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TRIGONOMETRICALLY FITTED MULTISTEP METHODS FOR SOLVING OSCILLATORY PROBLEMS

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By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Master of Science

December 2017

DEDICATIONS

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science

TRIGONOMETRICALLY FITTED MULTISTEP METHODS FOR SOLVING OSCILLATORY PROBLEMS

By

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Chairman : Fudziah binti Ismail, PhD
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At the beginning of the thesis, we trigonometrically fitted the first point of the existing block multistep method which was purposely derived for solving special second order ordinary differential equations (ODEs). Based on the original multistep method, we construct both explicit and implicit trigonometrically fitted multistep methods of step number k=4 and develop code using both constant and variable step size. The trigonometrically fitting technique has been applied to the original method in order to construct the new methods. The numerical results show that the trigonometrically fitted multistep method is more efficient compared to the existing methods in solving special second order ordinary differential equations (ODEs) which are oscillatory in nature.

Then, the 2-point explicit and implicit block multistep methods of step number k=3 and k=5 for solving special second order ODEs are derived using integration formula based on *Newton-Gregory backward interpolation polynomial*. The methods are implemented for constant step size by using the predictor-corrector technique, followed by the implementation using variable step size. The numerical results are given to show the efficiency of the new methods as compared to the existing methods.

The 2-point explicit and implicit block multistep methods of step number k=3 and k=5 are then trigonometrically fitted so that they are suitable for solving special second order ordinary differential equations, which are highly oscillatory in nature. We developed codes based on the trigonometrically fitted methods using constant step size in predictor-corrector mode. The numerical results obtained show that trigonometrically fitted the methods give more accurate solutions than the existing

methods.

In conclusion, trigonometrically fitted block and non-block multistep methods have been derived in this thesis for solving oscillatory problems. The illustrative examples are given in the form of both tables and graphs which clearly shown the advantage of the methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

KAEDAH MULTILANGKAH SUAI SECARA TRIGONOMETRI UNTUK MENYELESAIKAN MASALAH BERAYUN

Oleh

AINI FADHLINA BINTI MANSOR

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Pengerusi : Fudziah binti Ismail, PhD
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Pada permulaan tesis, kami suai secara trigonometri titik pertama kaedah multilangkah blok sedia ada yang diterbitkan dengan tujuan untuk menyelesaikan persamaan pembezaan biasa (PPB) peringkat kedua khas. Berdasarkan kepada kaedah multilangkah asal, kami membina kedua-dua kaedah multilangkah suai secara trigonometri secara tak tersirat dan tersirat bagi nombor langkah k=4 dan membangunkan kod menggunakan kedua-dua saiz langkah tetap dan boleh ubah. Kaedah suai secara trigonometri telah digunakan ke atas kaedah asal bagi membina kaedah baru tersebut. Keputusan berangka menunjukkan bahawa kaedah multilangkah suai secara trigonometri adalah lebih berkesan berbanding dengan kaedah sedia ada dalam menyelesaikan persamaan pembezaan biasa (PPB) peringkat kedua khas yang mana berayun secara semula jadi.

Kemudian, 2-titik kaedah multilangkah blok secara tak tersirat dan tersirat bagi nombor langkah k=3 dan k=5 untuk menyelesaikan PPB peringkat kedua khas telah diterbitkan menggunakan formula kamiran berdasarkan kepada *Newton-Gregory backward interpolation polynomial*. Kaedah tersebut dilaksanakan untuk saiz langkah tetap dengan menggunakan teknik peramal-pembetul, diikuti dengan perlaksanaan menggunakan saiz langkah boleh ubah. Keputusan berangka diberikan untuk menunjukkan keberkesanan kaedah baru tersebut berbanding dengan kaedah sedia ada.

2-titik kaedah multilangkah blok secara tak tersirat dan tersirat bagi nombor langkah k=3 dan k=5 kemudiannya disuaikan secara trigonometri supaya ianya sesuai untuk menyelesaikan persamaan pembezaan biasa peringkat kedua khas, yang

mana sangat berayun secara semula jadi. Kami membangunkan kod berdasarkan kepada kaedah suai secara trigonometri menggunakan saiz langkah tetap dalam mod peramal-pembetul. Keputusan berangka yang diperolehi menunjukkan bahawa suai secara trigonometri kaedah tersebut memberikan lebih banyak penyelesaian yang tepat daripada kaedah sedia ada.

Kesimpulannya, kaedah multilangkah blok dan bukan blok suai secara trigonometri telah diterbitkan di dalam tesis ini untuk menyelesaikan masalah berayun. Contoh-contoh illustrasi diberikan di dalam kedua-dua bentuk jadual dan graf yang dengan jelas menunjukkan kelebihan kaedah-kaedah tersebut.

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I certify that a Thesis Examination Committee has met on 21 December 2017 to conduct the final examination of AINI FADHLINA BINTI MANSOR on her thesis entitled ”TRIGONOMETRICALLY FITTED MULTISTEP METHODS FOR SOLVING OSCILLATORY PROBLEMS” in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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LIST OF ABBREVIATIONS

IVP	Initial value problem
LMMs	Linear multistep methods
ODE	Ordinary differential equation
LHS	Left-hand side
RHS	Right-hand side
LTE	Local truncation error
TOL	Tolerance
p	Order of the method
ω	Frequency weight dependent of the problem
h	Step size
k	Step number
xn	End of interval
MAXERR	Maximum error
FE	Total number of function evaluations
SSTEP	Total number of successful steps
FSTEP	Total number of fail steps
NOS	Total number of steps
TIME(s)	Time taken to compute the method in second
ETFMM	The four-step explicit trigonometrically fitted multistep method proposed in chapter 3
ITFMM	The four-step implicit trigonometrically fitted multistep method proposed in chapter 3
EMSM	The four-step explicit multistep method by Yap et al. (2011)
IMSM	The four-step implicit multistep method by Yap et al. (2011)
ETSHMs	The fourth order explicit two-step hybrid method by Franco (2006)
IRKNM	The fourth order improved Runge-Kutta-Nystrom method with three stages by Rabiei et al. (2012)
PFRKN	The fourth order phase fitted Runge-Kutta-Nystrom method by Papadopoulos et al. (2008)
MSHMs	The four-step multistep hybrid method by Li and Wang (2016)
ERKN6(4)	The sixth order optimized embedded Runge-Kutta-Nystrom pair by Anastassi and Kosti (2015)
ERKN4(3)	The fourth order embedded Runge-Kutta-Nystrom method by Van de Vyver (2005)
MERK4(3)	The fourth order Merson embedded Runge-Kutta method by Butcher (2008)
2PBM4	The fourth order 2-point block multistep method derived in chapter 4
2PBM6	The sixth order 2-point block multistep method derived in chapter 4

VSAM4	The fourth order variable step size Adams method
VSAM6	The sixth order variable step size Adams method
TF2PBM4	The fourth order trigonometrically fitted 2-point block multistep method derived in chapter 5
TF2PBM6	The sixth order trigonometrically fitted 2-point block multistep method derived in chapter 5
ETSHM6	The sixth order explicit two-step hybrid method by Franco (2006)
PFHM6	The sixth order phase fitted hybrid method by Senu et al. (2015)
NTM6	The sixth order explicit Numerov-type method by Tsitouras (2003)
ISCM3	The three-step implicit Stormer-Cowell method obtained from Dormand (1996)
ISCM5	The five-step implicit Stormer-Cowell method obtained from Dormand (1996)
DRKN4	The fourth order explicit Runge-Kutta-Nystrom method by Dormand et al. (1987)

CHAPTER 1

INTRODUCTION

1.1 Differential Equations

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change and the equation defines a relationship between the two. Differential equations play a prominent role in many disciplines including engineering, physics, economics and biology. In mathematics, differential equations are studied from several different perspectives but mostly concerned with their solutions. Ordinary differential equation (ODE) is a differential equation contains only derivatives of one or more unknown functions with respect to a single independent variable.

1.2 The Initial Value Problem

Differential equation together with its initial value is called initial value problem (IVP). The initial value problem for a special second order ordinary differential equation is defined in the form of

$$y'' = f(x,y), \quad y(a) = \eta, \quad y'(a) = \eta', \tag{1.2.1}$$

where the first derivative does not appear explicitly.

Theorem 1.1 *(Existence and Uniqueness)*
Let f(x,y) be defined and continuous for all points (x,y) in the region D defined by $a \leq x \leq b$, $-\infty < y < \infty$, where a and b finite, and let there exist a constant L such that for any $x \in [a,b]$ and any two numbers y and y,*

$$|f(x,y) - f(x,y*)| \leq L|y - y*|. \tag{1.2.2}$$

Then, if η is any given number, there exists a unique solution y(x) of the initial value problem (1.2.1), where y(x) is continuous and differentiable for all (x,y) in D.

The condition (1.2.2) is known as a *Lipschitz condition*, and the constant L as a *Lipschitz constant*. For proof, see (Henrici (1962)). In this work, we shall assume the theorem establishes the existence of a unique solution of (1.2.1).

Such problems often arise in many scientific areas of engineering and applied science such as celestial mechanics, molecular dynamics and quantum mechanics. The solution of (1.2.1) also often exhibits a pronounced oscillatory character. It is well known that it is rather difficult to get the accurate numerical results if the initial value problems are oscillatory in nature. A lot of research has been focused on developing methods to address the problem.

1.3 Numerical Methods for Ordinary Differential Equations

Numerical analysis involves the study of methods of computing numerical data. In many problems this implies producing a sequence of approximations by repeating the procedure again and again. Many differential equations cannot be solved analytically. For practical purposes, however, such as in engineering, a numeric approximation to the solution is often sufficient. Numerical methods for ODEs are methods used to find numerical approximations to the solutions of ODEs where this tecnique usually used by scientists and engineers to solve their problems. The implementation of a numerical method with an appropriate convergence check in a C Programming Language is called as a numerical algorithm.

Conceptually, a numerical method starts from an initial point and then takes a short step forward in time to find the next solution point. Generally, numerical methods often fall into one of two large categories that are single step and multistep methods. The methods also can be divided into explicit and implicit methods.

1.4 Single Step and Multistep Methods

Single step methods refer to only one previous point and its derivative to determine the current value. There are several well-known single step methods such as Euler method and Runge-Kutta method. Methods such as Runge-Kutta methods take some intermediate steps to obtain a higher order method and discard all the previous information before taking a second step.

Methods that require more than one previous point to compute the approximation solution at the next point we called them as linear multistep methods (LMMs). Based on Lambert (1973), linear multistep method for special second order ODEs is defined by,

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j}, \tag{1.4.1}$$

where α_j and β_j are constant and assume $\alpha_k = 1$. Linear multistep method (1.4.1) is said to be explicit if $\beta_k = 0$ and implicit if $\beta_k \neq 0$. In this research, we are going to focus on the linear multistep method in the form of equation (1.4.1). Both Adams Bashforth and Adams Moulton methods are well-known explicit and implicit LMMs respectively.

Definition 1.1 [Lambert (1973)]
The linear multistep method (1.4.1) is said to be convergent if, for all initial value problems (1.2.1) subject to the hypothesis of the Theorem 1.1, we have that

$$\lim_{\substack{h\to 0\\ nh=x-a}} y_n = y(x) \tag{1.4.2}$$

holds for all $x \in [a,b]$, and for all solutions y_n of the difference equation (1.4.1) satisfying starting conditions $y_\mu = \eta_\mu(h)$ for which $\lim_{h\to 0} \eta_\mu(h) = \eta, \mu = 0,1,2,\ldots,k-1$.

1.5 Block Method

Block method is the approximation of solution for initial value problems at r points simultaneously. There are two types of block methods called as one-step block and multi-block. In one-step block method, the new block $y_{n+1}, y_{n+2}, \ldots, y_{n+r}$ is computed from the value y_n meanwhile in multi-block method, the new block is computed using the information from one or more previous blocks. Block method is an efficient method compared to the non-block method in terms of accuracy, total number of steps and execution times since it computes more than one value of the solutions at one time. Thus, the time period used and cost in solving the problems can be reduced.

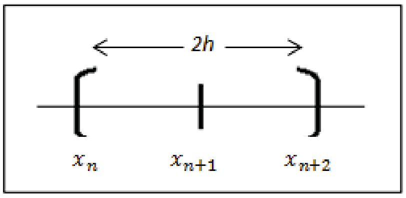

Figure 1.1: 2-point block multistep method.

In Figure (1.1), the approximation solutions of y_{n+1} and y_{n+2} are simultaneously computed at the points x_{n+1} and x_{n+2} where x_n becomes the starting point and x_{n+2} is the last point in the block that has the step size $2h$. The final values of y_{n+2} at the point x_{n+2} will then be taken as the initial values for the next iteration. The interval $[a,b]$ is divided into a series of blocks that contained two points at each block.

1.6 Trigonometrically Fitted Method

In order to find efficient methods for solving oscillatory problems, a lot of research has been focused on developing methods with reduced phase-lag and amplification error. Phase-lag or dispersion error is the angle between the true and the approximated solution while dissipation (amplification) error is the distance of the computed solution from the cyclic solution. The performance of numerical methods for solving oscillatory problems can be enhanced by phase fitting the method. Trigonometrically fitting the method means we assumed that the true solution is in the form of trigonometric functions and the methods are derived based on this assumption.

Using the ideas of trigonometrically fitting as in Fang and Wu (2008), the linear multistep method (1.4.1) is required to integrate exactly the linear combination of the functions $\left\{e^{-i\omega t}, e^{i\omega t}\right\}$ or equivalently with $\{\sin(\omega t), \cos(\omega t)\}$ for $\omega \in R$ and consider the test equation $y'' = -\omega^2 y$ with $\omega > 0$. Let $y_n = e^{i\omega t}$, we have $y_{n+1} = e^{i\omega(t+h)}$ and by substituting $e^{iH} = \cos(H) + i\sin(H)$ where $H = \omega h$ to the (1.4.1), the fitted values will be obtained.

1.7 Order Conditions

Order conditions of the method are important to determine the order of the method. Based on linear multistep method (1.4.1), the linear difference operator L is defined as:

$$\mathscr{L}[y(x);h] = \sum_{j=0}^{k} [\alpha_j y(x+jh) - h^2 \beta_j y''(x+jh)], \tag{1.7.1}$$

where $y(x)$ is an arbitrary function, continuously differentiable on [a,b]. By expanding $y(x+jh)$ and $y''(x+jh)$ as Taylor series about point x, it gives

$$\mathscr{L}[y(x);h] = C_0 y(x) + C_1 h y^{(1)}(x) + \ldots + C_q h^q y^{(q)}(x) + \ldots, \tag{1.7.2}$$

where C_q are constants.

Based on equation (1.7.2), the order conditions for special second order ODEs are defined as follows:

$$
\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j \\
C_1 &= \sum_{j=0}^{k} (j\alpha_j) \\
C_2 &= \sum_{j=0}^{k} (\frac{j^2}{2!}\alpha_j - \beta_j) \\
C_3 &= \sum_{j=0}^{k} (\frac{j^3}{3!}\alpha_j - j\beta_j) \\
&\vdots \\
C_q &= \sum_{j=0}^{k} (\frac{j^q}{q!}\alpha_j - \frac{j^{q-2}}{(q-2)!}\beta_j), \quad q = 3, 4, \ldots
\end{aligned} \tag{1.7.3}
$$

Following Henrici (1962), the method is said to has order p if
$C_0 = C_1 = \cdots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0;$
C_{p+2} is then the error constant of the method.

1.8 Stability of the Method

Based on Lambert (1973), the first and second characteristic polynomial of linear multistep method (1.4.1) are defined as $\rho(\zeta)$ and $\sigma(\zeta)$ respectively:

$$\rho(\zeta) = \sum_{j=0}^{k} \alpha_j \zeta_j, \quad \sigma(\zeta) = \sum_{j=0}^{k} \beta_j \zeta_j. \tag{1.8.1}$$

Based on Lambert (1973), the test equation used for LMM (1.4.1) is,

$$y'' = \lambda y. \tag{1.8.2}$$

Next, definitions on certain types of stability of LMM will be given.

Definition 1.2 [Lambert (1973)]
The linear multistep method (1.4.1) is said to be zero-stable if no root of the first characteristic polynomial $\rho(\zeta)$ has modulus greater than one, and if every root with modulus one has multiplicity not greater than two.

Definition 1.3 [Fatunla (1991)]
The block method is zero stable provided the roots $R_j, j = 1(1)k$ of the first charac-

teristic polynomial $\rho(R)$ specified as

$$\rho(R) = det[\sum_{i=0}^{k} A^{(i)} R^{k-i}] = 0, A^{(0)} = -I \tag{1.8.3}$$

satisfies $|R_j| \leq 1$, and for those roots with $|R_j| = 1$, the multiplicity must not exceed 2.

Definition 1.4 [Lambert (1973)]
The linear multistep method (1.4.1) is said to be absolutely stable for a given $\hbar$ if, for that $\hbar$, all the roots r_s of stability polynomial, $\pi(r,\hbar) = \rho(r) - \hbar^2\sigma(r) = 0$, where $\hbar = h\lambda$ satisfy $|r_s| < 1, s = 1,2,\ldots,k$, and to be absolutely unstable for that $\hbar$ otherwise.

Definition 1.5 [Yap et al. (2011)]
The linear multistep method (1.4.1) is said to have region of absolute stability $\mathscr{D}$ where $\mathscr{D}$ is a region of the complex $\hbar$-plane, if it is absolutely stable for all $\hbar \in \mathscr{D}$. The intersection of $\mathscr{D}$ with the real axis is called the interval of absolute stability.

1.9 The Objectives of the Thesis

Objectives of this research are as follows:

1. To construct the trigonometrically fitted explicit and implicit multistep methods of step number *k*=4 based on the existing methods derived by Yap et al. (2011) and to develop constant and variable step size based on the methods for solving oscillatory problems.
2. To derive 2-point block multistep method of step number *k*=3 and *k*=5 and developing codes based on the methods using both constant and variable step size techniques for solving special second order ODEs.
3. To construct the trigonometrically fitted 2-point block multistep methods of step number *k*=3 and *k*=5 for solving oscillatory problems.
4. To determine the zero stability and absolute stability of the new methods.

1.10 Outline of the Thesis

This thesis is arrange as follows:

In Chapter 1, a brief introduction to the differential equation, initial value problem and numerical methods for ordinary differential equations are given. Definition

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