UNIVERSITI PUTRA MALAYSIA

COMBINATORIAL STRUCTURE ASSOCIATED WITH LOW-DIMENSIONAL FILIFORM LEIBNIZ ALGEBRAS

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IPM 2018 2

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By

AYU AMELIATUL SHAHILAH BINTI AHMAD JAMRI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Master of Science

April 2017

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DEDICATIONS

To my parents
Sarimah Kantong and Ahmad Jamri Mohd Piah
for raising me to believe that
everything was possible
. . .

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

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April 2017

Chairman: Sharifah Kartini Said Husain, PhD
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This thesis is concerned on the studying a graph representation of $(n+1)$-dimensional filiform Leibniz algebras. The filiform Leibniz algebras contain three subclasses called first, second and third class that are denoted in dimension n over a field K, by $FLb_n(K)$, $SLb_n(K)$ and $TLb_n(K)$, respectively.

This research deals with combinatorial structures associated with $FLb_n(K)$ and $SLb_n(K)$. Therefore, an algorithm is defined in order to construct such structures associated with filiform Leibniz algebras. By using the table of multiplication of filiform Leibniz algebras, an algorithm for the combinatorial structures associated with filiform Leibniz algebras will be obtained.

Next, the structural properties of the combinatorial structure will be constructed to show the non-isomorphism between two classes of filiform Leibniz algebras in such a way of graph theory. Hence, some propositions on combinatorial structures regarding number of vertices and edge, components, degree of vertices, diameter and degree sequences are given.

Besides that, an algorithm will be used on association the combinatorial structures with the isomorphism classes of $FLb_n(K)$ and $SLb_n(K)$. Thus, any two isomorphism classes of $FLb_n(K)$ or $SLb_n(K)$ are non-isomorphic using combinatorial structures.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Sarjana Sains

STRUKTUR KOMBINATORIK BERKAITAN DENGAN DIMENSI RENDAH BAGI ALJABAR LEIBNIZ FILIFORM

Oleh

AYU AMELIATUL SHAHILAH BINTI AHMAD JAMRI

April 2017

Pengerusi: Sharifah Kartini Said Husain, PhD
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Tesis ini adalah berkenaan dengan kajian terhadap perwakilan graf bagi subkelas $(n+1)$-dimensi aljabar Leibniz filiform. Aljabar Leibniz filiform mengandungi tiga subkelas dipanggil kelas pertama, kedua dan ketiga yang diwakili, dalam dimensi n pada medan K, dengan masing- masing $FLb_n(K)$, $SLb_n(K)$ dan $TLb_n(K)$.

Kajian ini dilaksanakan dengan struktur kombinatorik yang berkaitan dengan $FLb_n(K)$ dan $SLb_n(K)$. Oleh itu, algoritma adalah ditakrifkan untuk membina struktur kombinatorik yang berkaitan dengan aljabar Leibniz filiform. Dengan menggunakan jadual pendaraban untuk aljabar Leibniz filiform, algoritma bagi struktur kombinatorik yang berkaitan dengan aljabar Leibniz filiform akan diperolehi.

Seterusnya, sifat-sifat struktural bagi struktur kombinatorik akan dibina untuk membuktikan bukan isomorfisma diantara dua kelas aljabar Leibniz filiform dalam cara teori graf. Oleh itu, beberapa sifat-sifat bagi struktur kombinatorik mengenai bilangan bucu dan sisi, komponen, bucu darjah, diameter dan jujukan darjah adalah diberi.

Selain itu, algoritma akan digunakan bagi kaitan struktur kombinatorik dengan kelas isomorfisma untuk $FLb_n(K)$ dan $SLb_n(K)$. Oleh itu, mana-mana dua kelas isomorfisma bagi $FLb_n(K)$ atau $SLb_n(K)$ adalah bukan isomorfisma dengan menggunakan struktur kombinatorik.

ACKNOWLEDGEMENTS

Alhamdulillah. I praise the Almighty Allah providing me the opportunity for making this Master thesis entitled ”Combinatorial Structure Associated with Low-Dimensional Filiform Leibniz Algebras” success.

I would like to express my special thanks of gratitude to my supervisor, Dr. Sharifah Kartini Said Husain who gave me the golden opportunity to do this project, which helped me in doing a lot of research and I came to know about so many new things. Without his guidance this thesis would not have been possible.

I also want to thank my committee supervisor, Professor Dr. Isamiddin S. Rakhimov for his guidance, contributions, and constant support throughtout this research. He has been very patient in contributing his time to improve my research.

I would like to thank all lecturers and staffs of Department of Mathematics and Institute of Mathematical Research, Universiti Putra Malaysia for their assistance through out my study. Next, these acknowledgements would not be completed without mentioning my main financial support during my studies, School of Graduate Studies, Universiti Putra Malaysia for choosing me to be one of the grantees of the Graduate Research Fellowship (GRF) and MyBrain program.

Last but not least, I express my sincere thanks to my entire family for their love, encouragements and supports both financially and mentally to complete my study. Besides, an immeasurable gratitude goes to my dear friends and colleagues for sharing valuable knowledge and always give support to me. Over and above all, I thank Allah who saw me through it all.

I certify that a Thesis Examination Committee has met on 20 April 2017 to conduct the final examination of Ayu Ameliatul Shahilah binti Ahmad Jamri on her thesis entitled "Combinatorial Structure Associated with Low-Dimensional Filiform Leibniz Algebras" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

$\mathbb{C}$	The field of complex numbers
$\mathbb{N}$	The field of natural numbers
$\mathbb{R}$	The field of real numbers
$\mathbb{Z}$	The field of integer numbers
Lb_n	n-dimensional filiform Leibniz algebras
FLb_{n+1}	First class of $(n+1)$-dimensional filiform Leibniz algebras
SLb_{n+1}	Second class of $(n+1)$-dimensional filiform Leibniz algebras
TLb_{n+1}	Third class of $(n+1)$-dimensional filiform Leibniz algebras
NGF_i	Natural Gradation of filiform Leibniz algebra
grL	Natural Gradation of Leibniz algebra
γ_{ij}^k	Structural constants
$M_n(K)$	Set of all $n \times n$ matrices over a field K
$End(V)$	Set of all endormorphism of vector space V
$dim_{\mathbb{R}}(\mathbb{C})$	Dimension of complex numbers over real numbers
$dim_K(M_n(K))$	Dimension of $n \times n$ matrices over a field K
$dimL^i$	Dimension of i-th degree of L
$G = (V,E)$	A graph
$V(G), V_G$	Set of vertices of graph G
$E(G), E_G$	Set of edges of graph G
$\lvert V(G)\rvert$	Cardinality of $V(G)$
$\lvert E(G)\rvert$	Cardinality of $E(G)$
$N_G(v)$	Set of all vertices in $V(G)$ that are adjacent to v
$deg(v), d_v$	Degree of a vertex v
$\{\}$	Empty set
P_n	Path of n vertices
$H \subseteq G$	H is a subgraph of G
$G-\{v\}$	Graph obtained from G by deleting the vertex v and all edges incident with v
$G-\{e\}$	Graph obtained from G by deleting the edge e
$d(u,v)$	Length of shortest path from u to v
$diam(G)$	Diameter of G
a_{ij}	(i,j)-entry matrix A where row i and column j
A_G	Adjacency matrix of a graph G
$G(Lb_{n+1})$	Graph of Lb_{n+1}
$G(FLb_{n+1})$	Graph of FLb_{n+1}
$G(SLb_{n+1})$	Graph of SLb_{n+1}
$W_G(a_k b_{ij})$	Weight of edge $(a_k b_{ij})$ in graph G

CHAPTER 1

INTRODUCTION

1.1 Introduction

This chapter is organized to review some basic concepts of algebra and Leibniz algebra which have been used in this research. Some literatures that have been referred to are also mentioned in this chapter followed by research objectives.

1.2 Basic Concepts

In this section, we state some definitions and examples which are used throughout this research. We also provide a structure of Leibniz algebra to give a clearer overview as shown in Figure 1.2.

In 1979, Jacobson introduced the definitions of algebra, associative algebra and Lie algebra, which are mentioned in Definitions 1.1, 1.2 and 1.3, respectively.

Definition 1.1 *An algebra A over a field K is a vector space over K equipped with a bilinear map $f : A \times A \to A$ such that*

i) $f(x,y+z) = f(x,y) + f(x,z)$,

ii) $f(x+y,z) = f(x,z) + f(y,z)$,

iii) $f(\alpha x,z) = \alpha f(x,z)$,

iv) $f(x,\alpha z) = \alpha f(x,z)$,

for all $x,y,z \in A$ and $\alpha \in K$.

Example 1.1 *(Girard, 1984). Let H be the set of all numbers of the form:*

$$H = \{a+bi+cj+dk \mid \text{ where } a,b,c,d \in R\}.$$

Suppose H is a quaternion of 4-dimensional vector space V over the field of real numbers $\mathbb{R}$ with a basis $\{1,i,j,k\}$. Then, the multiplication conditions are imposed:

i) $1 \cdot a = a$, *for all* $a \in H$,

ii) $i^2 = j^2 = k^2 = ijk = -1$,

iii) $ij = k, jk = i, ki = j, ji = -k, kj = -i, ik = -j$.

By considering a multiplication of any two quaternions, $H_1 = a_1 + b_1 i + c_1 j + d_1 k$ *and* $H_2 = a_2 + b_2 i + c_2 j + d_2 k$,

$$\begin{aligned} H_1 H_2 &= (a_1 + b_1 i + c_1 j + d_1 k)(a_2 + b_2 i + c_2 j + d_2 k) \\ &= a_1 a_2 - a_1 b_2 i + a_1 c_2 j + a_1 d_2 k + b_1 a_2 i + b_1 b_2 i^2 + b_1 c_2 ij + b_1 d_2 ik \\ &\quad + c_1 a_2 j + c_1 b_2 ji + c_1 c_2 j^2 + c_1 d_2 jk + d_1 a_2 k + d_1 b_2 ki + d_1 c_2 kj + d_1 d_2 k^2 \\ &= (a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2) + (a_1 b_2 + a_2 b_1 - c_1 d_2 - d_1 c_2) i \\ &\quad + (a_1 c_2 - b_1 d_2 + c_1 a_2 - d_1 b_2) j + (a_1 d_2 + b_1 c_2 - c_1 b_2 + d_1 a_2) k \end{aligned}$$

Thus, H with three operations: vector addition, scalar multiplication and quaternion multiplication is said to be quaternion algebra.

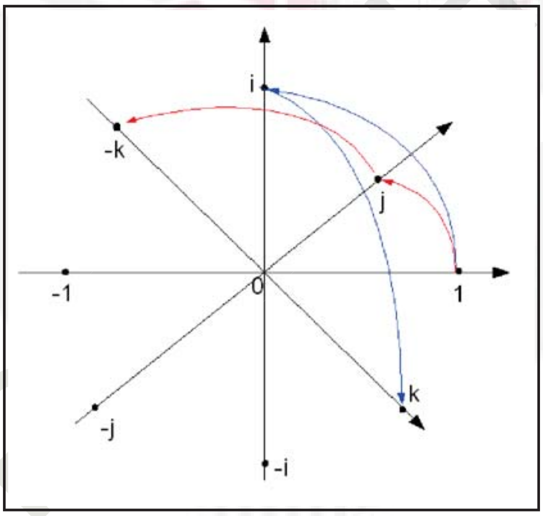

Figure 1.1: Graphical representation of quaternion units in 4D-space, $ij = k, ji = -k, ij = -ji$

Example 1.2 *(Mohamed, 2014). The set of all complex numbers* $\mathbb{C}$ *is an algebra over the set of all real numbers* $\mathbb{R}$ *with* $dim_{\mathbb{R}}(\mathbb{C}) = 2$. *One set of basis is* $\{a, bi\}$ *where* $a, b \in \mathbb{R}$.

Definition 1.2 *An associative algebra A over a field K is a vector space over a field K with a bilinear map* $f : A \times A \to A$ *satisfying the associative law*

$$f(f(x,y),z) = f(x, f(y,z)), \text{ for all } x, y, z \in A.$$

Example 1.3 *(Jacobson, 1979) The vector space of n-square matrices, $M_n(K)$ over a field K is an associative algebra over a field K where $dim_K(M_n(K)) = n^2$.*

Example 1.4 *(Abdulkareem, 2014). Let V be an n-dimensional vector space over a field K. The set of all endormorphism, $End(V)$ forms a vector space is a linear transformation from V to V. The multiplication of two elements $f, g \in End(V)$ is defined by*

$$(f \circ g)(v) = f(g(v)), \text{ for all } v \in V.$$

This product of $End(V)$ is an associative algebra.

Definition 1.3 *Let L be an algebra over a field K is called Lie algebra if its bilinear operation $[\cdot,\cdot]$ satisfies the following properties:*

i) Anti-symmetry: $[x,y] = -[y,x]$, for all $x, y \in L$,

ii) Jacobi identity: $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$, for all $x,y,z \in L$.

The following two examples of Lie algebra can be found in Goze and Khakimdjanov (2013).

Example 1.5 *Let V be the 3-dimensional Euclidean $\mathbb{R}^3$ and define the bracket $[\cdot,\cdot] : \mathbb{R}^3 \to \mathbb{R}^3$ by $[x,y] = x * y$ for all $x, y \in V$, the cross-product of the vector becomes a 3-dimensional Lie algebra.*

Example 1.6 *Let A be an associative algebra over a field K, then the bilinear operation $[\cdot,\cdot]$ on A defined by $[x,y] = xy - yx$, for all $x, y \in A$. Therefore, A together with $[\cdot,\cdot]$ is a Lie algebra.*

Example 1.7 *(Abdulkareem, 2014). Let the product of two elements in Example 1.4 be defined by $[f,g] = f \circ g - g \circ f$. The bracket $[\cdot,\cdot]$ is called Lie bracket. The Jacobi identity was verify as shown below.*

$$\begin{aligned}
[[f,g],h] + [[g,h],f] + [[h,f],g] &= (f \circ g - g \circ f) \circ h - h \circ (f \circ g - g \circ f) \\
&\quad + (g \circ h - h \circ g) \circ f - f \circ (g \circ h - h \circ g) \\
&\quad + (h \circ f - f \circ h) \circ g - g \circ (h \circ f - f \circ h) \\
&= 0
\end{aligned}$$

Therefore, $(End(V), [\cdot,\cdot])$ is a Lie algebra.

Now, we give the main definitions that we focus in this research.

Definition 1.4 *(Rikhsiboev and Rakhimov, 2012). An algebra L over a field K is called a Leibniz algebra, if it statisfies the following Leibniz identity:*

$$[[x,y],z] = [[x,z],y] + [x,[y,z]], \textit{ for all } x,y,z \in L,$$

where $[\cdot,\cdot]$ *is the multiplication in L.*

Examples of Leibniz algebras can be found in Demir et al. (2014).

Example 1.8 *Let L be a 2-dimensional algebra with the following multiplications*

$$[y,x] = x, [y,y] = x, \textit{ for all } x,y \in L,$$

then L is a Leibniz algebra.

Example 1.9 *Let A be any associative algebra over a field K equipped with a linear operator* $D : A \to A$ *such that* $D(xD(y)) = DxDy = D((Dx)y)$*, for any* $x,y \in A$*. Define the multiplication* $[\cdot,\cdot] : A \times A \to A$ *by*

$$[x,y] = (Dx)y - y(Dx), \textit{ for all } x,y \in A.$$

The Leibniz identity can rewrite as follows:

$[[x,y],z] - [[x,z],y] - [x,[y,z]] = 0.$

Then, substitute the definition $[x,y] = (Dx)y - y(Dx)$*, its become*

$$\begin{aligned}
&= [(Dx)y - y(Dx), z] - [(Dx)z - z(Dx), y] - [x, (Dy)z - z(Dy)] \\
&= D((Dx)y)z - zD((Dx)y) - D(y(Dx))z + z(D(y(Dx))) \\
&\quad - (D((Dx)z)y) - y(D((Dx)z)) - D(zD(x))y + yD(zD(x))) \\
&\quad - ((Dx)(Dy)z - (Dy)z(Dx) - (Dx)z(Dy) + z(Dy)(Dx)) \\
&= DxDyz - zDxDy - DyDxz + zDyDx - DxDzy + yDxDz + DzDxy \\
&\quad - yDzDx - (Dx)(Dy)z + (Dy)z(Dx) + (Dx)z(Dy) - z(Dy)(Dx) \\
&= z(DxDy - DxDy) + z(DyDx - DyDx) + y(DxDz - DxDz) \\
&\quad + y(DzDx - DzDx) + z((Dx)(Dy) - (Dx)(Dy)) + z((Dy)(Dx) - (Dy)(Dx)) \\
&= 0
\end{aligned}$$

Thus, $(A, [\cdot,\cdot])$ *is a Leibniz algebra.*

If a Leibniz algebra L has the property of antisymmetricity

$$[x,y] = -[y,x], \text{ for all } x,y \in L.$$

Then substitute antisymmetric property to Leibniz identity we will obtain as follow:

$$[[x,y],z] - [[x,z],y] - [x,[y,z]] = 0$$
$$[[x,y],z] - [-[z,x],y] - [-[[y,z],x]] = 0$$
$$[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$$

The last expression is shown that the Leibniz identity can be easily simplified into Jacobi identity: $[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$. Therefore, Leibniz algebras are generalizations of Lie algebras.

For a given finite-dimensional Leibniz algebra L, take

$$L^1 = L, L^2 = [L^1, L], L^3 = [L^2, L], \cdots, L^{k+1} = [L^k, L], \text{ where } k \in \mathbb{N}.$$

Then the descending series of L can be write as follows:

$$L^1 \supset L^2 \supset \cdots \supset L^s \supset \cdots.$$

From the descending series, Ayupov and Omirov (2001) introduce the following definitions.

Definition 1.5 *A Leibniz algebra L is said to be nilpotent if there exists an integer $s \in \mathbb{N}$, such that*

$$L^1 \supset L^2 \supset \cdots \supset L^s = 0.$$

The smallest integer s for that $L^s = 0$ is called the nilindex of L.

Definition 1.6 *An n-dimensional Leibniz algebra L is said to be filiform, if*

$$dim L^i = n - i, \text{ where } 2 \le i \le n.$$

By Definitions 1.5 and 1.6, it is clear that filiform Leibniz algebra is nilpotent but nilpotent Leibniz algebra is not necessarily to be filiform.

Example 1.10 *(Abdulkareem, 2014) Let L be a 4-dimensional Leibniz algebra with a basis $\{e_1, e_2, e_3, e_4\}$ and let the table of multiplication be given as follows:*

$$[e_1, e_1] = e_2, [e_2, e_1] = e_3, [e_1, e_2] = e_4.$$

Then,

- $L^1 = L = span\{e_1, e_2, e_3, e_4\}$, *where* $dimL^1 = 4$,
- $L^2 = [L, L] = span\{e_2, e_3, e_4\}$, *where* $dimL^2 = 3$,
- $L^3 = [L^2, L] = span\{e_3\}$, *where* $dimL^3 = 1$,
- $L^4 = [L^3, L] = span\{0\}$, *where* $dimL^4 = 0$.

Hence, $L^1 \supset L^2 \supset L^3 \supset L^4 = 0$. *Therefore, L is a nilpotent Leibniz algebra.*

The Example 1.10 is an example of nilpotent but not filiform Leibniz algebra since $dimL^2 = 3 \neq n - 2 = 4 - 2 = 2$.

Example 1.11 *(Abdulkareem, 2014) Let L be a 4-dimensional Leibniz algebra with a basis* $\{e_1, e_2, e_3, e_4\}$. *Let the multiplication table for L be given as follows:*

$$[e_1, e_1] = e_2, [e_1, e_2] = e_2, [e_2, e_1] = e_4, [e_3, e_1] = e_4.$$

Then, the dimension of degree of L given as follows:

- $L^1 = L = span\{e_1, e_2, e_3, e_4\}$, *where* $dimL^1 = 4$,
- $L^2 = [L, L] = span\{e_2, e_4\}$, *where* $dimL^2 = 2$,
- $L^3 = [L^2, L] = span\{e_4\}$, *where* $dimL^3 = 1$,
- $L^4 = [L^3, L] = span\{0\}$, *where* $dimL^4 = 0$.

Thus, L is filiform Leibniz algebra since $dimL^2 = n - i = 4 - 2 = 2$. *Its clear that filiform Leibniz algebra L is a nilpotent Leibniz algebra since* $L^1 \supset L^2 \supset L^3 \supset L^4 = 0$.

Leibniz algebras have been introduced as a “non-antisymmetric” analogue of Lie algebras (Loday, 1993). Hence, any Lie algebra is a Leibniz algebra. In Leibniz algebras, some difficulties arise when considering the nilpotent Leibniz algebras up to dimension 5. Therefore, Gomez and Omirov (2015) proposed a subset of nilpotent Leibniz algebras called filiform Leibniz algebras. The class of filiform Leibniz algebras in dimension n is denoted by Lb_n. This n-dimensional filiform Leibniz algebras is split into three subclasses denoted by FLb_n, SLb_n and TLb_n. It is shown in Theorem 2.2 of Chapter 2. In general, $Lb_n = FLb_n \cup SLb_n \cup TLb_n$. The following figure illustrates the Leibniz algebras structures.

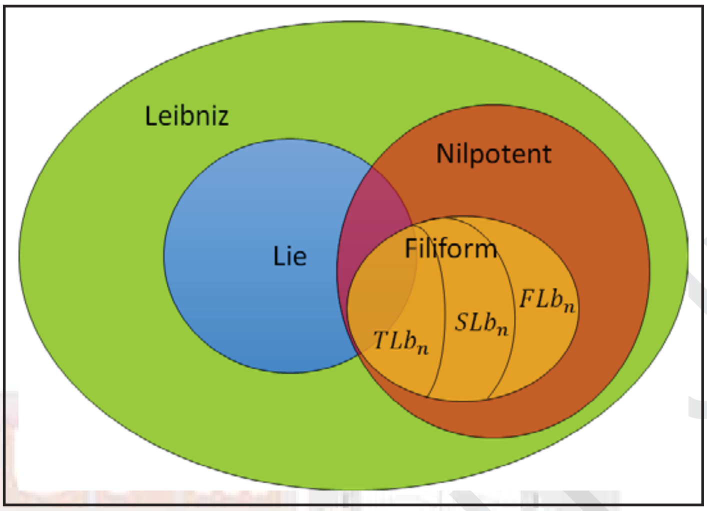

Figure 1.2: Leibniz algebras structures

Let $\{e_i\}_{i=1}^{n}$ be a basis of a n-dimensional Leibniz algebra L over the field K. Then, the table of multiplication of L is defined by $[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k$, for $1 \leq i, j \leq n$. Here, γ_{ij}^{k} is said to be structure constants of L, where the set of n^3 constants, $\gamma_{ij}^{k} \in K$. The n-dimensional Leibniz algebra L is specified by the following system of equations with respect to structure constants γ_{ij}^{k}:

$$\sum_{l=1}^{n} (\gamma_{jk}^{l}\gamma_{il}^{m} - \gamma_{ij}^{l}\gamma_{lk}^{m} + \gamma_{ik}^{l}\gamma_{lj}^{m}) = 0, \quad \text{where } i, j, k, m = 1, 2, \cdots, n.$$

When a basis is fixed and the set of structure constants relative to this basis are given, a Leibniz algebra with respect to this basis and the structure constants was described. This description is called table of multiplication of the Leibniz algebra. In this study, all algebras considered are finite-dimensional over the field of complex number $\mathbb{C}$.

1.3 Literature Review

The concept of Leibniz algebra was introduced by Loday (1993). It has been introduced as a non-associative algebra and a anticommutative generalization of Lie algebra. In fact, Leibniz algebras inherit some properties of Lie algebras. Many well-known results on Lie algebras can be extended to Leibniz algebras. For instance, Skjelbred and Sund (1978) used the central extension in the classification problem of nilpotent Lie algebras in a given dimension. Like in the Lie algebras case, the Leibniz central extension play an important role in the structural theory of Leibniz algebras. Therefore, Rakhimov and Langari (2010) applied the Skjelbred-Sund classification method to the classification problem of complex nilpotent

Leibniz algebras in low dimensional cases. The Leibniz identity and Jacobi identity are similar when the multiplication is skew-symmetric.

In Leibniz algebras, results that focus on nilpotency, classification of low-dimensional Leibniz algebras were considered in works of Ayupov and Omirov (1991, 2001). In 2001, Ayupov and Omirov described the filiform complex Leibniz algebra and classification of Leibniz algebras with maximal nilpotency index are given as well. In 2006, the classification of 4-dimensional nilpotent complex Leibniz algebras was studied by Albeverio et al. (2006). Also in 2015, a method of simplification of the basis transformations of filiform Leibniz algebra that obtained from the naturally graded filiform non-Lie filiform Leibniz algebras had been proposed (by Gomez and Omirov).

The filiform Leibniz algebras in dimension n over a field K were split into three subclasses denoted by $FLb_n(K)$, $SLb_n(K)$ and $TLb_n(K)$ (see, Rakhimov and Bekbaev (2010) and Gomez and Omirov (2015)). The classification problem of low dimensional for filiform Leibniz algebras over complex field have been solved by Sozan et al. (2010), Hassan et al. (2010), Rakhimov and Said Husain (2011a,b), Deraman et al. (2012), Mohd Kasim (2014), Mohamed (2014) and Abdulkareem et al. (2015).

On the other hand, graph theory is a useful mathematical tool to deal with the study of other topics. Many researchers concern more on the study of the interaction between Lie algebras and some kind of graphs. Carriazo et al. (2004) introduced the relation between Lie algebras of finite dimension from a selected basis and certain types of combinatorial structures. They authors characterized the types of combinatorial structures(or graphs) associated with Lie algebras and this association leads to a complete characterization according to the graphs have 3-cycles or not. Later, Fernández and Martín-Martínez (2005) studied such association if the combinatorial structure is only formed by triangles of weighted and non-directed edges so called triangular configurations. They considered triangular configurations which are plane and connected sets of triangles such that any two non-disjoint of those triangles only share either one edge or one vertex.

In 2011, Ceballos et al. proposed a method that associated the n-dimensional Lie algebras with complete simple graph. By using this combinatorial structure, they showed some properties that related to Lie algebras. They also showed the types of Lie algebras associated with combinatorial structures in order to get their classification of Lie algebras. Ceballos et al. (2012) proposed the complete triangular structures and several families of digraphs that associated with Lie algebras. The properties of these structures are used as a tool for classifying the types of Lie algebras. The result showed that using complete graphs, cycle and bipartite digraphs are very useful in determining the families of Lie algebras. Later, Cáceres et al. (2012) dealt with several operations on graphs that linking them with their associated Lie algebras. They obtained some criterias to determine when there exists a Lie algebra associated with combinatorial structures and an algorithmic method for one of those operations was shown.

Núñez et al. (2010) classified the family n-dimensional Lie algebras over the finite field $\mathbb{Z}/3\mathbb{Z}$ by using certain kind of graphs called directed pseudographs as a tool. They introduced properties of such graphs by using graph theory. In 2011, Falcón et al. used the same approach to classify low-dimensional filiform Lie algebras over a finite field. As a result, there exist six isomorphism classes of 6-dimensional filiform Lie algebras over $\mathbb{Z}/p\mathbb{Z}$ for $p = 2, 3, 5$.

The idea lies in the representation of each Lie algebras by a certain type of graph. Similarly, the algorithm to classify the low dimensional filiform Leibniz algebras over a complex field using graph as a tool is proposed. Until now, many results for classification on these algebras defined over complex field have been obtained by researchers. The classification up to isomorphism of any class of algebras is a fundamental and very difficult problem. Therefore, graph theory is used to make it easier to classify these family. In addition, we also apply this procedure to obtain the isomorphism classes of low dimensional filiform Leibniz algebras by using combinatorial structures.

In this research, some ideas from graph theory are applied to study structural properties of algebras. All algebras are assigned with a graph as follows:

1. Vertices of the graph correspond to the basis vectors of the algebra.
2. Two vertices are connected according to the table of multiplication of the algebra.

The concept of this research comes from Falcón et al. (2011) as mentioned above. The main purpose of this research is to establish the combinatorial structure that associated with Leibniz algebras. The focus of this research is the subclass of Leibniz algebras called filifrom Leibniz algebras.

1.4 Research Objectives

In this thesis, we deal with two mathematical fields: Leibniz algebra and graph theory. Our main goal was motivated by the paper by Carriazo et al. (2004), where a mapping between Lie algebras and combinatorial structures was proposed in order to convert some properties of Lie algebras into the language of graph theory and vice versa. In this research, the class of complex filiform Leibniz algebras arising from naturally graded non-Lie filiform Leibniz algebras that can be divided into two subclasses denoted by FLb_n and SLb_n in dimensions n are considered. The following are objectives of this research:

1. to propose an algorithm for the combinatorial structure associated with first and second class of non-Lie filiform Leibniz algebras.
2. to describe some properties and characterization of combinatorial structure that has been obtained.
3. to use combinatorial structure to find out any two filiform Leibniz algebras to be isomorphic or not.

1.5 Outline of Thesis

This thesis consists of five chapters. In Chapter 1, some basic definitions of algebra, Lie algebra and Leibniz algebra are reviewed. The literature reviews and research objectives for this research are also included in this chapter.

Chapter 2 reviews about a classification of filiform Leibniz algebras over complex field. The definition of nilpotent and filiform Leibniz algebra and followed by naturally graded non-Lie filiform Leibniz algebras was described. The concepts of graph theory also are described in this chapter.

The main results of this thesis are presented in Chapters 3 and 4. An algorithm for the combinatorial structure associated with filiform Leibniz algebra is provided in Chapter 3 while in Chapter 4, some structural properties on the combinatorial structures are given.

Chapter 5 contains the summary of this research and suggests some ideas for future research works.

REFERENCES

Abdulkareem, A. O. (2014). *Classification of a Subclass of Filiform Leibniz Algebras*. PhD thesis, Universiti Putra Malaysia.

Abdulkareem, O. A., Rakhimov, I. S., and Said Husain, S. K. (2015). Isomorphism classes and invariant of low dimensional filiform Leibniz algebras. *Linear and Multilinear Algebra*, 63(11):2254–2274.

Albeverio, S., Omirov, B. A., and Rakhimov, I. S. (2006). Classification of 4-dimensional nilpotent complex Leibniz algebras. *Extracta Mathematicae*, 21(3):197–210.

Ayupov, S. A. and Omirov, B. A. (1991). On 3-dimesional Leibniz algebras. *Uzbek Mathematical Journal*, 42(1):15–24.

Ayupov, S. A. and Omirov, B. A. (2001). On some classes of nilpotent Leibniz algebras. *Siberian Mathematical Journal*, 42(1):15–24.

Cáceres, J., Ceballos, M., Núñez, J., Puertas, M. L., and Tenorio, Á. F. (2012). Combinatorial structures of three vertices and Lie algebras. *International Journal of Computer Mathematics*, 89(13-14):1879–1900.

Carriazo, A., Fernández, L., and Núñez, J. (2004). Combinatorial structures associated with Lie algebras of finite dimension. *Linear algebra and its applications*, 389:43–61.

Ceballos, M., Nunez, J., and Tenorio, Á. F. (2011). Complete triangular structures and Lie algebras. *International Journal of Computer Mathematics*, 88(9):1839–1851.

Ceballos, M., Núñez, J., and Tenorio, Á. F. (2012). Study of Lie algebras by using combinatorial structures. *Linear Algebra and its Applications*, 436(2):349–363.

Chartrand, G. and Zhang, P. (2005). *Introduction to Graph Theory*. The Walter Rudin student series in advanced mathematics. McGraw-Hill Higher Education.

Demir, I., Misra, K. C., and Stitzinger, E. (2014). On some structures of Leibniz algebras. *American Mathematical Society*, 623(1):15–24.

Deraman, F., Rakhimov, I. S., and Said Husain, S. K. (2012). Isomorphism classes and invariants for a subclass of nine-dimensional filiform Leibniz algebras. *AIP Conference Proceedings*, 1450(1):326–331.

Falcón, O. J., Núñez, J., Pacheco, A. M., and Villar, M. (2011). Low-dimensional filiform Lie algebras over finite fields. In *Proceedings of the 2011 international conference on Applied & computational mathematics*, pages 66–71. World Scientific and Engineering Academy and Society (WSEAS).

Fernández, L. M. and Martín-Martínez, L. (2005). Lie algebras associated with triangular configurations. *Linear algebra and its applications*, 407:43–63.

Girard, P. R. (1984). The quaternion group and modern physics. *European Journal of Physics*, 5(1):25.

Gomez, J. and Omirov, B. (2015). On classification of filiform Leibniz algebras. *Algebra Colloquium*, 22:757–774.

Goze, M. and Khakimdjanov, Y. (2013). *Nilpotent Lie Algebras*. Springer Science & Business Media, Dordrecht.

Gustafson, R., Frisk, P., and Hughes, J. (2008). *College Algebra*. Available 2010 Titles Enhanced Web Assign Series. Cengage Learning.

Hassan, M. A., Said Husain, S. K., and Rakhimov, I. S. (2010). On isomorphisms and invariants of 5th dimensional complex filiform Leibniz algebras. *AIP Conference Proceedings*, 1309(1):430–438.

Jacobson, N. (1979). *Lie Algebras*. Dover books on advanced mathematics. Dover.

Loday, J.-L. (1993). Une version non commutative des algbres de Lie: les algbres de Leibniz. *Enseign. Math.*, 39:269–293.

Malik, D., Sen, M., and Ghosh, S. (2014). *Introduction to Graph Theory*. Cengage Learning Asia, Singapore.

Mohamed, N. (2014). Classification of first class of filiform Leibniz algebras in dimension 10. Master's thesis, Universiti Putra Malaysia.

Mohd Kasim, S. (2014). Classification of second class 10-dimensional complex filiform Leibniz algebras. Master's thesis, Universiti Putra Malaysia.

Neusel, M. (2007). *Invariant Theory*. Student mathematical library. American Mathematical Society.

Núñez, J., Pacheco, A., and Villar, M. (2010). Study of a family of Lie algebra over Z/3Z. *International Journal of Mathematics and Statistics*, 7(W10):40–45.

Rakhimov, I. S. and Bekbaev, U. D. (2010). On isomorphisms and invariants of finite dimensional complex filiform Leibniz algebras. *Communications in Algebra*, 38(12):4705–4738.

Rakhimov, I. S. and Langari, S. J. (2010). A cohomological approach for classifying nilpotent Leibniz algebras. *International Journal of Algebra*, 4(4):153–163.

Rakhimov, I. S. and Said Husain, S. K. (2011a). On isomorphism classes and invariants of a subclass of low-dimensional complex filiform Leibniz algebras. *Linear and Multilinear Algebra*, 59(2):205–220.

Rakhimov, I. S. and Said Husain, S. K. (2011b). Classification of a subclass of low-dimensional complex filiform Leibniz algebras. *Linear and Multilinear Algebra*, 59(3):339–354.

Rikhsiboev, I. M. and Rakhimov, I. S. (2012). Classification of three dimensional complex Leibniz algebras. *AIP Conference Proceedings*, 1450(1):358–362.

Skjelbred, T. and Sund, T. (1978). Sur la classification des algèbres de Lie nilpotentes. *CR Acad. Sci. Paris*, 286(1):978.

Sozan, J., Rakhimov, I., and Mohd Atan, K. (2010). Classification of first class of complex filiform Leibniz algebras. *Lambert Academic Publishing*.