations and fuzzy Volterra integro-differential equations are given. The definitions and basic theorems of fuzzy theory are presented as well.

In Chapter 3, we give the introduction and derivation of a third order GLM using B-series and a technique known as rooted trees. This leads to the order conditions of GLM being obtained. We derive new sets of coefficients from the obtained order conditions and show the effectiveness of this new set of coefficients, some test problems are carried out. The stability region of GLM is proposed as well.

In Chapter 4, the FDEs are interpreted based on the generalized Hukuhara differentiability concept and by using the characterization theorem, the FDEs are converted into systems of ODEs. We develop the third order GLM based on the generalized Hukuhara differentiability and the convergence of the GLM is showed. Several test problems of FDEs are applied using the developed GLM. Numerical results and graphical illustrations are presented and compared with existing numerical method.

In Chapter 5, fuzzy Volterra integro-differential equations which are part of FDEs are explored. The FVIDEs appeared by having both the differential and integral operators in its equations. By using the same concept of generalized Hukuhara differentiability, the third order GLM is developed to solve the FVIDEs by adapting composite Simpson's rule and Lagrange interpolation polynomial. These numerical integration methods are applied to achieve the solutions from the integral operator accurately. Numerical results and graphical illustrations are included and compared with existing numerical method. Apart from the GLM, we develop the Improved Runge-Kutta method of order four for solving FVIDEs. IRK method requires lower function evaluations than the existing Runge-Kutta method. To illustrate the performance of IRK for solving FVIDEs, numerical results are provided and discussed.

Lastly, Chapter 6 covers the summary of this thesis and future works that could extend the research this study.

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