UNIVERSITI PUTRA MALAYSIA

GROUP ACTIONS AND THEIR APPLICATIONS IN ASSOCIATIVE ALGEBRAS AND ALGEBRAIC STATISTICS

NADIA FAIQ MOHAMMED

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GROUP ACTIONS AND THEIR APPLICATIONS IN ASSOCIATIVE ALGEBRAS AND ALGEBRAIC STATISTICS

By

NADIA FAIQ MOHAMMED

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia,
in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

September 2017

DEDICATIONS

To

my lovely country

Iraq

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

GROUP ACTIONS AND THEIR APPLICATIONS IN ASSOCIATIVE ALGEBRAS AND ALGEBRAIC STATISTICS

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September 2017

Chair: Professor Isamiddin S. Rakhimov, PhD
Faculty: Science

The study of group action is an exciting and very intensive research that takes place in Mathematics and Physics. There can be different ways for a group to act on different kinds of objects.

This dissertation is mostly concerned with group actions on vector spaces and affine algebraic varieties. It is mainly comprised of three parts. In the first part, we consider an action of associative algebra A on a vector space to study low-dimensional cohomology groups of associative algebras. The origin of cohomology groups is found in algebraic topology. The dimension of the cohomology groups is considered one of the important invariant to study properties of algebras. Particularly, this invariant plays a rigorous role in geometric classification of associative algebras. We focus on the applications of low dimensional cohomology groups $H^i(A,A)$. We start with the zero order cohomology groups of two and three dimensional complex associative algebras. Then, we consider an important special case of derivations so-called inner derivation mapping which can be roughly interpreted as $1-$coboundaries on vector space where the algebra A acting. We study some of their properties and give an algorithm to obtain the inner derivations of associative algebras. Another main result of this part is precisely formulated two algorithms for describing the $2-$cocycles and $2-$coboundaries of A. Using an existing classification result of low dimensional associative algebras, we apply all these algorithms to complex associative algebras up to dimension three.

In the second part, general linear group acts on an affine algebraic variety over an algebraically closed field. This part is devoted to the study of the rigidity of associative algebras. We give necessary invariance arguments for the existence of degenerations which is helpful in finding out for associative algebras to be rigid. Subsequently, applications of the invariance arguments to the varieties of low-dimensional complex associative algebras are described.

In the last part of the thesis, we consider an action of dihedral group D_p on the rational vector space $\mathbb{Q}^p$ to study an invariance group of Markov basis for some contingency tables. We define a Markov basis B for a $\frac{p(v-1)(p-v)}{2v} \times v \times \frac{p}{v}$ - contingency table with fixed two-dimensional marginals, where p is a multiple of v and greater than or equal to $2v$. Using this action, we find the invariance subgroup H of Markov basis B with respect to the dihedral group D_p. Moreover, an algorithm to obtain the set of all independence models of two-way contingency tables with the same row sums and column sums is proposed. Finally, we introduce a new class of algebras which is closely related to the Markov basis in algebraic statistics.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

TINDAKAN KUMPULAN DAN APLIKASINYA DALAM ALJABAR BERSEKUTU DAN STATISTIK BERALJABAR

Oleh

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Kajian tindakan kumpulan adalah satu penyelidikan yang menarik dan sangat intensif yang berlaku dalam Matematik dan Fizik. Terdapat cara yang berbeza bagi kumpulan untuk bertindak pada pelbagai jenis objek.

Disertasi ini kebanyakannya berkaitan dengan tindakan kumpulan pada ruang vektor dan pelbagai afin aljabar. Ianya terdiri daripada tiga bahagian utama. Dalam bahagian pertama, kami mempertimbangkan tindakan terhadap aljabar bersekutu A pada suatu ruang vektor untuk mengkaji dimensi rendah kumpulan kohomologi bagi aljabar-aljabar bersekutu. Asal-usul kumpulan kohomologi terdapat dalam topologi aljabar. Dimensi bagi kumpulan-kumpulan kohomologi dianggap sebagai salah satu tak varian yang penting untuk mengkaji sifat-sifat aljabar. Terutamanya, tak varian ini memainkan peranan yang ketat dalam pengelasan bergeometri bagi aljabar-aljabar bersekutu. Kami menumpukan pengaplikasian bagi dimensi rendah kumpulan-kumpulan kohomologi $H^i(A,A)$. Kami mulakan dengan kumpulan kohomologi berperingkat sifar bagi dimensi dua and tiga aljabar-aljabar bersekutu kompleks. Kemudian, kami mempertimbangkan satu kes penting bagi pembezaan yang berasal dari pemetaan pembezaan dalaman yang boleh ditakrifkan secara kasar sebagai $1-$ kosempadan dalam ruang vektor di mana aljabar tersebut A bertindak. Kami mengkaji beberapa sifat mereka dan memberikan satu algoritma untuk mendapatkan pembezaan-pembezaan dalaman bagi aljabar-aljabar bersekutu. Hasil utama lain dari bahagian ini adalah merumuskan dua algoritma dengan tepat untuk menerangkan $2-$ kokitar dan $2-$ kosempadan bagi A. Dengan menggunakan pengelasan sedia ada bagi aljabar-aljabar bersekutu berdimensi rendah, kami menggunakan semua algoritma-algoritma ini ke atas aljabar-aljabar bersekutu kompleks sehingga dimensi empat.

Dalam bahagian kedua, kumpulan linear terturun am bertindak ke atas pelbagai afin aljabar terhadap medan aljabar tertutup. Bahagian ini dikhaskan untuk mengkaji ketegaran aljabar-aljabar bersekutu. Kami memberikan hujah-hujah tak varian yang diperlukan untuk kewujudan kemerosotan yang membantu dalam mencari aljabar-aljabar bersekutu untuk menjadi tegar. Selanjutnya, aplikasi hujah tak varian kepada pelbagai aljabar-aljabar bersekutu kompleks berdimensi rendah diterangkan.

Dalam bahagian akhir tesis ini, kami mempertimbangkan tindakan terhadap kumpulan dwihedron D_p pada ruang vektor nisbah $\mathbb{Q}^p$ untuk mengkaji suatu kumpulan ketak varianan bagi asas Markov untuk beberapa jadual kontingensi. Kami mentakrifkan satu asas Markov B untuk $\frac{p(v-1)(pv)}{2v} \times v \times \frac{p}{v}$ - jadual kontingensi dengan kekerapan dua dimensi yang tetap, di mana p adalah suatu gandaan daripada v dan lebih besar daripada atau sama dengan $2v$. Dengan menggunakan tindakan ini, kami mendapati satu ketak varianan subkumpulan H dari asas Markov B terhadap kumpulan dwihedron D_p. Selain itu, satu algoritma untuk mendapatkan satu set bagi semua model-model kebebasan untuk dua-cara jadual kontingensi dengan jumlah baris dan jumlah lajur yang sama dicadangkan. Akhir sekali, kami memperkenalkan satu kelas baru bagi aljabar-aljabar yang sangat berkait rapat dengan asas Markov dalam statistik beraljabar.

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I certify that a Thesis Examination Committee has met on 25 September 2017 to conduct the final examination of Nadia Faiq Mohammed on her thesis entitled "Group Actions and their Applications in Associative Algebras and Algebraic Statistics" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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LIST OF ABBREVIATIONS

As_p^q	q^{th} isomorphism class of associative algebras in dimension p
$Der(A)$	The algebra of all derivations of the algebra A
$In(A)$	The algebra of all inner derivations of the algebra A
IC	Isomorphism classes of algebras
$\mathbb{N}$	The set of natural numbers
$\mathbb{Z}$	The set of integer numbers
$\mathbb{Q}$	The set of rational numbers
$\mathbb{R}$	The set of real numbers
$\mathbb{C}$	The set of complex numbers
$\mathbb{K}$	Field
$Z^1(A,D)$	1-cocycles
$Z^2(A,D)$	2-cocycles
$B^1(A,D)$	1-coboundaries
$B^2(A,D)$	2-coboundaries
$\square$	The end of a proof

CHAPTER 1

INTRODUCTION

1.1 Introduction

Group action is very exciting and intensive research that takes place in Mathematics and Physics. Symmetries coming from a group action appearing in such a construction are a powerful tool for understanding the structure of geometric objects. There can be different ways for a group to act on different kinds of objects, such as vector spaces and affine algebraic varieties. It will be an interesting topic to study the applications of group actions on these special cases which will lead to resolve some of the issues in relation to cohomology of associative algebras, geometric classification of associative algebras and invariance groups of generating sets for a toric ideal.

In this thesis, an associative algebra A acts on a vector space to study low-dimensional cohomology groups of associative algebras $H^i(A,A)$, where $i = 0,1,2$. The dimension of the cohomology groups is considered one of the important invariant to study properties of algebras. Particularly, this invariant plays a rigorous role in geometric classification of associative algebras. Then, a general linear group acts on an affine algebraic variety over an algebraically closed field to investigate the affine algebraic varieties of low dimensional complex associative algebras in relation to the of geometric classification. Of course, it is interesting aspect but difficult problem to determine the number of irreducible components of an algebraic variety. In order to identify solutions to such a problem, a degeneration approach can be considered as helpful. Finally, a new area which has been given the name ”algebraic statistics” is concerned with independence models for contingency tables that can be described, in some way, via polynomials. Algebraic statistics is concerned with the development of techniques in algebraic geometry, commutative algebra, and combinatorics, to address problems in statistics and its applications. There is a wide array of problems for which this approach has been highly successful such as optimization, computational biology, the maximum likelihood estimation and parametric inference. One of the main objects which is considered in algebraic statistics is toric ideal. In this regard, a dihedral group acts on the rational vector space $\mathbb{Q}^p$ to study an invariance group of generating set for a toric ideal of the so-called Markov basis for some contingency tables.

1.2 Basic concepts

In this section, we introduce some well-known definitions along with associated concepts, which have implications for the present study. Readers who are familiar with related materials in the area may attest to the wealth of information in relation to Lie algebras, which can be found even in standard books such as Jacobson (1962). In the case of associative algebras, most of the notations and along with related concepts are referred to Pierce (1982). Finally, the main notions of algebraic geometry are provided in Hartshorne (2013).

Definition 1.1 *Let $\mathbb{K}$ be a field, V be a vector space over $\mathbb{K}$ with a binary operation $\lambda : V \times V \longrightarrow V$. If the binary operation is bilinear, i.e.,*

$$\lambda(\alpha_1 x + \alpha_2 y, z) = \alpha_1 \lambda(x,z) + \alpha_2 \lambda(y,z),$$

$$\lambda(z, \alpha_1 x + \alpha_2 y) = \alpha_1 \lambda(z,x) + \alpha_2 \lambda(z,y),$$

where $x,y,z \in V$ and $\alpha_1, \alpha_2 \in \mathbb{K}$ then $A = (V, \lambda)$ is said to be an algebra over $\mathbb{K}$.

For ease of reference, we first introduce some classes of algebras, which are closely related to the ones reported in the present study. The definitions and examples reported in the section are mostly well-known. In this regard we attempt to help the readers to have a better understanding of these definitions and examples in relation to the present study.

Definition 1.2 *An associative algebra A is a vector space over a field $\mathbb{K}$ equipped with bilinear map*

$$\lambda : A \times A \to A$$

satisfying the associative law:

$$\lambda(\lambda(x,y),z) = \lambda(x, \lambda(y,z)), \ \ \forall x,y,z \in A.$$

Further the notations $x \cdot y$ (even just xy) will be used for $\lambda(x,y)$. Note that fields, polynomial algebras, set of linear transformations on a vector space (quadratic matrices over a fixed field) are simple examples of associative algebras.

Let A be an n-dimensional algebra. By taking a basis $\{e_1, e_2, \cdots, e_n\}$ in A, elements x and y of A can be represented as follows:

$$x = \sum_{i=1}^{n} x^i e_i \quad \text{and} \quad y = \sum_{j=1}^{n} y^j e_j.$$

Notice that the product $x \cdot y$ of any two elements x and y in A, is completely determined by the product $e_i \cdot e_j$ of pairs of basis elements, that is:

$$x \cdot y = \sum_{i,j=1}^{n} x^i y^j e_i \cdot e_j.$$

Since $e_i \cdot e_j \in A$, the product $e_i \cdot e_j$ is expanded with respect to the basis $\{e_1, e_2, \cdots, e_n\}$:

$$e_i \cdot e_j = \sum_{k=1}^{n} \gamma_{ij}^{k} e_k, \qquad i, j = 1, 2 \cdots, n.$$

The numbers γ_{ij}^{k} are called the structure constants of A with respect to the basis $\{e_1, e_2, \cdots, e_n\}$. It is worth highlighting that any algebra is completely determined by its structure constants.

Definition 1.3 *A Lie algebra L is a vector space over a field $\mathbb{K}$ equipped with bilinear map*

$$[\cdot,\cdot] : L \times L \to L$$

satisfying the following conditions:

(1) $[x,x] = 0, \; \forall x \in L$

(2) $\Big[[x,y],z\Big] + \Big[[y,z],x\Big] + \Big[[z,x],y\Big] = 0, \; \forall x,y,z \in L$

Condition (2) is known as the Jacobi identity. As the Lie bracket $[.,.]$ is bilinear, we have

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x].$$

The condition (1) implies

$$(1') \; [x,y] = -[y,x] \;\; \forall\, x,y \in L \qquad \text{(anti-symmetry).}$$

Example 1.1 *Any vector space V has a Lie bracket defined by $[x,y] = 0$ for all $x,y \in V$. In particular, the field $\mathbb{K}$ may be regarded as a one-dimensional abelian Lie algebra.*

Most of the readers are aware of the following fact which plays a fundamental role in the classification.

Let A be a associative algebra with a bilinear map λ. Then, the law μ defined by

$$\mu(x,y) = \lambda(x,y) - \lambda(y,x), \;\; \forall\, x,y \in A$$

is a Lie law.

Let $\mathbb{K}[X] = \mathbb{K}[x_1, \cdots, x_n]$ be the polynomial ring with n indeterminants over algebrically closed field $\mathbb{K}$. Let $\mathbb{A}^n = \{P = (a_1, \cdots, a_n) : a_i \in \mathbb{K}\}$ be the affine n-space over $\mathbb{K}$. We will interpret the elements of $\mathbb{K}[X]$ as functions from the affine n-space to $\mathbb{K}$, by defining $f(P) = f(a_1, \cdots, a_n)$, where $f \in \mathbb{K}[X]$ and $P \in \mathbb{A}^n$.

Definition 1.4 *Let S be any subset of* $\mathbb{K}[X]$*, we define the zero set of S to be the common zeros of all elements of S:*

$$V(S) = \{P \in \mathbb{A}^n : f(P) = 0, \ \forall f \in S\}.$$

A subset Y is called an algebraic set if there exists a subset $S \subset \mathbb{K}[X]$ *such that* $Y = V(S)$.

Upon the discussion, it is necessary to introduce as earlier as possible a topology on the algebraic set in the present study, which is called Zariski topology.

Proposition 1.1

(i) *The affine n-space* $\mathbb{A}^n$ *and the empty set are algebraic sets.*
(ii) *The union of two algebraic sets is algebraic.*
(iii) *The intersection of any family of algebraic sets is algebraic.*

The meaning of the proposition is that if we take the algebraic sets as closed sets, we get a topology on $\mathbb{A}^n$, which is referred to as Zariski topology. If X is any algebraic set, the Zariski topology on X is the topology induced on it from $\mathbb{A}^n$. Closed/open sets in X are intersections of X with closed/open sets in $\mathbb{A}^n$.

Definition 1.5 *A nonempty subset Y of a topological space X is irreducible if it cannot be expressed as a union* $Y = Y_1 \cup Y_2$ *of two proper subsets, each of them is closed in* Y.

Definition 1.6 *An affine algebraic variety or simply affine variety is an irreducible closed subset of* $\mathbb{A}^n$ *(with the induced topology).*

The next step is to explore the correspondence identified by subsets of $\mathbb{A}^n$ and ideals in $\mathbb{K}[X]$.

Definition 1.7 *Let Y be any subset of* $\mathbb{A}^n$. *Let*

$$I(Y) = \{f \in \mathbb{K}[X] : f(a_1, \cdots, a_n) = 0, \ \forall (a_1, \cdots, a_n) \in Y\}.$$

It is easy to see that $I(Y)$ *is an ideal of* $\mathbb{K}[X]$. *If Y is affine variety, then this ideal is called defining ideal of the affine variety* Y.

Definition 1.8 *Let Y be an affine variety in* $\mathbb{A}^n$. *A function* $f : Y \longrightarrow \mathbb{K}$ *is said to be regular if it coincides with the restriction on Y of some polynomial function, i.e., a function* $\mathbb{A}^n \longrightarrow \mathbb{K}$ *mapping a point to* $F(a)$, *where* $F \in \mathbb{K}[X]$.

Definition 1.9 *Let X and Y be affine varieties. The function* $\phi : X \longrightarrow Y$ *is called morphism (or regular map) if* $f \circ \phi : X \longrightarrow \mathbb{K}$ *is regular for each regular function* $f : Y \longrightarrow \mathbb{K}$.

Example 1.2 *Let* ϕ_i, *where* $1 \le i \le m$, *be polynomials in* $\mathbb{K}[X]$ *then a function* $\phi : A^n \longrightarrow A^m$ *defined by* $\phi(P) = (\phi_1(P), \phi_2(P), \cdots, \phi_m(P))$ *is a morphism.*

Definition 1.10 *An algebraic group is an algebraic variety G equipped with the structure of a group, such that the multiplication map and the inverse map are morphisms of varieties.*

Example 1.3 *The general linear group* $GL_n(\mathbb{C})$ *consisting of all invertible* $n \times n$ *matrices with complex coefficients, is the open subset of the space* M_n *of* $n \times n$ *complex matrices (an affine space of dimension* n^2*) whose determinant is nonzero. Then* $GL_n(\mathbb{C})$ *is an algebraic group.*

Definition 1.11 *An action of algebraic group G on a variety Z is a morphism* $\sigma : G \times Z \longrightarrow Z$ *with*

(i) $\sigma(e, z) = z$, *where e is the unit element of G and* $z \in Z$.

(ii) $\sigma(g, \sigma(h, z)) = \sigma(gh, z)$, *for any* $g, h \in G$ *and* $z \in Z$.

We shortly write gz for $\sigma(g, z)$, and all call Z a G-variety.

Definition 1.12 *Let* $A = (V, \lambda)$ *and* $A_1 = (V, \lambda_1)$ *be two algebras over a field* $\mathbb{K}$. *A linear mapping* $\psi : A \longrightarrow A_1$ *is a homomorphism if*

$$\psi(\lambda(x, y)) = \lambda_1(\psi(x), \psi(y)), \ \ \forall x, y \in A.$$

Remark 1.1

1. *The mapping* ψ *is an isomorphism if it is a bijective homomorphism (one to one and onto).*
2. *The mapping* ψ *is an endomorphism if* ψ *is a homomorphism and* $A = A_1$.
3. *The mapping* ψ *is an automorphism if* ψ *is an isomorphism and* $A = A_1$.

Let V be a vector space of dimension n over an algebraically closed field $\mathbb{K}$ ($char\mathbb{K} = 0$). Bilinear maps $V \times V \to V$ form a vector space $\mathrm{Hom}(V \otimes V, V)$ of dimension n^3, which can be considered together with its natural structure of an affine algebraic variety over $\mathbb{K}$ and denoted by $Alg_n(\mathbb{K}) \cong \mathbb{K}^{n^3}$. An n-dimensional algebra A over $\mathbb{K}$ can be an element $\lambda(A)$ of $Alg_n(\mathbb{K})$ via the bilinear mapping $\lambda : A \otimes A \to A$. Let $\{e_1, e_2, \cdots, e_n\}$ be a basis of the algebra A. Then the table of multiplication of A is represented by point (γ_{ij}^k) of affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k.$$

The general linear group $GL_n(\mathbb{K})$ acts on $Alg_n(\mathbb{K})$ by:

$$(g*\lambda)(x,y) = g(\lambda(g^{-1}(x), g^{-1}(y))).$$

Let $O(\lambda)$ be the set of laws isomorphic to λ. It is called an orbit of λ with respect to the actions of $GL_n(\mathbb{K})$. Two algebras λ_1 and λ_2 are isomorphic if and only if they belong to the same orbit under this action.

Let $A_n(\mathbb{K})$ be a subvariety of $Alg_n(\mathbb{K})$ consisting of all n-dimensional associative algebras over a field $\mathbb{K}$. It is stable under the above mentioned action of $GL_n(\mathbb{K})$. As a subset of $Alg_n(\mathbb{K})$ the set $A_n(\mathbb{K})$ is specified by the system of polynomial equations with respect to the structure constants $\{\gamma_{ij}^k\}$:

$$\sum_{l=1}^{n}(\gamma_{ij}^l\gamma_{lk}^s - \gamma_{il}^s\gamma_{jk}^l) = 0, \text{ where} i,j,k,s = 1,\cdots,n.$$

Definition 1.13 *A derivation of associative algebra A over* $\mathbb{K}$ *is a linear transformation* $d : A \to A$ *where*

$$d(x\cdot y) = d(x)\cdot y + x\cdot d(y), \qquad \forall x,y \in A.$$

We denote the set of all derivations of an associative algebra A by $Der(A)$. The $Der(A)$ is considered as a Lie algebra with respect to the bracket $[d_1,d_2] = d_1\circ d_2 - d_2\circ d_1$.

Moving on, we now define an important special case of the derivation mapping, which is widely referred to the inner derivations.

Definition 1.14 *Let A be an associative algebra over* $\mathbb{K}$ *and z be a vector in A. Let* $ad_z : A \to A$ *be a function defined as follows:*

$$ad_z(x) = xz - zx, \qquad \forall x \in A.$$

The function ad_z *is called an inner derivation of A.*

The set of all inner derivations of an associative algebra A is denoted by $Inn(A)$. The set $Inn(A)$ is an ideal of $Der(A)$.

Definition 1.15 *Let A be an associative algebra over* $\mathbb{K}$. *We define*

$$A^1 = A,\ A^k = \lambda(A, A^{k-1}) \qquad (k > 1).$$

The series

$$A^1 \supseteq A^2 \supseteq A^3 \supseteq \cdots$$

is called the descending central series of A. If there exists an integer $s \in \mathbb{N}$, such that

$$A^1 \supseteq A^2 \supseteq \cdots \supseteq A^s = \{0\},$$

then the associative algebra A is said to be nilpotent. The smallest integer s for which $A^s = \{0\}$ is called nilpotency rank of A and denoted by $r_n(A)$.

Example 1.4

(1) *Every abelian algebra is nilpotent with nilpotency rank equals to* 2.
(2) *The space of n-square matrices $M = [a_{ij}]$ on $\mathbb{K}$ such that $a_{ij} = 0$ for $i \geq j$, with the usual matrix multiplication is a nilpotent associative algebra over $\mathbb{K}$ with nilpotency rank equals to n.*

1.3 Motivation

The motivation to carry out the present study was sparked by the applications of group actions on vector spaces, affine algebraic varieties and generating set of toric ideal to resolve some of the issues in relation to cohomology of associative algebras, geometric classification of associative algebras and invariance groups of generating sets for a toric ideal, respectively. These problems can be summarized as follows:

On cohomology of associative algebras:

In algebra, the dimension of the cohomology groups is considered as one of the important invariants in studying the geometric classification. It is for such a reason that the researcher in the present study, consider an action of associative algebra A on a vector space to study low-dimensional cohomology groups $H^i(A,A)$ of associative algebras, where $i = 0, 1, 2$.

On geometric classification of associative algebras:

It is interesting aspect but difficult problem to determine the number of irreducible components of an algebraic variety. However, if one interested in finding the dimension of the algebraic variety then degeneration approach can be considered as helpful. In order to identify solutions to such a problem, a general linear group reportedly acts on an affine algebraic variety over an algebraically closed field.

On invariance groups of generating set for a toric ideal:

As it has been indicated in the literature, one of the main objects, which is considered in algebraic statistics is toric ideal. In this regard, we of the present study look into the invariance groups of generating set for a toric ideal of the so-called Markov basis for some contingency tables. In doing so, we consider an action of dihedral group D_p on the rational vector space $\mathbb{Q}^p$ to describe these invariance groups.

1.4 Objectives of the research

The main objectives of this research are:

1. To propose an algorithm to describe the inner derivations of associative algebras, which can be roughly interpreted as 1-coboundaries of the first cohomology groups, and give another two algorithms for describing the 2-cocycles and 2-coboundaries of the second cohomology groups. The implementation of these algorithms is then provided in low dimensional complex associative algebras.
2. To investigate the affine algebraic varieties of low dimensional complex associative algebras in relation to the geometric classification.
3. To describe the invariance groups of generating set for a toric ideal so-called Markov basis for some contingency tables.
4. To propose an algebraic method to obtain the set of all independent models of two-way contingency tables with the same row sums and column sums, which is referred to as fiber in algebraic statistics.
5. To define and study a new class of algebras which is closely related to the Markov basis in algebraic statistics.

1.5 Methodology

- First Objective: The algorithm to describe the inner derivations of associative algebras, which can be roughly interpreted as 1-coboundaries of the first cohomology groups, involves on solving a system of linear algebraic equations:

$$d_{ji} = \sum_{t=1}^{n} a_t \gamma_{it}^{j} - \sum_{t=1}^{n} a_t \gamma_{ti}^{j} \qquad \text{for} \quad i, j, t = 1, 2 \cdots, n. \tag{1.1}$$

  To derive this system, fix a basis $\{e_1, e_2, \cdots, e_n\}$ of A and a vector $z = a_1e_1 + a_2e_2 + \cdots + a_ne_n$ in A, where A is an n-dimensional associative algebra. Suppose $ad_z : A \to A$ be a linear transformation defined as follows:

$$ad_z(x) = xz - zx, \qquad \forall x \in A. \tag{1.2}$$

On other hand, ad_z can be considered as a linear transformation of A that is represented in a matrix form

$$ad_z = (d_{ij}),\ i,j = 1,2,\cdots,n. \tag{1.3}$$

Combining and simplify the equations (1.2) and (1.3), then we get the system of linear algebraic equations (1.1).

The remaining algorithms of the first objective can be done in a similar manner as shown above.

- Second Objective: We take a complete list of nonisomorphic associative algebras of a fixed dimension. For each member of this list we calculate invariance argument that concern necessary criteria of degeneration. Then, for each pair of algebras from the list we test possible existence of degeneration with the necessary criteria of degeneration via comparing the calculated invariance argument.

- Third Objectives: We consider a dihedral group D_p of degree p, where p is a positive integer greater than or equal to 3, acting on the p- dimensional vector space $\mathbb{Q}^p$ over the field of rational numbers $\mathbb{Q}$. Suppose $B \subset \mathbb{Q}^p$, where B is a Markov basis for $\frac{p(v-1)(p-v)}{2v} \times v \times \frac{p}{v}$ - contingency tables with fixed two dimensional marginal. Let H be the largest invariance subgroup of the dihedral group D_p for the Markov basis B, i.e., $gB = B$ for all $g \in H$. Then, for each subgroup of D_p we check possible existence of a largest invariance subgroup H of the dihedral group D_p for the Markov basis B. As a result, it is worthy of note that the largest invariance subgroup H of the dihedral group D_p is generated by $\langle sr, r^d \rangle$ where d equals to the number of columns of two-way contingency tables. Additionally, the order of H is $2\times$ (the number of rows of two-way contingency tables).

- Fourth Objectives: We intend to solve the following non-homogeneous linear system of equations:

$$A\mathbf{x} = \mathbf{t} \tag{1.4}$$

where A is a configuration matrix and has size $v \times p$, $\mathbf{x}$ is a $I \times J$ two-way contingency table that can be represented as p -dimensional column frequency vector and $\mathbf{t}$ is a v- dimensional column vector. This system is regarded underdetermined system because it consists of a set of $(I+J-1)$ linear equations in $(I \times J)$ variables. Let $x_{ij},\ i = 1,2,\cdots,I,\ j = 1,2,\cdots,J$, be the observed cell. Furthermore, let the leading variables be the frequency of cells in the first row and in the first column denoted as x_{11}, x_{k1}, x_{1l} and the free variables be the other frequency of cells in the contingency table $\mathbf{x}$ and denoted by

$$x_{kl} = d_{(k-1)(l-1)}, \qquad k = 2,\cdots,I, \quad l = 2,\cdots,J. \tag{1.5}$$

To obtain the set of all independent models of two-way contingency tables with the same row sums and column sums, we propose an algebraic method that is consisted of four steps. The summarized steps for this method can be found in section 5.4.

- Fifth Objectives: To define a new class of algebras which is closely related to the Markov basis in algebraic statistics, let a fiber $F_{\mathbf{t}}$ be the set of all $I \times J$ two-way contingency tables with the same row sums and column sums such that $\mid F_{\mathbf{t}} \mid$ represents the number of tables in a fiber $F_{\mathbf{t}}$. Using the Markov basis B, we can construct a connected, aperiodic and symmetric Markov chain with the transition probability matrix $R = (r_{ij})$ of size $\mid F_{\mathbf{t}} \mid \times \mid F_{\mathbf{t}} \mid$. The new algebra A is called Markov algebra if their structure constants determine the multiplication table in A via the following rule:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} p_{ij,k}\, e_k, \qquad p_{ij,k} = \frac{(r_{ik} + r_{jk})}{2}, \qquad \forall i,j,k = 1, \cdots, \mid F_{\mathbf{t}} \mid.$$

1.6 Organization of thesis

The thesis contains six chapters. Now, we briefly describe the layout of the thesis.

In Chapter 1, as it was highlighted earlier, we discuss some basic notations, affine variety and Zarisk topology which is used throughout the thesis.

In Chapter 2, we provide a brief review on algebraic and geometric classification of associative algebras and some basic introduction on algebraic statistics are given.

In Chapter 3, we focus on applications of low dimensional cohomology groups $H^i(A,A)$. It begins with the zero order cohomology groups of low dimensional complex associative algebras. Moving on, we consider an important special case of derivations, so-called inner derivation mapping, which can be interpreted as $1-$coboundaries on vector space where the algebra A acting. We look into some of their properties and give an algorithm to obtain the inner derivations of associative algebras. Another significant aspect here is the precisely formulated two algorithms for describing the $2-$cocycles and $2-$coboundaries of A. We apply all these algorithms on low dimensional complex associative algebras.

In Chapter 4, we investigate the affine algebraic varieties of low-dimensional complex associative algebras in the sense of geometric classification. For this reason, we consider an action of general linear group on an affine algebraic variety over an algebraically closed field. We provide necessary invariance arguments for the existence of degenerations, which is helpful to discover that the associative algebras are rigid.

Moreover, applications of the invariance arguments to the varieties of low-dimensional complex associative algebras are described.

In Chapter 5, we deal with a generating sets for a toric ideals so-called Markov basis. Therefore, an action of dihedral group on the rational vector space to describe the invariance group of some contingency tables is considered. Additionally, we propose an algebraic method, which serves to obtain the set of all independence models of two-way contingency tables with the same row sums and column sums, which is referred as fiber in algebraic statistics. Finally, we introduce a new class of algebras which is closely related to the Markov basis in algebraic statistics.

In Chapter 6, we summarize and conclude our research. We also give future research problems in these areas.

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LIST OF PUBLICATIONS

The following are the list of publications that arise from this study.

Journal Articles:

N. F. Mohammed, I.S. Rakhimov and M. Shitan, 2017, Algebraic Method for Independence Model of Two-Way Contingency Tables, *Malaysian Journal of Mathematical Sciences*, 11(S), February: 5372.

N. F. Mohammed, I.S. Rakhimov and Sh.K. Said Husain, 2017, On Cohomology Groups of Three-Dimensional Complex Associative Algebras, *Far East Journal of Mathematical Sciences*, 102(4), 669-686.

N. F. Mohammed, I.S. Rakhimov and Sh.K. Said Husain, 2017, On Contractions of Three-Dimensional Complex Associative Algebras, *Journal of Generalized Lie Theory and Applications*, 11(3).

M. A. Fiidow, **N. F. Mohammed**, I.S. Rakhimov and Sh.K. Said Husain, 2017, On Inner Derivations of Finite Dimensional Associative Algebras, *Far East Journal of Mathematical Sciences*, ((In Press).

Conferences:

N. F. Mohammed, I.S. Rakhimov and M. Shitan, *Markov Bases and Toric Ideals for Some Contingency Tables*, The 2nd International Conference of Mathematical Sciences and Statistics (ICMSS 2016). Sunway Putra Hotel, Kuala Lumpur, Malaysia. January 26-28, 2016.

N. F. Mohammed, I.S. Rakhimov and M. Shitan, *Toric Ideals and Markov Bases for an Agriculture Data Set*, Conference on Agriculture Statistics 2015 (CAS2015), Faculty of Agriculture and Food Sciences, UPM Bintulu Sarawak Campus, Malaysia. October 6-8, 2015.

N. F. Mohammed, I.S. Rakhimov and Sh.K. Said Husain, *Contractions of Low Dimensional Complex Associative Algebras*, The 2nd International Conference and Workshop of Mathematical Analysis (ICOWMA 2016), Century Langkawi Beach Resort, Langkawi, Malaysia. August 2-4, 2016.

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