UNIVERSITI PUTRA MALAYSIA

ITERATIVE METHODS FOR SOLVING SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

LAWAN BULAMA MOHAMMED

FS 2016 74

ITERATIVE METHODS FOR SOLVING SPLIT COMMON FIXED POINT PROBLEMS IN HILBERT SPACES

By

LAWAN BULAMA MOHAMMED

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

December 2016

DEDICATIONS

To my parents;
Late Malam Bulama Mohammed (May his soul rest in perfect peace, Ameen) &
Hajjiya Zulai Mohammad.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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December 2016

Chair: Professor Adem Kılıçman, PhD
Faculty: Science

The split common fixed point problems (SCFPP) attracted and continued to attract the attention of many researchers; this is due to its applications in many branches of mathematics both pure and applied. Further, SCFPP provides us with a unified structure to study large number of nonlinear mappings. Our interest here is to apply these mappings to propose some algorithms for solving split common fixed point problems and its variant forms, in the end, we prove the convergence results of these algorithms.

In other words, we construct parallel and cyclic algorithms for solving the split common fixed point problems for strictly pseudocontractive mappings and prove the convergence results of these algorithms. We also suggest some iterative methods for solving the split common fixed point problems for the class of total quasi asymptotically nonexpansive mappings and prove the convergence results of the proposed algorithms. As a special case of this split common fixed point problems, we consider the split feasibility problem and prove its convergence results.

To solve the split common fixed point problems, one needs to estimate the norm of the bounded linear operator. To determine the norm of this bounded linear operator is a tough task. In this regard, we consider an algorithm for solving such a problem which does not need any prior information on the norm of the bounded linear operator and establish the convergence results of the proposed algorithm. These were done by considering the class of demicontractive mappings.

We also formulate and analyse algorithms for solving the split common fixed point equality problems for the class of finite family of quasi-nonexpansive mappings. Furthermore, we propose another problem namely split feasibility and fixed point equality problems and suggest some new iterative methods and prove their convergence results for the class of quasi-nonexpansive mappings.

Finally, as a special case of the split feasibility and fixed point equality problems, we consider the split feasibility and fixed point problems and propose Ishikawa-type extra-gradients algorithms for solving these split feasibility and fixed point problems for the class of quasi-nonexpansive mappings in Hilbert spaces. In the end, we prove the convergence results of the proposed algorithms.

Results proved in this thesis continue to hold for different types of problems, such as; convex feasibility problem, split feasibility problem and multiple-set split feasibility problems. For more details, see Chapter 3 Corollaries 3.4.1, 3.4.2, 3.4.3 and 3.4.4.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH LELARAN UNTUK MENYELESAIKAN MASALAH TITIK TETAP PISAH SEPUNYA DALAM RUANG HILBERT

Oleh

LAWAN BULAMA MOHAMMED

Disember 2016

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Masalah titik tetap sepunya yang terpisah (SCFPP) menarik minat pengkaji, dahulu dan akan datang, kerana aspek gunaanya dalam cabang matematik tulen dan matematik gunaan. Selanjutuya, SCFPP memberikan struktur yang sepadu untuk kajian pemetaan bukan linear dengan nombor yang besar. Tumpuan kami di sini ialah untuk menggunakan pemetaan tersebut berkaitan algoritma penyelesaian masalah titik tetap sepunya yang terpisah dan bentuk variannya, dan kami buktikan keputusan penumpuan algoritma berkenaan.

Dalam kata lain, kami bina algoritma selari dan berkala dalam penyelesaian masalah titik tetap sepunya yang terpisah untuk pemetaan psedukecutan dan membuktikan keputusan penumpuan algoritma tersebut. Kami turut mencadangkan kaedah lelaran untuk penyelesaian masalah titik tetap sepunya yang terpisah bagi kelas pemetaan bukan mengembang yang berkuasi total asimptot, dan membuktikan keputusan penumpuan algoritma. Sebagai kes khusus masalah titik tetap sepunya, kami pertimbangkan masalah terpisah yang tersaur serta membuktikan keputusan penumpuan.

Untuk menyelesaikan masalah titik tetap sepunya, kita diperlukan untuk menganggar norma peng-operasi linear yang terbatas. Menetukan norma pengoperasi linear terbatas, merupakan usaha yang sukar. Dalam hubungan ini, kami mempertimbangkan algoritma bagi penyelesaian masalah yang tidak memerlukan maklumat awalan di atas norma pengoperasi linear serta membukti keputusan penumpuan yang dicadangkan. Hal ini dilakukan dengan pertimbangan kelas pemetaan demikecutan. Kami merumus dan menganalisis algoritma penyelesesian masalah kesamaan titik tetap sepunya bagi kelas tak terhingga untuk keluarga pemetaan quasi - bukankembangan yang terhingga. Seterusnya kami cadangkan masalah lain, iaitu masalah tersaur terpisah dan titik tetap kesamaan serta mencadangkan kaedah letaran yang baru dan membuktikan keputusan sifat penumpuannya bagi kelas pemetaan quasi-bukan kembangan. Akhirnya, kami

pertimbangkan untuk kes khusus masalah pemisahan tersaur dan kesamaan titik tetap, masalah pemisahan tersaur dan titak tetap kesamaan dan cadangkan algoritma jenis-Ishikawa kecerunan lebih untuk menyelesaikan masalah pemisahan tersaur dan masalah titik tetap untuk keluarga pemetaan quasi-bukan kembangan dalam ruang Hilbert. Akhirnya, kami buktikan keputusan penumpuan algoritma yang dicadangkan.

Keputusan yang dibuktikan di dalam tesis ini boleh digunakan untuk beberapa jenis masalah yang berbeza, seperti; masalah boleh laksana cembung, masalah boleh laksana pisah, dan masalah boleh laksana pisah set-berganda. Untuk lebih mendalam, lihat Bab 3 Korolari 3.4.1, 3.4.2, 3.4.3 and 3.4.4.

ACKNOWLEDGEMENTS

My first gratitude goes to Almighty Allah for his guidance and protection throughout my life and studies. May His peace and blessings continue to shower on the best creature, the seal of prophet-hood, Muhammad (SAW) and his descendants.

My sincere gratitude and appreciation goes to my parents and my wife for their immense prayers, support and encouragement at all times, I pray that they live long to reap the fruits of their labour, and I pray that Almighty Allah will reward them abundantly.

I sincerely appreciate the tremendous effort of my supervisor, an internationally renowned mathematician, **Prof. Dr. Adem Kilicman**, the mastermind of this research work and for his guidance, encouragement and patience at all time. I am very grateful for his careful and constructive criticisms, helpful suggestion and his untiring efforts coupled with his accessibility at a time of needs. My special appreciation and thanks goes the members of my supervisory committee; **Associate Prof. Ibragimov Gafurjan** and **Associate Prof. Zarina Bibi bt Ibrahim** for their valuable contribution, observation and assistance up to the completion of this thesis. I equally commend all staff and student of the Department of Mathematics for their help in one way of the other.

Special thanks to my family, especially my mother for her prayers, love, and sacrifices. I would like to express my deepest appreciation to Bauchi State University, Gadau (BASUG) for their support throughout the entire process. And I would also like to thanks **Prof. H.N. Aliyu, Prof. Bashir Ali, Assc. Prof. S.I. Bala, Dr. A.J. Badakaya, Dr. Y. I. Fagge, Dr. S.L. Bichi, Aminu Bala, Faruq Umar, Umar Mohammed, Yusuf Mohammed, Kabir Usman, Mujittafa Mohammed and Ibrahim Saminu.** I appreciate your support, encouragement and prayers.

This acknowledgement will be incomplete if I fail to recognize the tremendous efforts of **Alh Sani Bala, Malam Idris Sulaiman** and my brother **Umar Bulama Mohammed.** I am grateful and to each goes my sincere thanks.

Finally, I wish to express my thanks to **Adamu Adamu Muhammad, Arzuka Ibrahim, Abba Auwalu, Yufuf Dauda Jikan-Toro** to mention but a few. I appreciate your encouragement.

I certify that a Thesis Examination Committee has met on 14 December 2016 to conduct the final examination of Lawan Bulama Mohammed on his thesis entitled "Iterative Methods for Solving Split Common Fixed Point Problems in Hilbert Spaces" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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Declaration by graduate student

I hereby confirm that:

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- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
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This is to confirm that:
- the research conducted and the writing of this thesis was under the supervision;
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$\forall x \in X$	For all x in X
$x \in X$	x is an element of the set X
$x \notin X$	x is not in X
$\exists x \in X$	There exists x in X
$A \subset X$	A is a subset of X
$f : X \to Y$	Mapping (function) from X to Y
$\emptyset$	Empty set
iff	If and only if
$\mathbb{N}$	Set of natural numbers (positive integers)
$\mathbb{Z}$	Set of integer numbers
$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set positive real numbers
(V ,.,+)	Vector space
$\mathbb{C}$	Set of complex numbers
$\mathbb{F}$	Field ($\mathbb{R}$ or $\mathbb{C}$)
H, H_1, H_2, H_3	Hilbert Spaces
E	Banach Space
C,Q	Nonempty, Closed Convex Subset of H_1 and H_2
FPP	Fixed Point Problem
CFP	Convex Feasibility Problem
SFP	Split Feasibility Problem
MSSFP	Multiple Set Split Feasibility Problems
CFPP	Common Fixed Point Problems
SCFPP	Split Common Fixed Point Problems
SFFPP	Split Feasibility Fixed Point Problems
SFFPEP	Split Feasibility Fixed Point Equality Problems
SCFPEP	Split Common Fixed Point Equality Problems
Γ	The solution set of the SCFPP
Ω	The solution set of the SFP
Δ	The solution set of the SFFPP
Φ	The solution set of the SFFPEP
Ψ	The solution set of the SCFPEP
$\to$	Strong Convergence
$\rightharpoonup$	Weak Convergence
$\omega_\omega(x_n)$	The set of the cluster point of $\{x_n\}$ in the weak topology i.e., { there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x\}$.

CHAPTER 1

INTRODUCTION

1.1 Background

Functional analysis is an abstract branch of mathematics that originated from classical analysis. The impetus came from; linear algebra, problems related to ordinary and partial differential equations, calculus of variations, approximation theory, integral equations, and so on. Functional analysis can be defined as the study of certain topological-algebraic structures and of the methods by which the knowledge of these structures can be applied to analytic problems (Rudin (1973)).

Fixed point theory (FPT) is one of the most powerful and fruitful tools of modern mathematics and may be considered a core subject of nonlinear analysis. It has been a nourishing area of research for many mathematicians. The origins of the theory, which date to the later part of the nineteenth century, rest in the use of successive approximations to establish the existence and uniqueness of the solutions, particularly to differential equations, for example, see Arino et al. (1984); Yamamoto (1998); Taleb and Hanebaly (2000); Nieto and Rodríguez-López (2005); Pathak et al. (2007); Sestelo and Pouso (2015) and references therein.

The classical importance of fixed point theory in functional analysis is due to its usefulness in the theory of ordinary and partial differential equations. The existence or construction of a solution to a differential equation often reduces to the existence or location of a fixed point for an operator defined on a subset of a space of functions. Fixed point theory had also been used to determine the existence of periodic solutions for functional differential equations when solutions are already known to exist, for example, see Chow (1974); Grimmer (1979); Torres (2003); Kiss and Lessard (2012) and references therein.

Related to the FPT, we have the split common fixed point problems (SCFPP). The SCFPP was introduced and studied by Censor and Segal (2009b) as a generalization of many existing problems in nonlinear sciences, both pure and applied. Moreover, Censor and Segal (2009b) had shown that the problem of fixed point, convex feasibility, multiple-set split feasibility, split feasibility and much more can be studied more conveniently as SCFPP. The results and conclusions that are true for the SCFPP continue to hold for these problems, and it shows the significance and range of applicability of the SCFPP. One of the important applications of SCFPP can be seen in intensity modulation radiation therapy (IMRT), for more details, see Censor et al. (2006).

This research work falls within the general area of "Nonlinear Functional Analysis", an area with the vast amount of applicability in the recent years, as such becoming the object of an increasing amount of study. We focus on an important topic within this area "**Iterative Methods for Solving Split Common Fixed Point Problems in Hilbert Spaces**."

In this regard, we discuss the SCFPP and its variant forms. We show that already known problems are special cases of the split common fixed point problems (SCFPP). We use approximation methods to suggest different iterative algorithms for solving SCFPP and its variant forms. In the end, we give the convergence results of these algorithms.

1.2 Problem Formulations

Mathematically, the convex feasibility problem (CFP) consist as finding a vector x^* such that

$$x^* \in \bigcap_{i=1}^{N} C_i, \tag{1.1}$$

where $C_i, i = 1,2,3,...,N$ are closed and convex subset of H_1.

The multiple set split feasibilty problems (MSSFP) was introduced by Censor et al. (2005), and is formulated as:

$$\text{Find } x^* \in \bigcap_{i=1}^{N} C_i \text{ such that } Ax^* \in \bigcap_{j=1}^{M} Q_j, \tag{1.2}$$

where $C_i, i = 1,2,3,...,N$ and $Q_j, j = 1,2,3,...,M$ are closed convex subset of H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator.

If $M = N = 1$, Equation (1.2) reduces to

$$\text{find } x^* \in C \text{ such that } Ax^* \in Q. \tag{1.3}$$

Equation (1.3) is known as the split feasibility problem (SFP).

Since every nonempty closed convex subset of Hilbert space is a fixed point of its associating projection, then, Problem (1.1) and (1.2) becomes:

$$x^* \in \bigcap_{i=1}^{N} Fix(T_i), \text{ and} \tag{1.4}$$

$$x^* \in \bigcap_{i=1}^{N} Fix(T_i) \text{ such that } Ax^* \in \bigcap_{j=1}^{M} Fix(U_j). \tag{1.5}$$

Equation (1.4) and (1.5) are called common fixed point problem (CFPP) and split common fixed point problem (SCFPP), respectively, where $T_i : H_1 \to H_1$ (i = 1,2,3,...,N) and $U_j : H_2 \to H_2$ (j= 1,2,3,...,M) are some nonlinear operators.

1.3 Objectives of the Thesis

The main objectives are:

- To construct parallel and cyclic algorithms for solving the split common fixed point problems and prove the convergence results of the proposed algorithms.
- To propose iterative algorithms for solving the split common fixed point problems for total quasi asymptotically nonexpansive mappings and prove the convergence results of these algorithms.
- To study the split feasibility problems for total quasi asymptotically nonexpansive mappings in Hilbert spaces. To discuss the solutions to these problems, we will suggest some algorithms and prove the convergence results of these algorithms.
- To propose Ishikawa-type extra-gradient algorithms for solving the split feasibility and fixed point problems for quasi-nonexpansive mappings in Hilbert spaces and prove the convergence results of the proposed algorithms.
- To propose the split feasibility and fixed point equality problems (SFFPEP) and split common fixed point equality problems (SCFPEP). To discuss the solutions to these problems, we will suggest some algorithms and study their convergence for the class of quasi-nonexpansive mappings.

1.4 Scope and Limitation

1.4.1 Scope

This research work will focus on an approximation of the split common fixed point problems for the class of; quasi-nonexpansive mappings, strictly pseudocontractive mappings, demicontractive mappings and total quasi-asymptotically nonexpansive mappings. Also to consider these mappings and suggest some algorithms for solving split common fixed point problems and its variant forms. At the end to give the weak and strong convergence results of the proposed algorithms.

1.4.2 Limitation

This research work is theoretically in nature, and will focus mainly on an approximation of the split common fixed point problems and its variant forms in the general content of Hilbert spaces.

1.5 Thesis Outlines

This thesis organizes as follows:

In Chapter 1, we give the background of the research, objectives, scope and limitation, outlines of the thesis, fundamental concepts and preliminary results from the literature. Furthermore, we include some useful properties of Hilbert space and also consider useful properties of different classes of nonlinear mappings. These properties are utilized in the proof of the main results of this thesis.

Chapter 2 gives an overview of the split common fixed point problems and its variant forms.

In Chapter 3, we study parallel and cyclic algorithms for solving the split common fixed point problems for the finite family of strictly pseudocontractive mappings in Hilbert spaces and prove the weak and strong convergence theorems of these algorithms. Also, we give some special cases of our suggested methods.

In Chapter 4, we study the SCFPP for total quasi asymptotically nonexpansive mappings in Hilbert spaces, this class of mapping generalizes the class of quasi-nonexpansive and asymptotically quasi-nonexpansive mappings. To discuss the solution of this type of mapping, we suggest some iterative algorithms and discuss the convergence of these algorithms. Furthermore, we consider an algorithm for solving the split common fixed point problems which does not need any prior information on the norm of the bounded linear operator and establish the convergence results of the proposed algorithm. These were done by considering the class of demicontractive mappings. In the end, we give some special cases of our suggested methods.

In Chapter 5, we study the split feasibility problem for total quasi asymptotically nonexpansive mappings in Hilbert spaces. To discuss the solutions of this problem, we suggest some algorithms and prove their convergence results. In the end, we give some special cases of our suggested methods

In Chapter 6, we study Ishikawa-type extra-gradient algorithms for solving split feasibility and fixed point problems for quasi-nonexpansive mappings in Hilbert spaces. Under some mild conditions imposed on the parameters and operators involved, we prove the convergence results of the proposed algorithms.

Chapter 7 deal with the following problems:

- Split feasibility and fixed point equality problems (SFFPEP);
- Split common fixed point equality problems (SCFPEP).

We study these problems for the class of quasi-nonexpansive mappings. Furthermore, we suggest new iterative algorithms and study their convergence results for the proposed problems. Finally, we give some special cases of our proposed methods.

Finally, in Chapter 8, we give the summary and conclusion of our research work. Some future works are also presented to provide a future research direction.

1.6 Basic Concepts and Definitions

1.6.1 Introduction

In this section, we give some definitions and basic results. We start from the definition of vector space and end with some results from Hilbert spaces. Those results that are commonly used in all the chapters are given in this section, and those results that are relevant to a particular chapter are provided at the beginning of each chapter. In short, this section works as a foundation for the structure of this thesis.

1.6.2 Vector Spaces

Vector spaces play a vital role in many branches of mathematics. In fact, in various practical (and theoretical) problems we have a set V whose elements may be vectors in three-dimensional space, or sequences of numbers, or functions, and these elements can be added and multiplied by constants (numbers) in a natural way, the result being again an element of V. Such concrete situations suggest the concept of a vector space as defined below. The definition will involve a general field $\mathbb{F}$, but in functional analysis, $\mathbb{F}$ will be $\mathbb{R}$ or $\mathbb{C}$. The elements of $\mathbb{F}$ are called scalars, while in this thesis they will be real or complex numbers.

Definition 1.1 A vector space over a field $\mathbb{F}$ is a nonempty set denoted by V together with addition (+) and scalar multiplication (.) satisfies the following conditions:

(i) x+y=y+x, for all $x, y \in V$;

(ii) x+(y+w)=(x+y)+w, for all $x, y, w \in V$;

(iii) there exists a vector denoted by θ such that $x + \theta = x$, for all $x \in V$;

(iv) for all $x \in V$, there exists a unique vector denoted by (-x) such that $x + (-x) = \theta$;

(v) $\alpha.(\beta.x) = (\alpha.\beta).x$, for all $\alpha, \beta \in \mathbb{F}$ and $x \in V$;

(vi) $\alpha.(x+y) = \alpha.x + \alpha.y$, for all $x, y \in V$ and $\alpha \in \mathbb{F}$;

(vii) $(\alpha + \beta).x = \alpha.x + \beta.x$, for all $\alpha, \beta \in \mathbb{F}$ and $x \in V$;

(viii) there exists $1 \in \mathbb{F}$ such that $1.x = x$, $\forall x \in V$.

Remark 1.1 From now we will drop the dot (.) in the scalar multiplication and denote $\alpha.\beta$ as $\alpha\beta$.

Let $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, ..., \mathrm{v}_\mathrm{n} \in \mathrm{V}$ and $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_\mathrm{n}$ be scalars. Consider the equation:

$$\alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + \alpha_3 \mathrm{v}_3 + ... + \alpha_\mathrm{n} \mathrm{v}_\mathrm{n} = 0. \tag{1.6}$$

Trivially, $\alpha_1 = \alpha_2 = \alpha_3 = ... = \alpha_n = 0$ solves Equation (1.6). If it is possible to have the solution of Equation (1.6) with at least one of the $\alpha_i{\prime}s$ non zero, then the vectors $\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, ..., \mathrm{v}_\mathrm{n}$ are called **Linearly Dependent** otherwise they are called **Linearly Independent**.

If $\mathbb{M} \subseteq \mathrm{V}$ consist of a linearly independent set of vectors; we say that $\mathbb{M}$ is a linearly independent set.

Definition 1.2 Span of $\mathbb{M}$ (Span$\mathbb{M}$) is defined as the set of all linear combination of $\mathbb{M}$, i.e., Span$\mathbb{M}$ = $\{\alpha_1 \mathrm{v}_1 + \alpha_2 \mathrm{v}_2 + \alpha_3 \mathrm{v}_3 + ..., \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, ... \in \mathrm{V}$, where $\alpha_1, \alpha_2, ...$ are scalars$\}$.

Definition 1.3 Let $\mathbb{M} \subseteq \mathrm{V}$. $\mathbb{M}$ is said to be basis for the space V, if

(i) $\mathbb{M}$ is a linearly independent set,

(ii) Span$\mathbb{M}$ = V.

Definition 1.4 Let V be a vector space, the dimension of V (dimV) is the number of vectors of the basis of V. V is of finite dimension if its dimension is finite. Otherwise, it is said to be of infinite dimensional space.

Definition 1.5 Let C be a subset of V. C is said to be convex, if for all $x, y \in C$, $\gamma \in [0,1]$, $(1-\gamma)x + \gamma y \in C$. In general, for all $x_1, x_2, x_3, ..., x_n \in C$ and for $\gamma_j \geq 0$ such that $\sum_{j=1}^n \gamma_j = 1$, the combination $\sum_{j=1}^n \gamma_j x_j \in C$ is called the convex combination.

Definition 1.6 A mapping $T : V_1 \to V_2$ is said to be linear, if $\forall u, v \in V_1$ and α, β scalars,

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v).$$

Limits (of convergent sequences), differentiation and integration, are examples of a linear map.

Remark 1.2 If in Definition 1.6, the linear space V_2 is replaced by a scalar field $\mathbb{F}$, then the linear map T is called linear functional on V_1.

1.6.3 Hilbert Space and its Properties

Definition 1.7 Let Y be a linear space. An inner product on Y is a function $\langle .,. \rangle$: $Y \times Y \to \mathbb{F}$ such that the following conditions are satisfies:

(i) $\langle y, y \rangle \geq 0 \ \forall y \in Y$;

(ii) $\langle y, y \rangle = 0$ iff $y = 0$, $\forall y \in Y$;

(iii) $\langle y, z \rangle = \overline{\langle z, y \rangle}$, $\forall y, z \in Y$, where the "bar" indicates the complex conjugation;

(iv) $\langle \alpha x + \beta y, z \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle y, z \rangle$, for all $x, y, z \in Y$ and $\alpha, \beta \in \mathbb{C}$.

Remark 1.3 The pair $(Y, \langle .,. \rangle)$ is called an inner product space. We shall simply write Y for the inner product space $(Y, \langle .,. \rangle)$ when the inner product $\langle .,. \rangle$ is known. Furthermore, if Y is a real vector space, then condition (iii) above reduces to $\langle x, z \rangle = \langle z, x \rangle$ (Symmetry).

Definition 1.8 Let Y be a linear space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). A norm on Y is a real-valued function $\|.\| : Y \to \mathbb{R}$ such that the following conditions are satisfies:

(i) $\|x\| \geq 0$, $\forall x \in Y$;

(ii) $\|x\| = 0$ iff $x = 0$, $\forall x \in Y$;

(iii) $\|\alpha x\| = |\alpha| \, \|x\|$, $\forall x \in Y$ and $\alpha \in \mathbb{R}$;

(iv) $\|x + z\| \leq \|x\| + \|z\|$, $\forall x, z \in Y$.

Remark 1.4 A linear space Y with a norm defined on it i.e., $(Y, \|.\|)$ is called a normed linear space. If Y is a normed linear space, the norm $\|.\|$ always induces a metric d on Y given by $d(z, x) = \|z - x\|$ for each $x, z \in Y$, with this, (Y, d) become a metric space. For a quick review of metric space the reader may consult Dunford et al. (1971).

Lemma 1.1 Let Y be an inner product space. For arbitrary $x, z \in Y$,

$$|\langle x, z \rangle|^2 \leq \langle x, x \rangle \langle z, z \rangle . \tag{1.7}$$

If x and z are linearly dependent, then Equation (1.7) reduces to

$$|\langle x, z \rangle|^2 = \langle x, x \rangle \langle z, z \rangle .$$

This lemma is known as Cauchy-Schwartz Inequality. For more details about the proof, one is referred to Chidume (2006).

Lemma 1.2 A mapping $\|.\| : Y \to \mathbb{R}$ defined by

$$\|x\| = \sqrt{\langle x, x\rangle}, \forall x \in Y$$

is a norm on Y.

Remark 1.5 As the consequence of Lemma 1.2, Equation (1.7) reduces to the following inequality:

$$|\langle x, z\rangle| \leq \|x\| \|z\|, \forall x, z \in Y.$$

Definition 1.9 A sequence $\{y_n\}$ in a normed linear space Y is said to converge to $y \in Y$, if $\forall \varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$, such that $\|y_n - y\| < \varepsilon$, $\forall n \geq N_\varepsilon$. The vector $y \in Y$ is called the limit of the sequence $\{y_n\}$ and is written as $\lim_{n\to\infty} y_n = y$ or $y_n \to y$, as $n \to \infty$.

Definition 1.10 A sequence $\{y_n\}$ in a normed linear space Y is said to converge weakly to $y \in Y$, if for all $h \in Y^*$ such that $\lim_{n\to\infty} h(y_n) = h(y)$, where Y^* denote the dual space of Y.

Next, we give some results regards to the weak convergence of a sequence. For more details about the proof, see Chidume (2006).

Lemma 1.3 Let $\{y_n\} \subseteq E$ (Banach space). Then the following results are satisfies:

(i) $y_n \rightharpoonup y \Leftrightarrow h(y_n) \to h(y)$ for each $h \in E^*$;

(ii) $y_n \to y \Rightarrow y_n \rightharpoonup y$;

(iii) $y_n \rightharpoonup y \Rightarrow \{y_n\}$ is bounded and

$$\|y\| \leq \liminf_{n\to\infty} \|y_n\|;$$

(iv) $y_n \rightharpoonup y$ (in E), $h_n \to h$ (in E^*) $\Rightarrow h_n(y_n) \to h(y)$ (in $\mathbb{R}$).

Remark 1.6 Lemma 1.3 (ii) Shows that strong convergence implies weak convergence. However, the converse may not necessarily be true, that is, in an infinite dimensional space, weak convergence does not always imply strong convergence, while they are the same if the dimension is finite. For the example of weak convergence which is not strong convergence, see Chidume (2006) and the references therein.

Definition 1.11 Let C be a subset of H. A sequence $\{y_n\}$ in H is said to be Fejer monotone, if

$$\|y_{n+1} - z\| \leq \|y_n - z\|, \forall n \geq 1, z \in C.$$

Definition 1.12 A sequence $\{y_n\}$ in a normed linear space Y is said to be Cauchy, if $\forall \varepsilon > 0$, $\exists N_\varepsilon \in \mathbb{N}$ such that $\|y_n - y_m\| < \varepsilon$, $\forall n, m \geq N_\varepsilon$.

Definition 1.13 A normed linear space Y is said to be complete if and only if every Cauchy sequence in Y converges.

Remark 1.7 With respect to the norm defined in Lemma 1.2, we can define the Cauchy sequence in an inner product space Y. A sequence $\{y_n\}$ in Y is said to be Cauchy if and only if $\langle y_n - y_m, y_n - y_m \rangle^{1/2} := \|y_n - y_m\| \to 0$ as $n, m \to \infty$.

Definition 1.14 An inner product space Y is said to be complete if and only if every Cauchy sequence converges.

Definition 1.15 A complete inner product space is called a Hilbert Space and that of normed linear space is known as a Banach Space.

1.6.4 Bounded Linear Map and its Properties

Definition 1.16 Let $T : H \to H$ be a linear map. T is said to be bounded, if there exists a constant $M \geq 0$ such that

$$\|Ty\| \leq M \|y\|, \forall y \in H.$$

Next, we give some results of a linear map that are continuous. For more details about the proof, see Chidume (2006).

Lemma 1.4 Let X and Y be normed linear spaces and $T : X \to Y$ be a linear operator. Then the following results are equivalent:

(i) T is continuous;

(ii) T is continuous at the origin i.e., if $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} x_n = 0, \text{ then } \lim_{n\to\infty} Tx_n = 0 \text{ in } Y;$$

(iii) T is Lipschitz, i.e., in the sense that there exists $M \geq 0$ such that

$$\|Tx\| \leq M \|x\|, \forall x \in X;$$

(iv) $T(\Delta)$ is bounded $\Big($in the sense that there exists $M \geq 0$ such that $\|Tx\| \leq M$ for all $x \in \Delta$, where $\Delta := \{x \in X : \|x\| \leq 1\}\Big)$.

Remark 1.8 In the light of Lemma 1.4, we have that a linear map $T : X \to Y$ is continuous iff it is bounded.

Definition 1.17 Let $A : H \to H$ be a bounded linear map. Define a mapping $A^* : H \to H$ by

$$\langle Ay, z \rangle = \langle y, A^* z \rangle, \forall y, z \in H.$$

The mapping A^* is called the adjoint of A.

The following results are fundamental for the adjoint operator on Hilbert space. For the proof, see Chidume (2006).

Lemma 1.5 Let $A : H \to H$ be a bounded linear map with its adjoint A^*. Then the following hold:

(i) $(A^*)^* = A$;

(ii) $\|A\| = \|A^*\|$;

(iii) $\|AA^*\| = \|A\|^2$.

1.6.5 Some Nonlinear Operators

Let $T : H \to H$ be a map. A point $x \in H$ is called a **fixed point** of T provided $Tx = x$. We denote the set of fixed point of T by $Fix(T)$, that is

$$Fix(T) = \{x \in H : Tx = x\}.$$

The $Fix(T)$ is closed and convex, for more details, see Goebel and Kirk (1990).

T is said to be $\eta-$**strongly monotone**, if there exists a constant $\eta > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \eta \|x - y\|, \forall x, y \in H,$$

and it is said to be **contraction**, if

$$\|Tx - Tz\| \leq k \|x - z\|, \forall x, z \in H, \tag{1.8}$$

where $k \in (0, 1)$.

Remark 1.9 If $T : H \to H$ is a contraction mapping with coefficient $k \in (0,1)$, then $(I-T)$ is $(1-k)-$strongly monotone, that is

$$\langle (I-T)w-(I-T)z,\ w-z\rangle \geq (1-k)\left\|w-z\right\|^2, \forall w,z \in H.$$

Proof:

$$\begin{aligned}\langle (I-T)w-(I-T)z,\ w-z\rangle &= \langle\ w-z,\ w-z\rangle + \langle Tz-Tw,\ w-z\rangle \\ &= \langle\ w-z,\ w-z\rangle - \langle Tw-Tz,\ w-z\rangle. \qquad (1.9)\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle Tw-Tz,\ w-z\rangle &\leq \|Tw-Tz\|\,\|w-z\| \\ &\leq k\,\|w-z\|,\ \text{ since f is a contraction mapping.} \qquad (1.10)\end{aligned}$$

By (1.9) and (1.10), we deduce that

$$\langle (I-T)w-(I-T)z,\ w-z\rangle \geq (1-k)\left\|w-z\right\|^2.$$

And the proof completed.

Equation (1.8) reduces to the following equation as $k=1$.

$$\|Tx-Tz\| \leq \|x-z\|, \forall x,z \in H.$$

This is known as nonexpansive mapping. As a generalization of nonexpansive mapping, we have **asymptotically nonexpansive** (see Goebel and Kirk (1972)), this mapping is defined as:

$$\|T^n x-T^n z\| \leq k_n \|x-z\|, \forall n \geq 1 \text{ and } x,z \in H,$$

where $k_n \subset [1,\infty)$ such that $\lim_{n\to\infty} k_n = 1$.

The map T is said to be **total asymptotically nonexpansive** (see Alber et al. (2006)), if

$$\|T^n x-T^n z\|^2 \leq \|x-z\|^2 + \nu_n \eta(\|x-z\|) + \mu_n, \forall n \geq 1 \text{ and } x,z \in H.$$

where $\{\nu_n\}$ and $\{\mu_n\}$ are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} \nu_n = 0$, $\lim_{n\to\infty} \mu_n = 0$, and $\eta : \Re^+ \to \Re^+$ is a strictly increasing continuous function with $\eta(0)=0$. This class of mapping generalizes the class of nonexpansive and asymptotically nonexpansive mappings (for more details see Chidume and Ofoedu (2007, 2009) and references therein). And it is said to be $(k,\{\mu_n\},\{\xi_n\},\phi)$- total asymptotically strict pseudocontraction, if there exists a constant $k \in [0,1)$, $\mu_n \subset [0,\infty)$, $\xi_n \subset [0,\infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, and continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0)=0$ such that

$$\|T^n x-T^n y\|^2 \leq \|x-y\|^2 + k\,\|(I-T^n)x-(I-T^n)y\|^2 \quad +\mu_n \phi(\|x-y\|) + \xi_n, \forall x,y \in H.$$

T is said to be **strictly pseudocontractive** (see Browder and Petryshyn (1967)), if

$$\|Tx - Tz\|^2 \leq \|x - z\|^2 + k\,\|(I - T)x - (I - T)z\|^2, \forall x, z \in H,$$

where $k \in [0, 1)$. And it is said to **pseudocontractive** if

$$\|Tx - Tz\|^2 \leq \|x - z\|^2 + \|(I - T)x - (I - T)z\|^2, \forall x, z \in H.$$

It is obvious that all nonexpansive mappings and strictly pseudocontractive mappings are pseudocontractive mappings but the converse does not hold.

T is said to be **quasi-nonexpansive** (see Diaz and Metcalf (1967)), if $Fix(T) \neq \emptyset$ and

$$\|Tx - z\| \leq \|x - z\|, \;\; \forall x \in H \text{ and } z \in Fix(T).$$

This is equivalent to

$$2\langle x - Tx, z - Tx\rangle \leq \|Tx - x\|^2, \forall x \in H \text{ and } z \in Fix(T). \tag{1.11}$$

Remark 1.10 Every nonexpansive mapping with $Fix(T) \neq \emptyset$ is a quasi-nonexpansive; however, the converse may not necessarily be true. Thus, the class of quasi- nonexpansive mapping generalizes the class of nonexpansive mapping.

The following is an example of a quasi-nonexpansive mapping which is not nonexpansive mapping, for more details, see He and Du (2012) and references therein.

Example 1.1 Let $H = \mathbb{R}$, defined $T : Q := [0, \infty) \to \mathbb{R}$ by

$$Ty = \frac{y^2 + 2}{1 + y} \text{ for all } y \in Q.$$

T is said to be $k-$demicontractive, if

$$\|Ty - z\| \leq \|y - z\| + k\,\|Ty - z\|, \forall y \in H \text{ and } z \in Fix(T), \tag{1.12}$$

where $k \in [0, 1)$. Trivially, the class of demicontractive mapping generalizes the class of quasi-nonexpansive mapping for $k \geq 0$.

The following is an example of a demicontractive mapping which is not quasi-nonexpansive mapping, for more details, see Chidume et al. (2015) and references therein.

Example 1.2 Define a map $T : l_2 \to l_2$ by

$$T(x_1, x_2, x_3, \ldots) = -\frac{5}{2}(x_1, x_2, x_3, \ldots), \text{ for arbitrary vector } (x_1, x_2, x_3, \ldots) \in l_2.$$

Remark 1.11 If $k = -1$, Equation (1.12) reduces to

$$\|Ty - z\| \leq \|y - z\| - \|Ty - y\|, \forall y \in H \text{ and } z \in Fix(T).$$

This is known as **firmly quasi-nonexpansive mapping**. Every strictly pseudocontractive mapping with $Fix(T) \neq \emptyset$ is a demicontractive mapping; however, the converse may not necessarily be true. Thus, the class of demicontractive mapping is more general than the class of strictly pseudocontractive mapping.

The following is an example of demicontractive mapping which is not strictly pseudocontractive mapping, for more details, see Browder and Petryshyn (1967) and references therein.

Example 1.3 Let $C = [-1, 1]$ be a sub set of a real Hilbert space H. Define T on C by

$$T(x) = \begin{cases} \frac{2}{3}x\sin(\frac{1}{x}), \text{ if } x \neq 0, \\ 0, \ x = 0. \end{cases}$$

Clearly, 0 is the only fixed point of T. For $x \in C$, we have

$$\begin{aligned} |Tx - 0|^2 &= |Tx|^2 \\ &= \left|\frac{2}{3}x\sin(\frac{1}{x})\right|^2 \\ &\leq \left|\frac{2x}{3}\right|^2 \\ &\leq |x|^2 \\ &\leq |x-0|^2 + k|Tx - x|^2, \text{ for any } k < 1. \end{aligned}$$

Thus, T is demicontractive mapping. Next, we see that T is not strictly pseudocontractive mapping. Let $x = \frac{2}{\pi}$ and $z = \frac{2}{3\pi}$, then $|Tx - Tz|^2 = \frac{256}{81\pi^2}$. However,

$$|x - z|^2 + |(I - T)x - (I - T)z|^2 = \frac{160}{81\pi^2}.$$

T is said to be **asymptotically quasi-nonexpansive**, if $Fix(T) \neq \emptyset$ such that for each $n \geq 1$,

$$\|T^n x - z\|^2 \leq t_n \|x - z\|^2, \forall z \in Fix(T) \text{ and } x \in H,$$

where $\{t_n\} \subseteq [1, \infty)$ with $\lim_{n\to\infty} t_n = 1$. It is clear from this definition that every asymptotically nonexpansive mapping with $Fix(T) \neq \emptyset$ is asymptotically quasi-nonexpansive mapping.

Also T is said to be $(\{r_n\},\{k_n\},\eta)$-**total quasi-asymptotically nonexpansive mapping,** if

$$\begin{aligned} \|T^n y - z\|^2 &\leq \|y-z\|^2 + r_n \eta(\|y-z\|) \\ &+ k_n, \forall n \geq 1, z \in Fix(T) \text{ and } y \in H, \end{aligned} \tag{1.13}$$

where $\{r_n\},\{k_n\}$ are sequences in $[0,\infty)$ such that $\lim_{n\to\infty} r_n = 0$, $\lim_{n\to\infty} k_n = 0$ and $\eta : \Re^+ \to \Re^+$ is a strictly continuous function with $\eta(0) = 0$. This class of mapping, generalizes the class of; quasi-nonexpansive, asymptotically quasi-nonexpansive and total asymptotically nonexpansive mapping.

T is said to be K-Lipschitzian, if

$$\|Ty - Tz\| \leq K\|y-z\|, \forall y,z \in H.$$

It is said to be uniformly K-Lipschitzian, if

$$\|T^n y - T^n z\| \leq K\|y-z\|, \forall y,z \in H.$$

Definition 1.18 A mapping $T : H \to H$ is said to be class$-\tau$ operator, if

$$\langle z - Ty, y - Ty\rangle \leq 0, \forall z \in Fix(T) \text{ and } y \in H.$$

It is important to note that, class$-\tau$ operator is also called directed operator, see Zaknoon (2003) and Censor and Segal (2009b), separating operator, see Cegielski (2010) or cutter operator, see Cegielski and Censor (2011) and references therein.

Definition 1.19 A self mapping T on H_1 is said to be semi-compact if for any bounded sequence $\{x_n\} \subset H$ with $(I-T)x_n$ converges strongly to 0, there exists a sub-sequence say $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to x.

Definition 1.20 A self mapping T on C is said to be demiclosed, if for any sequence $\{y_n\}$ in C such that $y_n \rightharpoonup y$ and if the sequence $Ty_n \to z$, then $Ty = z$.

Remark 1.12 In Definition 1.20, if $z = 0$, the zero vector in C, then T is called demiclosed at zero, for more details, see Moudafi (2011) and references therein.

Lemma 1.6 (Goebel and Kirk (1990)) If a self mapping T on C is a nonexpansive mapping, then T is demiclosed at zero.

Lemma 1.7 (Acedo and Xu (2007)) If a self mapping T on C is a $k-$strictly pseudo-contractive, then $(T-I)$ is demiclosed at zero.

Lemma 1.8 Let C be a subset of H_1, and P_C be a metric projection from H_1 onto C. Then $\forall y \in C$ and $x \in H_1$,

$$\|x - P_C(x)\| \leq \|y - x\| - \|y - P_C(x)\|.$$

For the proof of this lemma, see Li and He (2015) and references therein.

Lemma 1.9 For each $x, y \in H_1$, the following results hold.

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$,

(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \forall\, \alpha \in [0,1]$.

For the proof of this lemma, see Acedo and Xu (2007) and references therein.

Lemma 1.10 Let $\{a_n\}$ be a sequence of nonnegative real number such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n, n \geq 0,$$

where γ_n is a sequence in $(0,1)$ and σ_n is a sequence of real number such that;

(i) $\lim\limits_{n\to\infty} \gamma_n = 0$ and $\sum \gamma_n = \infty$;

(ii) $\lim\limits_{n\to\infty} \frac{\sigma_n}{\gamma_n} \leq 0$ or $\sum |\sigma_n| < \infty$. Then $\lim\limits_{n\to\infty} a_n = 0$.

For the proof, see Xu (2002).

Lemma 1.11 Let $\{x_n\}, \{y_n\}, \{z_n\}$ be sequences of nonnegative real numbers satisfying

$$x_{n+1} \leq (1+z_n)x_n + y_n.$$

If $\sum z_n < \infty$ and $\sum y_n < \infty$, then $\lim\limits_{n\to\infty} x_n$ exist.

For the proof of this lemma, see Xu (1993).

Lemma 1.12 Let $\{x_n\}$ be a Fejer monotone with respect to C, then the following are satisfied:

(i) $x_n \rightharpoonup x^* \in C$ if and only if $\omega_\omega \subset C$;

(ii) $\{P_C x_n\}$ converges strongly to some vector in C;

(iii) if $x_n \rightharpoonup x^* \in C$, then $x^* = \lim\limits_{n\to\infty} P_C x_n$.

For the proof, see Bauschke and Borwein (1996).

1.7 Summary

In this chapter, we discussed the background of the research study, the objectives of the research are given, and we briefly gave the outlines of the thesis and also provided some preliminary definitions and basic results which are very useful in understanding the study area of the research.

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