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MULTISTEP BLOCK METHODS FOR SOLVING HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

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MULTISTEP BLOCK METHODS FOR SOLVING HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

By

HOO YANN SEONG

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To my late parents for making me be who I am

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Doctor of Philosophy

MULTISTEP BLOCK METHODS FOR SOLVING HIGHER ORDER DELAY DIFFERENTIAL EQUATIONS

By

HOO YANN SEONG

May 2016

Chair: Professor Zanariah binti Abdul Majid, PhD
Faculty: Science

Delay differential equations (DDEs) play an important role in the investigated system which depends on the position of the system in the past and current time. The analytical solution of DDEs is hard to be found. Numerical methods provide an alternative way of constructing solutions to the problems.

This thesis describes the development of numerical algorithms for solving higher order DDEs. One-point and two-point multistep block method based on the Adam-Bashforth-Moulton methods for solving higher ordinary differential equation are adapted to solve the higher order DDEs. The proposed methods are based on constant step size and variable step size approach. Two types of DDEs are considered, namely retarded and neutral DDEs. Only the DDEs with constant delays and pantograph type are considered in this thesis. The delay term in DDEs with constant delays is approximated using Hermite interpolation. Linear and Hermite interpolators are used to approximate the delay terms in DDEs of pantograph type. The derivatives of the delay terms are approximated by using difference formula.

The thesis discusses the stability of the method when applied to DDEs with constant delays and pantograph type. The region of the stability is presented. Several problems are considered for illustrative purposes and the numerical approximations of their solutions are obtained using C-language. Numerical results of the proposed methods are compared with the existing numerical methods. Comparison among the methods indicated that the proposed methods achieve the desired accuracy. Block method are efficient when compare with the non-block method as the total steps taken can be reduced.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH MULTI LANGKAH BAGI MENYELESAIKAN PERSAMAAN PEMBEZAAN LENGAH PERINGKAT TINGGI

Oleh

HOO YANN SEONG

Mei 2016

Pengerusi: Profesor Zanariah binti Abdul Majid, PhD
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Persamaan pembezaan lengah (PPL) memainkan peranan yang penting dalam sistem kajian yang bergantung kepada kedudukan sistem tersebut dalam masa lalu dan sekarang. Penyelesaian analitik PPL sukar dicari. Kaedah berangka menyediakan kaedah alternatif bagi membentuk penyelesaian kepada masalah. Tesis ini menghuraikan proses pembangunan algoritma berangka bagi menyelesaikan PPL peringkat tinggi. Satu titik multi langkah dan blok dua titik multi langkah berdasarkan kaedah-kaedah Adam-Bashforth-Moulton bagi menyelesaikan persamaan pembezaan biasa disesuaikan bagi menyelesaikan PPL peringkat tinggi. Kaedah-kaedah cadangan berdasarkan pendekatan saiz langkah tetap dan berubah. Dua jenis PPL dipertimbangkan iaitu persamaan pembezaan lengah lewat (PPLL) dan persamaan pembezaan lengah neutral (PPLN). Hanya PPL jenis malar dan jenis pantograf akan dipertimbangkan di tesis ini. Sebutan lengah di dalam PPL jenis malar adalah dianggarkan menggunakan interpolasi Hermite. Interpolasi linear dan interpoasi Hermite digunakan bagi menganggarkan sebutan lengah di dalam PPL jenis pantograf. Terbitan bagi sebutan lengah dianggarkan menggunakan formula pembezaan.

Tesis ini membincangkan kestabilan kaedah apabila diaplikasikan ke PPLL dan PPLN. Rantau kestabilan dibentangkan. Beberapa masalah dipertimbangkan bagi tujuan ilutrasi dan penghampiran berangka bagi penyelesaian mereka adalah diperolehi menggunakan bahasa pengaturcaraan C. Perbandingan di antara kaedah-kaedah yang dibangunkan menunjukkan bahawa semua kaedah-kaedah tersebut mencapai tahap kejituan yang dikehendaki. Keputusan berangka bagi kaedah-kaedah dibangunkan adalah dibandingkan dengan kaedah berangka sedia ada. Kaedah-kaedah blok adalah cekap apabila dibandingkan dengan kedah tanpa blok kerana jumlah bilangan langkah yang diambil dapat dikurangkan.

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I certify that a Thesis Examination Committee has met on 25 May 2016 to conduct the final examination of Hoo Yann Seong on her thesis entitled "Multistep Block Methods for Solving Higher Order Delay Differential Equations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

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Declaration by graduate student

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LIST OF ABBREVIATIONS

1PMo1CS	1-point multistep method with constant step size for solving first order DDEs
1PMo1VS	1-point multistep method with variable step size for solving first order DDEs
2PMo1CS	2-point multistep block method with constant step size for solving first order DDEs
2PMo1VS	2-point multistep method with variable step size for solving first order DDEs
BVPs	boundary value problems
D1PMo2CS	Direct 1-point multistep method with constant step size for solving second order DDEs
D1PMo2VS	Direct 1-point multistep method with variable step size for solving second order DDEs
D1PMo3CS	Direct 1-point multistep method with constant step size for solving third order DDEs
D1PMo3VS	Direct 1-point multistep method with variable step size for solving third order DDEs
D1PMO4CS	Direct 1-point multistep method of order four with constant step size for solving second order DDE
D2PMo2CS	Direct 2-point multistep block method with constant step size for solving second order DDEs
D2PMo2VS	Direct 2-point multistep block method with variable step size for solving second order DDEs
D2PMo3CS	Direct 2-point multistep block method with constant step size for solving third order DDEs
D2PMo3VS	Direct 2-point multistep block method with variable step size for solving third order DDEs
DDE	delay differential equation
DDEs	delay differential equations
DDER1PVS	direct 1-point multistep method with variable step size to solve second order RDDE of constant

DDER2PVS	direct 2-point multistep block method with variable step size to solve second order RDDE of constant
DDERPDVS	direct 2-point multistep block method with variable step size is developed to solve RDDEs with pantograph delay
DN2PVS	direct 2-point multistep block method with variable step size to solve NDDE
LMM	linear multistep methods
NDDE	neutral delay differential equation
ODE	ordinary differential equation
ODEs	ordinary differential equations
RDDE	retarded delay differential equation

CHAPTER 1

INTRODUCTION

1.1 Background

Time delay differential equations exist in many physical and engineering systems. Delay differential equations (DDEs) constitute basic mathematical models for real phenomena, for instance in engineering, applied sciences, and economics. DDEs play an important role in the investigated system which depends on the position of the system in the past and current time.(Kuang, 1993).

A *n*-th order retarded DDE (RDDE) is usually given in the form of

$$y^{(n)}(t) = f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y\big(t - \tau(t,y(t))\big)\right). \quad (1.1)$$

The function $\tau\big(t,y(t)\big)$, the argument $t - \tau(t,y(t))$, a value of the solution delay term $y(t - \tau(t,y(t)))$ and a value of the derivative delay term $y'(t - \tau(t,y(t)))$ are named as a delay, a delay argument, the (solution) delay value and derivative delay value, respectively.

The delay may be a constant value ($\tau \geq 0$), a function of the time ($\tau(t) \geq 0$), or a function of the solution y itself ($\tau(t,y(t)) \geq 0$). Accordingly, Eqn. (1.1) is called a differential equation with constant delay, time-dependent delay, or state-dependent delay, respectively.

When the right-hand side of the problem depends on the delay value and derivative delay value, that is,

$$y^{(n)}(t) = f\left(t, y(t), y'(t), \ldots, y^{(n-1)}(t), y\left(t - \big(\tau(t,y(t)) \geq 0\big)\right), \ldots, y^{(n)}\left(t - \big(\tau(t,y(t)) \geq 0\big)\right)\right), \quad (1.2)$$

we have a neutral DDE (NDDE). The term DDE refers to both a RDDE and a NDDE.

The thesis is divided into two main parts according to the type of delay $\tau\big(t,y(t)\big)$ occurring in Eqn. (1.1) and (1.2). In general, DDE can be classified into two categories, namely those with finite time delay, i.e.

$$\lim_{t\to\infty} \sup \tau(t) < \infty$$

and those with infinite time delay, i.e.

$$\lim_{t\to\infty} \sup \tau(t) = \infty.$$

Let's compare their typical representatives, which are the equations

$$y'(t) = ay(t) + by(t - \tau), t > 0, \tag{1.3}$$

and

$$y'(t) = ay(t) + by(qt), t > 0, \tag{1.4}$$

where $a, b, \tau > 0$ and $q \in (0,1)$ are real numbers. Clearly, Eqn. (1.3) has a finite delay and Eqn. (1.4) belongs to the class of equations with the infinite delay. The category of Eqn. (1.3) is also known as the differential equation of constant delay while the category of Eqn. (1.4) is known as the differential equation of pantograph type.

The solutions of DDEs can be obtained by using analytical, semi-analytical, numerical methods or hybrid between semi-analytical and numerical methods. Sometimes, the exact solutions are difficult to obtain through analytical methods even if the exact solutions exist. Semi-analytical methods approximate the solutions in a series form. In addition, the transformation formulas used in the calculations are complicated. Numerical methods find approximations to the solutions of the problem using estimation and calculation.

Numerical methods for solving DDEs are adapted from numerical methods for ordinary differential equations (ODEs). Two most popular methods are linear multistep and Runge-Kutta type of method.

1.2 Problem Statement

Ordinary and partial differential equations have played an important role in the development of mathematical modelling. However, the use of past states is able to approximate a true situation and a more realistic model. The theory and numerical analysis for such a system have not been developed much compared to ordinary and partial differential equations.

The current research is mainly focusing on solving the first order of DDEs. DMRODE (Neves, 1975), DKLAG6 (Corwin et al., 1997), dde23 (Shampine & Thompson, 2000) and DDVERK90 (Zivaripiran, 2005) are examples of DDE solvers which were designed to solve the system of first order DDEs. In order to solve higher order DDEs using these existing solvers, the higher order DDEs are transformed into a system of first order DDEs. This indirectly creates a not so user-friendly situation, besides this system of transformation is going to burden up the calculations. In some cases, the systems of transformation become an implicit system of first order DDEs where the existing solvers are unable to solve (Shampine & Thompson, 2000).

The analysis of the numerical methods is important to ensure the methods are suitable to solve the differential equations. There are some research mainly studied the stability of the different types of DDEs. (Drazkova, 2014; Hrabalová, 2013; Li, 1988). Only a few of them studied the stability properties of the numerical methods for solving different type of DDE. Normally they pursue a purely theoretical analysis. (Jánský & Kundrat, 2011; Huang, 2007; Xu, 2006)

1.3 Objective of the Thesis

The main objective of the thesis is to use 1-point multistep and 2-point multistep block method to solve higher order DDEs with constant delay and DDEs of pantograph type directly. The study is also numerically investigating the shapes of the stability regions of the 1-point multistep and 2-point multistep block method for solving different types of DDEs.

The objectives can be accomplished by

(i) extending the order of the 1-point multistep and 2-point multistep block method with constant step size that have been derived by Abdullah (2014) to the order of six and seven.
(ii) developing new algorithms for the 1-point multistep and 2-point multistep block method for solving higher order DDE with constant delay using constant and variable step size directly.
(iii) developing new algorithms for the 1-point multistep and 2-point multistep block method for solving higher order DDE of pantograph delay using constant and variable step size directly.
(iv) analysing the shape of the stability region of RDDE and NDDE in the 1-point multistep and 2-point multistep block method.

1.4 Scope and Limitation of the Study

The scopes of the work presented in this thesis were:

(i) To develop new algorithms based on the 1-point multistep and 2-point multistep block method with constant step size and variable step size approaches to solve higher order DDEs.
(ii) Second and third order DDEs with single constant delay, and first to third order DDEs of single pantograph delay are considered in this thesis.
(iii) The stability regions for first order RDDEs with pantograph delay and second order RDDEs and NDDEs with constant delay are analyse.

The following limitations were imposed on the work in this thesis:

(i) The representative set of test problems available is limited to a single DDE equation. So in this thesis, only single DDE equation is considered.
(ii) Standard codes for treating DDEs efficiently over a wide range of tolerances still lack. The comparison of the available experimental codes for DDEs remains to be done. Therefore, in this thesis, the comparison is only conducted between proposed methods and the Matlab solvers.

1.5 Outline of the Thesis

Brief descriptions of every chapter in this thesis are presented below.

The general idea of the whole thesis is provided in Chapter 1. The concept of DDEs is also introduced.

The introduction of DDEs and the discussion on numerical difficulties for solving DDEs are presented in Chapter 2. It also consists a review of the research on DDEs.

In Chapter 3, a brief description of the derivations of the 1-point multistep methods for solving third order DDEs is restated. Algorithms have been developed to solve the second and third order RDDE of constant type by using the adaptation of the 1-point multistep methods. The stability regions of the methods for solving the second order RDDE of constant delay are presented. Numerical results are presented and analysed.

In Chapter 4, the second and third order RDDE with constant delay are solve by using the adaptation of the 2-point multistep block method. A brief description of the derivations of the 2-point multistep block methods for solving third order DDEs is restated as well. Algorithms for the implementation of the adaptation methods are developed to solve this particular type of DDE. The numerical results by using the 2-point multistep block method and numerical comparisons with existing method are discussed. The stability regions of the methods are presented for solving second order RDDE of constant delay.

DDEs with pantograph delay are a special type of DDE. In Chapter 5, the algorithms are developed to solve the first and higher order RDDE of pantograph type by using the adaptation of the 1-point multistep and 2-point multistep block method. The stability region of first order RDDE of pantograph type is discussed. The tested problems of RDDE with pantograph delay are solved by using the adaptation of the 1-point multistep and 2-point multistep block method. Numerical comparisons with the existing method are presented.

The first and higher order NDDE with constant and pantograph delay are solved in Chapter 6 by using the adaptation of the 1-point multistep and 2-point multistep block method with constant and variable step size approach. The stability regions for the second order NDDE with constant delay are presented.

In Chapter 7, a summary of this thesis is presented and future investigations are discussed.

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