UNIVERSITI PUTRA MALAYSIA

PRICING CURRENCY OPTIONS BY GENERALIZATIONS OF THE MIXED FRACTIONAL BROWNIAN MOTION

FOAD SHOKROLLAHI

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PRICING CURRENCY OPTIONS BY GENERALIZATIONS OF THE MIXED FRACTIONAL BROWNIAN MOTION

By

FOAD SHOKROLLAHI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

March 2016

DEDICATIONS

My Sweetheart, Arezoo

My Father and My Mother

My Brothers and My Sisters

My Father and My Mother-in-law

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

PRICING CURRENCY OPTIONS BY GENERALIZATIONS OF THE MIXED FRACTIONAL BROWNIAN MOTION

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March 2016

Chairman: Prof Adem Kılıçman, PhD
Faculty: Science

Option pricing is an active area in financial industry. The value of option pricing is usually obtained by means of a mathematical option pricing model. Since fractional Brownian motion and mixed fractional Brownian motion processes have some important features in order to get typical tail behavior from financial markets, such as: self-similarity and long-range dependence, they can play a significant role in pricing European option and European currency options. In this thesis, some extensions of the mixed fractional Brownian motion model are proposed to wider classes of pricing options systems.

In Chapter 3, a new framework for pricing the European currency option is developed in the case where the spot exchange rate follows a mixed fractional Brownian motion with jumps. An analytic formula for pricing European foreign currency options is proposed using the equivalent martingale measure. For the purpose of understanding the pricing model, some properties of this pricing model are discussed in Chapter 3 as well. Furthermore, the actuarial approach to pricing currency options which transform option pricing into a problem of equivalent of fair insurance premium is introduced.

In addition, in Chapter 4, the problem of discrete time option pricing by the mixed fractional Brownian model with transaction costs using a mean self-financing delta hedging argument is considered in a discrete time setting. A European call currency option pricing formula is then obtained. In particular, the minimal pricing of an option under transaction costs is obtained, which shows that time step δt and Hurst exponent H play an important role in option pricing with transaction costs.

Finally, Chapter 5 considers the problem of discrete time option pricing by a mixed

fractional subdiffusive Black-Scholes model. Under the assumption that the price of the underlying stock follows a time-changed mixed fractional Brownian motion, a pricing formula for the European call option and European call currency option is derived in a discrete time setting with transaction costs.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

HOPSYEN MATA WANG HARGA OLEH GENERALISASI GERAKAN CAMPURAN PECAHAN BROWNIAN

Oleh

FOAD SHOKROLLAHI

Mac 2016

Pengerusi: Prof Adem Kılıçman, PhD
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Di dalam industry kewangan, opsyen harga adalah satu ciri aktif. Nilainya diperolehi dengan model harga opsyen matematik. Pergerakan pecahan Brownian dan campuran pecahan Brownian mempunyai beberapa ciri penting di dalam mendapatkan tingkah laku ekor daripada pasaran kewangan seperti persamaan pergantungan diri dan pergantungan jarak jauh. Di dalam tesis ini, kami mencadangkan beberapa sambungan model pergerakan Brownian pecahan bercampur kepada kelas-kelas yang lebih luas daripada sistem opsyen harga.

Di dalam Bab 3, rangka kerja baru bagi opsyen harga untuk matawang Eropah telah dimajukan di mana kadar pertukaran ini mengikut pergerakan pecahan campuran Brownian dengan lompatan. Ukuran Martingale boleh digunakan untuk analisis formula opsyen harga matawang asing Eropah. Untuk tujuan memahami model penentuan harga, ciri-ciri sesetengah model penentuan harga dibincangkan di bahagian akhir bab ini. Selain itu, pendekatan aktuari pilihan harga matawang menjadi masalah setaraf dengan kewajaran premium insurans.

Di samping itu, di dalam Bab 4 terdapat masalah harga opsyen masa diskret oleh campuran pecahan Brownian yang bercampur-campur dengan kos urusniaga. Dengan min delta diri pembiayaan lindung nilai hujah dikira dalam suasana masa diskret, formula harga opsyen matawang panggilan Eropah diperolehi. Khususnya, minimum harga opsyen di bawah kos transaksi diperolehi, yang menunjukkan bahawa masa langkah t dan exponen Hurst H memainkan peranan penting dalam opsyen harga dengan kos urusniaga.

Akhirnya di dalam Bab 5, kami mempertimbangkan masalah opsyen harga masa diskret oleh campuran sub dengan pecahan model Black-Scholes. Di bawah andaian

bahawa harga saham pendasar mengikut masa berubah bercampur gerakan pecahan Brownian, kami memperolehi formula penetapan opsyen harga panggilan Eropah dan matawang panggilan Eropah di dalam suasana masa diskret dengan kos urusniaga.

ACKNOWLEDGEMENTS

In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohamed His servant and messenger.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

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R	real value
R_+	positive real value
Z	integer value
N	natural value
T	time maturity
K	strike price
S_T, S_t	spot price
r	interest rate
r_d	domestic interest rate
r_f	foreign interest rate
B_d	domestic bond
B_f	foreign bond
D	a dividend payoff of the life
σ	volatility
SS	self similarity
$\mathscr{F}$	σ-algebra
$\mathscr{H}$	a Hilbert space
$\mathscr{F}_t, \mathscr{M}_t$	a filtration
P	a probability measure
Ω	a class of all possible outcomes of an experiment
$(\Omega, \mathscr{F}, P)$	a probability space
X_t	a stochastic process
$E[X]$	the mean of random variable X
$\Phi(.), N(.)$	a cumulative standard normal distribution
$(x)^+$	$\max(x, 0)$
$var(.,.)$	variance function
$cov(.,.)$	covariance function
$s \wedge t$	$\min(s, t)$
$x \doteq y, x \triangleq y$	x and y with the same law
$L^P(\Omega)$	the space of random variable X with $E(\|X\|^P) < \infty$
$E(.\|.)$	conditional expectation
$\langle .,. \rangle$	the inner product
$C_P^\infty(R^n)$	a set of all function whose all its partial derivatives of any order have polynomial growth
$\triangle$	delta
Γ	gamma
Θ	theta
υ	vega
ρ	rho
SDE	stochastic differential equation
PDE	partial differential equation
RAD	risk adjust joint

BS	Black-Scholes
GK	Garman and Kohlhagen
BM	Brownian motion
GBM	geometric Brownian motion
$B(.)$	a Brownian motion
FBM	fractional Brownian motion
FBS	fractional Black-Scholes
$B^H(.)$	a fractional Brownian motion
$MFBM$	mixed fractional Brownian motion
$M^H(.)$	a mixed fractional Brownian motion
$JFBM$	jump fractional Brownian motion
$JMFBM$	jump mixed fractional Brownian motion
$FFPE$	fractional Fokker-Planck Equations
CMB	China Merchants Bank
QCE	quasi conditional expectation

CHAPTER 1

INTRODUCTION

1.1 Options

Financial markets throughout the modern world trade in derivative products such as futures and options. A financial derivative product is so named because its value is derived from the price of some underlying asset: a foreign currency, a stock, or a stock index, for example. Options, in particular, are contracts to buy or sell a number of the underlying asset, or combinations of assets, and techniques for determining a fair price for these contracts are central to this thesis. There are mainly four kinds of options, including American option, European option, Asian option, and Barrier option, in current financial markets. In this thesis, we only focus on pricing a European option and European currency option.

There are many reasons why investors may prefer options to stocks or to other underlying securities. Options may provide a pattern of returns that could not be obtained with common stocks, and using special knowledge, a portfolio with higher expected return than other portfolios with the same degree of risk can be obtained. Option markets may provide a way of hedging against unanticipated changes in stock volatility. Imagine the following scenario.You buy a large amount of shares in a stock. If the volatility of the stock price unexpectedly increases, you might have to sell some of your shares to reduce the risk of your investment. Therefore, you might lose some of your potential profit. Alternatively, you might buy insurance on volatility changes using options.

Options can be used as risk management tools in international portfolios of foreign assets or currencies. Investors use options as speculative devices by using their own volatility expectation if it is different from the volatility implied by the market prices of options. In certain circumstances, options can be used as a hedge against uncertainties.

International financial markets are characterized by a flexible exchange system, capital mobility and the integration of many economic systems. As a result, exchange rates fluctuate drastically. These fluctuations have a strong impact on international financial transactions, all cash inflows or outflows to or from a foreign country are subjected to this transactional exposure. In response to foreign exchange risk, transnational investors have developed several hedging techniques, some of which use derivative instruments such as futures, forwards and currency options contracts.

A currency options refers an agreement that gives right to the holder in order to buy or sell a determined amount of foreign currency at a constant exercise price on option exercise. Currency options can be used to hedge against contingencies and transactions that are not certain to materialize. Banks can write custom option contracts and then use exchange traded contracts to balance their positions. Options can be used for payment of debt denominated in foreign currencies. For the past 15 years, both practitioners and academicians have been concerned with the study of the valuation of these securities.

The fundamental concepts of financial mathematics are presented in this introduction chapter.

Definition 1.1 *A European call (Put) option grants the right to purchase (sell) a stock at a specific time called maturity T for a specific amount K called the exercise price (Clark and Ghosh (2004)).*

The value of a European call option is denoted by $(S_T - K)^+$ where $(x)^+ = \max(x, 0)$. Similarly, the value of a European put option is $(K - S_T)^+$. This amount is called the option payoff.

Definition 1.2 *The payoff of European call and put option are denoted by $(S_T - K)^+$ and $(K - S_T)^+$, respectively (Clark and Ghosh (2004)).*

Definition 1.3 *Stock price is the payoff for a European call which is expressed in terms of the stock price at maturity and the strike price and is given by $(S_T - K)^+$; likewise, the payoff for a European put is given by $(K - S_T)^+$. The stock price denoted by S_0 (Clark and Ghosh (2004); Hull (2006)).*

Definition 1.4 *The strike (exercise) price is the price at which a derivative can be exercised, and refers to the price of the derivatives underlying asset. The strike price will be denoted by K (Clark and Ghosh (2004); Hull (2006)).*

Definition 1.5 *Expiration date (maturity time) is date on which the option can be exercised or date on which the option ceases to exist or give the holder any rights. This will be denoted by T (Clark and Ghosh (2004); Hull (2006)).*

Definition 1.6 *Volatility is a measure of uncertainty in stock price movements. A large volatility implies the potential for wide variation in the stock price. The volatility will be denoted by σ (Clark and Ghosh (2004); Hull (2006)).*

Definition 1.7 *A risk free interest rate is the rate of return on an asset that possess no risk is called risk free interest rate and denoted by r (Clark and Ghosh (2004); Hull (2006)).*

Definition 1.8 *A dividend payout during the life of an option will have the affect of decreasing the value of a call and increasing the value of a put, since the stock price typically falls by the amount of the dividend when it is paid. This will be denoted by D (Clark and Ghosh (2004); Hull (2006)).*

Definition 1.9 *In the money is an option with positive intrinsic value. A call option when the asset price is above the strike, a put option when the asset price is below the strike (Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).*

Definition 1.10 *Out of the money is an option with no intrinsic value, only time value. A call option when the asset price is below the strike, a put option when the asset price is above the strike (Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).*

Definition 1.11 *At the money is a situation where the spot price and strike price are equal (Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).*

Definition 1.12 *A stochastic process X is a collection of random variables*

$$(X_t, t \in T) = (X_t(w), t \in T, w \in \Omega), \tag{1.1}$$

define on some space Ω. For a fixed outcome $w \in \Omega$, it is function of time:

$$X_t = X_t(w), \quad t \in T. \tag{1.2}$$

This function is called a realization, a trajectories or a sample path of the process X (Hull (2006); Pliska (1997)).

Definition 1.13 *A filtration is a family $\mathscr{M} = \{\mathscr{M}_t\}_{t \geq 0}$ of σ-algebras $\mathscr{M}_t \subset \mathscr{F}$ such that*

$$0 \leq s < t \Rightarrow \mathscr{M}_s \subset \mathscr{M}_t, \tag{1.3}$$

($\mathscr{M}_t$ is increasing) (Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).

Definition 1.14 *Let $\{\mathscr{N}_t\}_{t \geq 0}$ be an increasing family of σ-algebras of subsets of Ω. A process $g(t,w) : [0,\infty) \times \Omega \to \mathbf{R}^n$ is called $\mathscr{N}_t$-adapted if for each $t \geq 0$ the function*

$$w \to g(t,w), \tag{1.4}$$

is $\mathscr{N}_t$- measurable (Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).

Definition 1.15 *An n-dimensional stochastic process $\{X_t\}_{t \geq 0}$ on $(\Omega, \mathscr{F}, P)$ is called a martingale with respect to filtration $\{\mathscr{M}_t\}_{t \geq 0}$ (and with respect to P) if*

(i) X_t is $\mathscr{M}_t$-measurable for all t,

(ii) $E[|X_t|] < \infty$ *for all t, and*

(iii) $E[M_s|\mathscr{M}_t] = M_t$ *for all* $s \geq t$ *(Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).*

Definition 1.16 *An* $\mathscr{N}_t$ *- adapted stochastic process* $Z(t) \in \mathbf{R}^n$ *is called a local martingale with respect to the given filtration* $\mathscr{N}_t$ *if there exists an increasing sequence of* $\mathscr{N}_t$ *- stopping times* τ_k *such that*

$$\tau_k \to \infty \quad k \to \infty, \tag{1.5}$$

and

$$Z(t \wedge \tau_k), \tag{1.6}$$

is an $\mathscr{N}_t$*- martingale for all k, where* $t \wedge \tau_k = \min(t, \tau_k)$ *(Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).*

Definition 1.17 *A process* $\{X_t, \mathscr{F}_t, t \geq 0\}$ *is called semi-martingale, if it admits the representation*

$$X_t = X_0 + M_t + A_t, \tag{1.7}$$

where M_t *is an* $\mathscr{F}_t$*- local martingale with* $M_0 = 0$*,* A_t *is a process of locally bounded variation,* X_0 *is* $\mathscr{F}_0$*-measurable (Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997)).*

Definition 1.18 *Holder continuous(Kolb and Overdahl (1997); Musiela and Rutkowski (2006); Pliska (1997))*

A function $f : [0,1) \to R$ *is said to be locally* α*-Holder continuous at* $x \geq 0$*, if there exists* $\varepsilon > 0$ *and* $c > 0$ *such that*

$$|f(x) - f(y)| \leq c|x-y|^{\alpha}, \quad \textit{for all} \quad y \geq 0 \quad \textit{with} \quad |y-x| < \varepsilon. \tag{1.8}$$

1.2 Trading strategy and arbitrage

Let $(\Omega, \mathscr{F}, P)$ denote a probability space (Mikosch (1998)). Let us consider a financial market consisting of n assets with prices $S_1(t), ..., S_n(t)$, which under probability measure P are governed by the following stochastic differential equations:

$$dS_i = \mu_i(t)dt + \sigma_i(t)dB_i(t), \quad i = 1, 2, ..., n, \tag{1.9}$$

where $B_i(t)$ for $i = 1, 2, ..., n$ is a BM.

Next, we denote an n-dimensional stochastic process $\theta(t) = (\delta_1(t), ..., \delta_n(t))$ as a trading strategy, where $\delta_i(t)$ denotes the holding in asset i at time t. The value $V(\delta, t)$ at time t of a trading strategy δ is given by

$$V(\delta,t) = \sum_{i=1}^{n} \delta_i(t) S_i(t). \tag{1.10}$$

Definition 1.19 *A self-financing trading strategy is a strategy δ with the property:*

$$V(\delta,t) = V(\delta,0) + \sum_{i=1}^{n} \int_0^t \delta_i(t) S_i(t), \quad t \in [0,T]. \tag{1.11}$$

Hence, a self-financing trading strategy is a trading strategy that requires nor generates funds between time 0 and time T. In other words, any profit/loss is generated by buying or selling one of the assets S_i.

Definition 1.20 *An arbitrage opportunity is a self-financing trading strategy δ with*

(i) $V(\delta,0) \geq 0$ *almost surely,*

(ii) $E[V(\delta,0)] \geq 0$.

In words, arbitrage is a situation where it is possible to make a profit without the possibility of incurring a loss.

The most important derivative is the European call option.

Definition 1.21 *A derivative security with pay-off $H(T)$ at time T is said to be attainable if there is a self-financing strategy δ such that $V(\delta,T) = H(T)$.*

Definition 1.22 *An economy is called complete if all the derivative securities are attainable.*

Definition 1.23 *An asset is called a numeraire if it has strictly positive prices for all $t \in [0,T]$.*

We can use numeraire to denominate all prices in an economy.

Now, consider a numeraire $N(t)$ and a probability measure P_N that is associated with $N(t)$.

Definition 1.24 *The measure P_N is called equivalent martingale measure if*

(i) P_N is equivalent to P,

(ii) For any self-financing portfolio $V(\delta,t), V(\delta,t)/N(t)$ is a martingale under P_N,

$$E^{P_N}\Big[\frac{V(\delta,t)}{N(t)}|\mathscr{F}_s\Big] = \frac{V(\delta,s)}{N(t)}, \quad s \leq t. \tag{1.12}$$

Definition 1.25 *Portfolio is a grouping of financial assets such as stocks, bonds and cash equivalents, as well as their mutual, exchange-traded and closed-fund counterparts. Portfolios are held directly by investors and/or managed by financial professionals.*

1.3 Brownian motion

Definition 1.26 *Let $X = (X_t, t \in T)$ be a stochastic process and $T \subset \mathbf{R}$ be an interval (Karatzas and Shreve (2012); Hida (1980)). X said to have stationary increments if the random variables $X_t - X_s$ and $X_{t+h} - X_{s+h}$ have the same distribution for all $t, s \in T$ and h with $t+h, s+h \in T$.*

X said to have independent increments if for every choice of $t_i \in T$ with $t_1 < \ldots < t_n$ and $n \geq 1$,

$$X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}} \tag{1.13}$$

are independent random variables.

Remark 1.1 *Let $(X_t)_{t\in R_+}$ and $(Y_t)_{t\in R_+}$ be two processes defined on the same probability space $(\Omega, \mathscr{F}, P)$. The notation $\{X_t\} \triangleq \{Y_t\}$ will mean that $(X_t)_{t\in R_+}$ and $(Y_t)_{t\in R_+}$ have the same law.*

Definition 1.27 *Naively, self-similarity is a typical property of fractals. A self-similar object is exactly or approximately similar to a part of itself, i.e., the whole has the same shape as one or more of the parts. Many objects in the real world are statistically self-similar, such as Sierpinski triangle and fern, see Figures (1.1) and (1.2). A real-valued stochastic process $X(t), t \in \mathbf{R}$ is self-similar with index $H > 0$ or H-self similarity, if, for any $a > 0$,*

$$X(at) \triangleq a^H X(t). \tag{1.14}$$

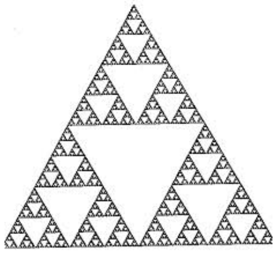

Figure 1.1: Sierpinski triangle

Figure 1.2: Fern

Definition 1.28 *A Gaussian process is a real valued stochastic process* $(X_t)_{t\in T}$, *if the random variables* $X_{t_1}, X_{t_2}, ..., X_{t_n}$ *are jointly normal for any* $t_1, t_2, ..., t_n$ *in* T.

Remark 1.2 *A Gaussian process* $(X_t)_{t\in T}$ *is called centered if* $E[X_t] = 0$ *for every* $t \in T$.

Definition 1.29 *Brownian Motion*

BM is a process $(B(t))_{t\geq 0}$ *with the following properties (Karatzas and Shreve (2012); Hida (1980)):*

(1) $B(0) = 0$,

(2) $B(t)$ *has independent increments:* $0 \leq t_1 < t_2 < ... < t_n$ *then* $(B(t_n) - B(t_{n-1}), ..., (B(t_2) - B(t_1))$ *are independent,*

(3) $B(t) - B(s) \sim N(0, t-s)$ *for* $s < t$.

Definition 1.30 *Long/Short-Range Dependence*

Let $(X_t)_{t\in T}$ *be a centered Gaussian process and let the auto-covariance between the n-th increment and the first increment of the process X be denoted as* $\gamma_n = E[(X_1 - X_0), (X_{n+1} - X_n)], n \geq 1$. *Then,*

(1) If $\gamma_n > 0$ *for all* $n \geq 1$*,the process has positively correlated increments. If* $\sum_{n=1}^{\infty} |\gamma_n| = \infty$ *we say that X has long-range dependence.*

(2) If $\gamma_n < 0$ *for all* $n \geq 1$*,the process has negatively correlated increments. If* $\sum_{n=1}^{\infty} |\gamma_n| = c < \infty, c \neq \infty$ *we say that X has short-range dependence.*

(3) If $\gamma_n = 0$ *for all* $n \geq 1$, $\sum_{n=1}^{\infty} |\gamma_n| = 0$ *we say that X is independent process.*

Definition 1.31 *Markov Process*

The process $(X_t)_{t \in T}$ *is a Markov process if*

$$E(f(X_t)|\mathscr{F}_s) = E[f(X_t)|X_s], \quad \forall t > s, \quad t, s \in T, \tag{1.15}$$

where $T \subseteq \mathbf{R}$, $\mathscr{F}_t = \sigma[X_{t,s \in T}|s \leq t]$, *and* f *is a bounded Borel function (Øksendal (2003); Seydel (2012)).*

Definition 1.32 *Levy process*

Levy process $(X_t)_{t>0}$ *is a process with the following properties*

(1) Independent increments,

(2) Stationary increments, and

(3) Continuous paths: That is $\lim_{h \to 0} P(|X_{t+h} - X_t| \geq \varepsilon) = 0$ *for any* $\varepsilon > 0$ *(Gylfadottir (2010)).*

Definition 1.33 *Poisson process* $(X_t, t \geq 0)$ *is a process which satisfies following conditions (Øksendal (2003); Seydel (2012)):*

(1) $X_0 = 0$,

(2) $X_t - X_s$ *are integer valued for* $0 \leq s < t < \infty$ *and*

$$P(X_t - X_s = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)} \quad \textit{for} \quad k = 0, 1, 2, \ldots \tag{1.16}$$

(3) The increment $X_{t_2} - X_{t_1}$ *and* $X_{t_4} - X_{t_3}$ *are independent for every* $0 \leq t_1 < t_2 < t_3 < t_4$.

1.4 Fractional Brownian motion

FBM has recently become a hot choice for modeling in mathematical finance and other sciences. On purely empirical data, some believe that FBM is an ideal candidate since it is a long-term dependent and self-similar process. Even with its popularity, our understanding of the properties and behaviour of FBM is limited.

Kolmogorov (Kolmogorov (1941)) was the first to introduce the Gaussian process which is now known as FBM in the theory of probability. This class of processes was studied by Kolmogorov in detail and it played an essential role in the series of problem s of the statistical theory of turbulence. Yaglom (Yaglom (1955)) discussed the spectral density and correlation function of FBM. A quadratic variation formula for FBM follows from a general result of Baxter (Baxter (1956)). Gladyshev (Gladyshev (1961)) extended Baxter's result and provided a theoretical result to determine the value of the Hurst effect denoted by H. However, most of the encomium to FBM has been given to Mandelbrot and Van Ness (Mandelbrot and Van Ness (1968)) who used FBM to model natural phenomena such as the speculative market fluctuations.

Definition 1.34 *FBM is a centered Gaussian process $(B^H(t))_{t\in\mathbf{R}}$ where $H \in (0,1)$ with the following properties (Hu and Øksendal (2003); Rodón (2006); Biagini et al. (2008)):*

(1) $B^H(0) = 0$,

(2) $B^H(t) - B^H(s)$ *is distributed as* $N(0, |t-s|^{2H})$,

(3) $t \to B^H(t)$ *is continuous.*

Figure (1.3) shows the sample path of the FBM for different parameter.

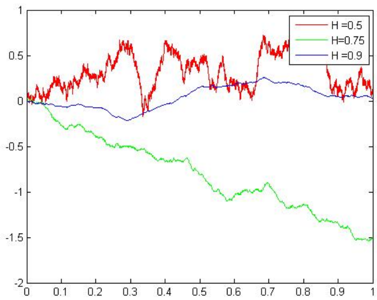

Figure 1.3: FBM with different Hurst parameter H

Corollary 1.1 *Since* $E[B^H(0)] = 0$ *and* $E[B^H(t)]^2 = t^{2H}$.

Then

$$\begin{aligned} E[B^H(t)B^H(s)] &= \frac{E[B^H(1)]^2}{2}\{E[B^H(t)]^2 + E[B^H(s)]^2 - E[B^H(|t-s|)]^2\} \\ &= \frac{E[B^H(1)]^2}{2}\{t^{2H} + s^{2H} - |t-s|^{2H}\}. \end{aligned} \tag{1.17}$$

Remark 1.3 *Throughout this thesis without loss of generality we assume a standard FBM, that is* $E[B^H(1)] = 1$.

Corollary 1.2 *FBM is a Gaussian H-self-similar process, that is*

$$E\left[B^H(t)\left(B^H(t)-B^H(s)\right)\right]=0, \quad \forall t>s. \tag{1.18}$$

FBM with $H=\frac{1}{2}$ satisfies the definition of BM (Definition 1.29). Additionally, when $H=1$ the process is degenerate since,

$$E[B^H(t)-tB^H(1)]=t^2-2t^2+t^2=0 \Rightarrow B^H(t)=^d tB^H(1). \tag{1.19}$$

Corollary 1.3 *FBM has stationary increments. Since FBM is a centered Gaussian processes, $\forall t>s,\tau>0$ we only need to consider the covariance function to prove the stationarity of increments,*

$$\begin{aligned} E\left[B^H_{t+\tau}B^H_{\tau}\right] &= E\left[B^H_{t+\tau}B^H_{s+\tau}\right]-E\left[B^H_{t+\tau}B^H_{\tau}\right]-E\left[B^H_{\tau}B^H_{s+\tau}\right]+E\left[B^H_{\tau}\right]^2 \\ &= \frac{1}{2}\{t^{2H}+s^{2H}-2(t-s)^{2H}\} \\ &= E[B^H_t B^H_s]. \end{aligned} \tag{1.20}$$

This proves that $(B^H_{t+\tau}-B^H_{\tau}, t\in T)\triangleq(B^H_t, t\in T)$ (Embrechts and Maejima (2002)).

Remark 1.4 *Using stationarity it can be shown that the auto-covariance function for FBM is given by*

$$\gamma_n=\frac{1}{2}\left[(n+1)^{2H}-2n^{2H}+(n-1)^{2H}\right], \tag{1.21}$$

therefore

$$\gamma_n \approx H(2H-1)n^{2H-2}, \quad as \quad n\to\infty, H\neq\frac{1}{2}. \tag{1.22}$$

Notice that when

(1) $H=\frac{1}{2},\gamma_n=0,\forall n$ therefore FBM has independent increments.

(2) $H>\frac{1}{2},\gamma_n>0$ and $\gamma_n\approx H(2H-1)n^{2H-2}$, as $n\to\infty$ therefore the increments of the FBM process are positively correlated and by p-series $\sum_{n=1}^{\infty}|\gamma_n|=\infty$, therefore has long-range dependence.

(3) $H<\frac{1}{2},\gamma_n<0$ and $\gamma_n\approx H(1-2H)n^{2H-2}$, as $n\to\infty$ therefore the increments of the FBM process are negatively correlated and by p-series $\sum_{n=1}^{\infty}|\gamma_n|=c<\infty$, therefore has short-range dependence.

Remark 1.5 *For $\frac{1}{2} < H < 1$, H measures the intensity of long-range dependence. The closer H is to 1 the stronger long-memory the process exhibits.*

1.5 Girsanov's Theorem

Assume we have the probability space $(\Omega, \mathscr{F}, P)$. Then a change of measure from P to Q means we have probability space $(\Omega, \mathscr{F}, Q)$.

Definition 1.35 *Two measures P and Q are equivalent if*

$$P(A) > 0 \Rightarrow Q(A) > 0, \quad \forall A \subset \Omega, \tag{1.23}$$

and

$$P(A) = 0 \Leftrightarrow Q(A) = 0, \quad \forall A \subset \Omega. \tag{1.24}$$

The Radon-Nikodym derivative can be defined by using two equivalent measures as follows:

$$M(t) = \frac{dQ}{dP}(t), \tag{1.25}$$

which enables us to change a measure to another. It follows that for any random variable X

$$E^P[XM] = \int_\Omega X(w)M(t,w)dP(w) = \int_\Omega X(w)dQ(w) = E^Q[X]. \tag{1.26}$$

This interchangeability of the expected values under two different measures confirms the important role of a Radon-Nikodym derivative as intermediate link between two measures (Tong (2012)).

To change the measures for stochastic processes we can use the Girsanov's theorem (Tong (2012)).

Theorem 1.1 *Girsanov's Theorem*

Let $\mathscr{F}_t$ be a a filtration on interval $[0,T]$ where $T < \infty$. Define a random process $M(t)$:

$$M(t) = \exp\Big[-\int_0^t \lambda(u)dB^P(u) - \frac{1}{2}\int_0^t \lambda^2(u)du\Big], \quad t \in [0,T]. \tag{1.27}$$

where $B^P(t)$ is a BM under probability measure P and $\lambda(t)$ is an $\mathscr{F}_t$- measurable *process that satisfies a condition*

$$E\Big\{\exp\Big[\frac{1}{2}\int_0^t \lambda^2(u)du\Big]\Big\} < \infty, \quad t \in [0,T]. \tag{1.28}$$

If we define B^Q by

$$B^Q(t) = B^P(t) + \int_0^t \lambda(u)du, \quad t \in [0,T]. \tag{1.29}$$

To change the measures for multidimensional stochastic process, we require a multidimensional Girsanov's theorem, which is very similar to the one dimensional.

Theorem 1.2 *Multidimensional Girsanov's Theorem*

Let $\mathscr{F}_t$ be a filtration on interval $[0,T]$ where $T < \infty$. Suppose $\Lambda(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t))$ be an n-dimensional process that is $\mathscr{F}_t$-adapted and satisfies a condition

$$E\Big\{\exp\Big[\frac{1}{2}\int_0^t \sum_{i=1}^{n} \lambda_i^2(u)du\Big]\Big\} < \infty, \quad t \in [0,T]. \tag{1.30}$$

We define a random process $M(t)$:

$$M(t) = \exp\Big[\sum_{i=1}^{n}\Big(-\int_0^t \lambda_i(u)dB_i^P(u) - \frac{1}{2}\int_0^t \lambda_i^2(u)du\Big)\Big], \quad t \in [0,T], \tag{1.31}$$

where $B_i^P(t)$ is an n-dimensional BM with respect to the probability measure P for $i = 1,...,n$. If we define B_i^Q by then the following outcomes holds:

(1) $M(t)$ defines a Radon-Nikodym derivative.

$$M(t) = \frac{dQ}{dP}(t). \tag{1.32}$$

(2) B_i^Q is a BM under $\mathscr{F}_t$ under the probability measure Q for $i = 1,...,n$ (Tong (2012)).

1.6 Ito Lemma

Let $X(t)$ be a stochastic process and suppose that there exists a real number $x(0)$ and two adapted processes $\mu(t)$ and $\sigma(t)$ such that the following relation holds for all

$t \geq 0$,

$$X(t) = x(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dB(s). \tag{1.33}$$

We can write the equation as follows

$$dX(t) = \mu(t)dt + \sigma(t)dB(t), \tag{1.34}$$

$$X(0) = x(0). \tag{1.35}$$

Then, we can say $X(t)$ satisfies the *SDE* given by (1.34) with the initial condition given by (1.35). Note that the formal notation $dX(t) = \mu(t)dt + \sigma(t)dB(t)$ has no particular meaning. It is simply a shorthand version of the expression (1.34) above.

In option pricing, we often take as given a *SDE* for some basic quantity such as stock price. Many other quantities of interest will be functions of that basic process. To determine the dynamics of these other processes, we shall apply Ito's Lemma, which is basically the chain rule for stochastic processes (Mikosch (1998); Tong (2012); Øksendal (2003); Hirsa and Neftci (2013)).

Theorem 1.3 *Ito's Lemma*

Assume the stochastic process $X(t)$ satisfies in the following equation

$$dX(t) = \mu(t)dt + \sigma(t)dB(t), \tag{1.36}$$

where $\mu(t)$ and $\sigma(t)$ are adapted processes to a filtration $\mathscr{F}_t$. Let $Y(t)$ be a new process defined by $Y(t) = f(X(t),t)$ where $f(x,t)$ is a function twice differentiable in its first argument and once in its second. Then $Y(t)$ satisfies the stochastic differential equation:

$$dY(t) = \Big(\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial X} + \frac{1}{2}\sigma^2(t)\frac{\partial^2 f}{\partial X^2}\Big)dt + \sigma(t)\frac{\partial f}{\partial X}dB(t), \tag{1.37}$$

where $\frac{\partial f}{\partial X} = \frac{\partial f}{\partial x}|\{x = X(t)\}$ *and* $\frac{\partial^2 f}{\partial X^2} = \frac{\partial^2 f}{\partial x^2}|\{x = X(t)\}$.

Now, we present some extended versions of the Ito lemma. Recall that a second order Taylor expansion yields that

$$\begin{aligned} f(t+dt, B_{t+dt}) - f(t, B_t) &= f_1(t,B_t)dt + f_2(t,B_t)dB_t \\ &+ \frac{1}{2}\Big[f_{11}(t,B_t)(dt)^2 + 2f_{12}(t,B_t)dtdB_t \\ &+ f_{22}(t,B_t)(dB_t)^2\Big] \\ &+ \ldots. \end{aligned} \tag{1.38}$$

Here, and what follows, we use the following notations for partial derivative of f.

$$\begin{aligned}
f_i(t,x) &= \frac{\partial}{\partial x_i} f(x_1,x_2)\big|_{x_1=t,x_2=x}, i=1,2 \\
f_{ij}(t,x) &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} f(x_1,x_2)\big|_{x_1=t,x_2=x}, i,j=1,2.
\end{aligned} \tag{1.39}$$

As in classical calculus, higher order terms in equation (1.38) are negligible, and so are the terms with factors $dtdB_t$ and $(dt)^2$. However, since we interpret $(dB_t)^2$ as dt, the term with $(dB_t)^2$ can not be neglected.

Theorem 1.4 *Extension I of Ito Lemma*

Let $f(t,x)$ be a function whose second order partial derivatives are continuous. Then

$$\begin{aligned}
f(t,B_t) &- f(s,B_s) \\
&= \int_s^t \left[f_1(x,B_x) + \frac{1}{2} f_{22}(x,B_x) \right] dx \\
&+ \int_s^t f_2(x,B_x) dB_x, \quad s<t.
\end{aligned} \tag{1.40}$$

An application of the Ito lemma 1.3 yields that the process X satisfies the following SDE

$$X_t - X_0 = c\int_0^t X_s ds + \sigma \int_0^t X_s dB_s. \tag{1.41}$$

For use we will need an even more general version of Ito lemma. We will consider a process of the form $f(t,X_t)$, where X is given by

$$X_t = X_0 + \int_0^t A_s^{(1)} ds + \int_0^t A_s^{(2)} dB_s, \tag{1.42}$$

and both, $A^{(1)}$ and $A^{(2)}$, are adapted to BM. Here it is assumed that the above integrals are well defined in the Riemann and Ito senses, respectively.

A process X, which has representation (1.42), is called an Ito process. One can show that the processes $A^{(1)}$ and $A^{(2)}$ are uniquely determined in the sense that, if X has representation (1.42), where the $A^{(i)}$s are replaced with adapted process $D^{(i)}$, then $A^{(i)}$ and $D^{(i)}$ necessarily coincide.

Now, using a similar argument with a Taylor expansion as above, one can show the following formula.

Theorem 1.5 *Extension II of Ito Lemma*

Let X be an Ito process with representation (1.42) and $f(t,x)$ be a function whose *second order partial derivatives are continuous. Thus*

$$\begin{aligned} f(t,X_t) \quad - \quad & f(s,X_s) \\ = \quad & \int_s^t \left[f_1(y,X_y) + A_y^{(1)} f_2(y,X_y) + \frac{1}{2}[A_y^{(2)}]^2 f_{22}(y,X_y) \right] dy \\ + \quad & \int_s^t A_y^{(2)} f_2(y,X_y) dB_y, \quad s < t. \end{aligned} \tag{1.43}$$

Formula (1.43) is frequently giving in the following form

$$\begin{aligned} f(t,X_t) \quad - \quad & f(s,X_s) \\ = \quad & \int_s^t \left[f_1(y,X_y) + \frac{1}{2}[A_y^{(2)}]^2 f_{22}(y,X_y) \right] dy \\ + \quad & \int_s^t f_2(y,X_y) dX_y, \quad s < t, \end{aligned} \tag{1.44}$$

where

$$dX_y = A_y^{(1)} dy + A_y^{(2)} dB_y. \tag{1.45}$$

Theorem 1.6 *Extension III of Ito Lemma*

Let X^1 and X^2 be two Ito process given by

$$X_t^{(i)} = X_0^{(i)} + \int_0^t A_s^{(1,i)} ds + \int_0^t A_s^{(2,i)} dB_s, \quad i = 1,2, \tag{1.46}$$

and $f(t,x_1,x_2)$ be a function whose second order partial derivatives are continuous. Then for $s < t$,

$$\begin{aligned} f(t,X_t^{(1)},X_t^{(2)}) \quad - \quad & f(s,X_s^{(1)},X_s^{(2)}) \\ = \quad & \int_s^t f(y,X_y^{(1)},X_y^{(2)}) dy + \sum_{i=1}^{2} \int_s^t f_i(y,X_y^{(1)},X_y^{(2)}) dX_y^{(i)} \\ + \quad & \frac{1}{2} \sum_{i=2}^{3} \sum_{j=2}^{3} \int_s^t f_{ij}(y,X_y^{(1)},X_y^{(2)}) A_y^{(2,i)} A_y^{(2,j)} dy. \end{aligned} \tag{1.47}$$

Here $f_i(t,x_1,x_2), f_{ij}(t,x_1,x_2)$ are the partial derivatives of $f(t,x_1,x_2)$ with respect to the ith, the ith and jth variables, respectively.

Theorem 1.7 *(Duncan et al. (2000)) Ito lemma for FBM*

If $f : \mathbf{R} \to \mathbf{R}$ is a twice continuously differentiable function with bounded derivatives *to order two, then*

$$f(B_T^H) - f(B_0^H) = \int_0^T f'(B_s^H) dB_s^H + H \int_0^T s^{2H-1} f''(B_s^H) ds. \qquad (1.48)$$

1.7 Mixed fractional Brownian motion

Let a and b be two real constants such that $(a,b) \neq (0,0)$.

Definition 1.36 *A MFBM with parameters* a, b, *and* H *is a process* $M^H = M_t^H(a,b), t \geq 0 = M_t^H t, t \geq 0$, *defined on the probability space* $(\Omega, \mathscr{F}, P)$ *by*

$$M_t^H = M_t^H(a,b) = aB_t + bB_t^H, \quad \forall t \in R_+ \qquad (1.49)$$

where $(B_t)_{t \in R_+}$ *is a BM and* $(B_t^H)_{t \in \mathbf{R}_+}$ *is an independent FBM with Hurst parameter H (Duncan et al. (2000); Cheridito (2001a); Mishura (2008); Zili (2006); Marinucci and Robinson (1999)).*

Lemma 1.1 *The MFBM has the following properties*

(i) M^H *is a centered Gaussian process,*

(ii) for all $t \in \mathbf{R}_+, E((M_t^H(a,b))^2) = a^2 t + b^2 t^{2H}$,

(iii) one has that

$$\begin{aligned} & Cov\Big(M_t^H(a,b), M_s^H(a,b)\Big) \\ = \; & a^2(t \wedge s) + \frac{1}{2} b^2 \Big[t^{2H} + s^{2H} - |t-s|^{2H}\Big], \forall s,t \in \mathbf{R}_+, \end{aligned} \qquad (1.50)$$

where $\quad t \wedge s = 1/2(t+s+|t-s|)$,

(iv) the increments of the MFBM are stationary (Zili (2006)).

Lemma 1.2 *For any* $h > 0, \{M_{ht}^H(a,b)\} \triangleq \{M_t^H(ah^{\frac{1}{2}}, bh^H)\}$. *This property will be called the mixed-self-similarity (Zili (2006)).*

Theorem 1.8 *For all* $H \in (0,1) - \{\frac{1}{2}\}, a \in \mathbf{R}$ *and* $b \in \mathbf{R} - \{0\}, (M_t^H(a,b))_{t \in \mathbf{R}}$ *is not a Markovian process (Zili (2006)).*

Remark 1.6 *Let X and Y be two random variables defined on the same probability space* $(\Omega, \mathscr{F}, P)$.*We denote the correlation coefficient* $\rho(X,Y)$ *by*

$$\rho(X,Y) = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}}. \tag{1.51}$$

Corollary 1.4 *For all* $a \in \mathbf{R}$ *and* $b \in \mathbf{R} - \{0\}$, *the increments of* $(M_t^H(a,b))_{t \in \mathbf{R}_+}$ *are positively correlated if* $\frac{1}{2} < H < 1$, *uncorrelated if* $H = \frac{1}{2}$, *and negatively correlated if* $0 < H < \frac{1}{2}$ *(Zili (2006)).*

Definition 1.37 *Let* $\{X_t, t \in \mathbf{R}_+\}$ *be a process with stationary trajectories and* $(r(n))_{n \in \mathbf{N}}$ *the sequence defined by*

$$\forall n \in \mathbf{N}, \quad r(n) = E(X_{n+1}X_1). \tag{1.52}$$

We recall that the process X is called long-range dependent if and only if

$$\sum_{n \in \mathbf{N}} r(n) = \infty. \tag{1.53}$$

Remark 1.7 *Since* $\{X_t, t \in \mathbf{R}_+\}$ *is a process with stationary trajectories*

$$\forall s \in \mathbf{R}_+, \forall n \in \mathbf{N}, \quad r(n) = E(X_{n+s}X_s). \tag{1.54}$$

Lemma 1.3 *For all* $a \in \mathbf{R}$ *and* $b \in \mathbf{R} - \{0\}$, *the increments of* $(M_t^H(a,b))_{t \in \mathbf{R}_+}$ *are long-range dependent if and only if* $H > \frac{1}{2}$ *(Zili (2006)).*

We see that $\sum_{n \in \mathbf{N}} r(n) = +\infty$ if and only if $2H - 2 > -1$; that is, if and only if $H > \frac{1}{2}$.

Lemma 1.4 *Holder continuity*

For all $T > 0$ *and* $\gamma < \frac{1}{2} \wedge H$, *the* $MFBM$ *has a modification which sample paths have a Holder-continuity, with order* γ, *on the interval* $[0,T]$ *(Zili (2006)).*

Definition 1.38 *A random function* $g(x)$ *is said to be* $O(f(x))$, *if there exists a fixed* $N > 0$ *such that* $\left|\frac{g(x)}{f(x)}\right| \leq N$ *for enough small x .*

1.8 Greeks

Greeks summarize how option prices change with respect to underlying variables and are critically important in asset pricing and risk management. It can be used to rebalance the portfolio to achieve desired exposure to a certain risk. More importantly, knowing the Greek, a particular exposure can be hedged from adverse changes in the market by using appropriate amount of other related financial instruments. Unlike option prices, which can be observed in the market, Greeks can not be observed and have to be calculated given a model assumption. Typically, the Greeks are computed using a partial differentiation of the price formula (Higham (2004); Cvitanić and Zapatero (2004); Lyuu (2001); Shokrollahi et al. (2015)).

Definition 1.39 *Delta*

Delta (Δ) *of an option defined as*

$$\Delta = \frac{\text{change in option price}}{\text{change in underlying}}. \tag{1.55}$$

The sensitivity of the option to the underlying finance is assessed by Delta.

Definition 1.40 *Gamma*

Gamma (Γ) *calculated the immediate changes of the delta in terms of partial alterations, which occur in the underlying stock price. It is the second derivative of the option value respect to the underlying asset.*

$$\Gamma = \frac{\text{change in delta}}{\text{change in underlying}}. \tag{1.56}$$

Definition 1.41 *Theta*

Theta (Θ) *is defined as*

$$\Theta = -\frac{\text{change in option price}}{\text{change in time to maturity}}. \tag{1.57}$$

Theta measures the sensitivity of the value of the option to the change of time to maturity. If the asset price is constant, consequently the option will change by theta with time.

Definition 1.42 *Vega*

The Vega (υ) *, assesses the sensitivity to volatility, which expresses as the amount of money per stock gain or lose as volatility increases or decreases* 1 *percent. It is the* *derivative of the value of the option in terms of the volatility of the stock price .*

$$\upsilon = \frac{\textit{change in option price}}{\textit{change in volatility}}. \tag{1.58}$$

Definition 1.43 *Rho*

Rho (ρ) *refers to the rate of option alteration with respect to the rate of interest.*

$$\rho = \frac{\textit{change in option price}}{\textit{change in interestrate}}. \tag{1.59}$$

1.9 Objectives of the thesis

The main objectives of this thesis are as follows:

- To propose a satisfactory model for currency options pricing to get discontinuous or jumps in financial markets that play a significant role in stocks markets.
- To achieve pricing currency options into a problem of equivalent of fair insurance premium by combine the $MFBM$ and jump processes.
- To valuate European currency options in discrete time setting case by using delta hedging strategy, and $MFBM$ model.
- To price European option by using the $MFBM$ when the physical time t are replaced by inverse subordinator process in the presence and absence of transaction costs.
- To create a new model for pricing currency options when the underlying asset follows time-changed $MFBM$ model.
- To obtain the Greeks for our proposed models.
- To show the impact of Hurst parameter H, transaction costs, and time-step on proposed pricing formulas.

1.10 Outline of thesis

Black-Scholes (Black and Scholes (1973)) put forward option pricing in 1973, which leads to be studied by different scholars (Dravid et al. (1993); Ho et al. (1995); Toft and Reiner (1997); Kwok and Wong (2000); Duan and Wei (1999)) claim that two issues in stock markets are not able to be presented clearly in this option pricing introduced by *BS* in accordance with *BM*. These concepts refer to asymmetric leptokurtic features and the volatility smile. In view of this, the *BS* model was improved by Garman and Kohlhagen (Garman and Kohlhagen (1983)) in order to assess European currency options by considering two prominent features;

(1) The market volatility estimation of an underlying as obvious as price and time functioning void of referring to the characteristics of a particular investor directly. These characteristics could be functions of utility, measures of risk aversion, or yield expecting.

(2) Strategy of self-replicating or hedging.

However, it is significant to note that the mispriced currency options by the *GK* model were also substantiated in some studies (Cookson (1992)). The most important reason of inappropriateness of this model for stock markets is the fact that the currencies are different from stocks so that the currency behavior is not captured by *GBM* (Ekvall et al. (1997)). To tackle this problem, regarding pricing currency options, various models were recommended by modifying the *GK* model (Rosenberg (1998); Sarwar and Krehbiel (2000); Bollen and Rasiel (2003); Jorion (1988)).

In view of this, the independency of logarithmic returns of the exchange rate was pointed out in all these studies along with the distribution of normal random variables. In addition, the empirical studies reveal that the logarithmic returns disseminations in the asset markets widely manifest excess kurtosis with high possibility of mass around the origin and in the tails, and indicate low possibility in the flanks in comparison with normal distribution of data. It means that financial return series include the properties, which are not normal, independent, linear and are self-similar, with heavy tails. Both autocorrelations and cross-correlations and also volatility clustering are considered as among these properties.

In this regard, two fundamental features are considered in FBM and $MFBM$ namely self-similarity and long-range dependence. Thus, employing these process is more feasible in terms of capturing the behavior from financial asset . Although, FBM is neither a semi-martingale nor a Markov process then, employing the conventional stochastic calculus for analyzing it is impossible. It is fortunate that the Wick product was utilized by Xiao et al. (Xiao et al. (2011)) instead of the path wise product for describing a fractional stochastic integral in which mean is zero. This was considered as an appropriate feature both theoretically and practically.

These motivate us to employ the FBM and the $MFBM$ to achieve the valuation European options and European currency options as stochastic models driven by *BM* and FBM processes.

The rest of this thesis divided in five chapters as follows.

In Chapter 2, some historical development of *BS* model will be pointed out and other models to be extended in this thesis will be described.

In Chapter 3, the combination of $MFBM$ process and the Poisson jump process to capture jumps or discontinuities, fluctuations will be considered and the long memory property will be take into account.

In Chapter 4, in return series the time scaling and long-range correlation which have an influence on currency option pricing with and without transaction costs are indicated. The option pricing problem on the $MFBM$ model with transaction costs is considered and a closed form representation of the currency option pricing formula is given.

In Chapter 5, in order to describe properly financial data exhibiting periods of constant values, the subdiffusive strategy based on the $MFBM$ is considered in order to identify financial data with the periods of the constant prices.

Conclusion and perspectives are finally outlined in Chapter 6.

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