UNIVERSITI PUTRA MALAYSIA

DIFFERENTIAL GAMES PROBLEMS DESCRIBED BY SYSTEM OF INFINITE DIFFERENTIAL EQUATIONS IN HILBERT SPACE

USMAN WAZIRI

FS 2018 35

DIFFERENTIAL GAMES PROBLEMS DESCRIBED BY SYSTEM OF INFINITE DIFFERENTIAL EQUATIONS IN HILBERT SPACE

By

USMAN WAZIRI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

January 2018

DEDICATIONS

To my parents
Alhaji Waziri Aliyu Madara (Head District of Lariski) and Maryam Hassan.
My wife
Fauziyya Hadi Adam.
My daughters
Amatullah Usman Waziri and Amatulrahman Usman Waziri.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

DIFFERENTIAL GAMES PROBLEMS DESCRIBED BY SYSTEM OF INFINITE DIFFERENTIAL EQUATIONS IN HILBERT SPACE

By

USMAN WAZIRI

January 2018

Chair: Associate Professor Ibragimov Gafurjan, PhD
Faculty: Science

This thesis deals with the solutions of differential game problems described by some infinite systems of ordinary differential equations in Hilbert space. The infinite system arises from the solution of some control and differential game of problems described by some partial differential equations. By using decomposition method, some of these problems can be reduced to the one described by some infinite system of ordinary differential equations.

Therefore, this thesis focuses on different types of infinite systems using various approaches in Hilbert space. The first system is an infinite system of first order differential equations, and the second system is an infinite system of 2-systems of first order differential equations. For all the systems, we study the existence and uniqueness, and then we consider control and differential game problems with some forms of constraints on controls of the players.

For the first system, we present solution of optimal pursuit problems with negative coefficients, where the controls of the players are subjected to integral constraints. Pursuer's goal is to force the state of the system toward the origin and the evader tries to avoid this.

Secondly, we extend the first system and introduce another state away from that of the initial state. In this game, pursuer attempts to bring the state of the system toward another the evader's purpose is opposite where we study pursuit game problems with negative coefficients.

Furthermore, the second game is improved with various constraints and the coefficients assumed to be any real numbers, the condition of completion of pursuit with geometric and integral constraints is proposed.

For the second system, we solve pursuit differential game problem of 2-system of first-order that involves a generalization of all considered games with conjugate complex, the case of integral constraints.

The main findings and contributions of this thesis is to study differential game described by infinite system of differential equations. For the first system, we propose an optimal pursuit time. For the second the third cases, we propose a new approach of completion of pursuit and for the second system, a guaranteed pursuit time is also proposed.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

MASALAH PERMAINAN PEMBEZAAN YANG DIHURAIKAN OLEH SISTEM PERSAMAAN PEMBEZAAN YANG TIDAK TERHINGGA DALAM RUANG HILBERT

Oleh

USMAN WAZIRI

Januari 2018

Pengerusi: Profesor Madya Ibragimov Gafurjan, PhD
Fakulti: Sains

Tesis ini mengendalikan penyelesaian kepada masalah permainan pembezaan yang diterangkan oleh beberapa sistem tak terhingga persamaan pembezaan biasa dalam ruang Hilbert. Sistem tak terhingga berkenaan muncul dari penyelesaian bagi beberapa kawalan dan permainan pembezaan bagi masalah yang diterangkan oleh beberapa persamaan pembezaan separa. Dengan menggunakan kaedah pemecahan, beberapa masalah ini boleh dimudahkan kepada masalah yang diterangkan oleh sistem tak terhingga bagi persamaan pembezaan biasa.

Oleh itu, tesis ini memfokuskan kepada jenis sistem tak terhingga yang berbeza dengan menggunakan berbagai pendekatan dalam ruang Hilbert. Sistem yang pertama adalah sistem tak terhingga persamaan pembezaan peringkat pertama, dan sistem kedua adalah sistem tak terhingga bagi sistem-2 bagi persamaan pembezaan peringkat pertama. Untuk semua sistem, kami mengkaji kewujudan dan keunikan, dan kemudian kami pertimbangkan kawalan dan masalah permainan pembezaan dengan beberapa bentuk kekangan terhadap kawalan pemain.

Untuk sistem pertama, kami mempersembahkan selesaian kepada masalah pengejaran optimal dengan pekali negatif, di mana kawalan pemain adalah bergantung kepada kekangan kamiran. Tujuan pengejar adalah untuk memaksa keadaan sistem kearah asalan dan pengelak cuba untuk mengelak dari ini berlaku.

Kedua, kami memperkembangkan sistem pertama dan memperkenalkan keadaan

yang lain, pada kedudukan yang lain dari keadaan permulaan. Dalam permainan ini, pengejar cuba untuk membawa keadaan sistem ke arah yang lain. Tujuan pengelak adalah bertentangan di mana kami mengkaji masalah permainan pengejaran dengan pekali negatif.

Di samping itu, permainan kedua diperbaiki dengan pelbagai kekangan dan pekali adalah dianggap sebagai sebarang nombor nyata, keadaan bagi penyempurnaan pengejaran dengan kekangan geometri dan kamiran dicadangkan.

Untuk sistem kedua, kami menyelesaikan masalah permainan pembezaan pengejaran 2-sistem peringkat pertama yang melibatkan pengitlakan semua masalah yang dipertimbangkan dengan konjugat kompleks, kes kekangan kamiran.

Hasil dan sumbangan utama tesis ini adalah untuk mengkaji permainan pembezaan yang diterangkan oleh sistem tak terhingga persamaan pembezaan. Untuk sistem yang pertama, kami mencadangkan masa pengejaran optimal. Untuk kes kedua dan ketiga , kami cadangkan pendekatan baru bagi penyempurnaan pengejaran dan untuk sistem kedua, dicadangkan juga satu masa pengejaran yang pasti.

ACKNOWLEDGEMENTS

All praise is to Almighty ALLAH, The Beneficent, The Merciful and The Lord of the Universe for his elegance and wisdom through out my study. May Allah's Mercy and peace be upon our noble Prophet Muhammad (S.A.W), his family and Companions.

I am sincerely grateful to my supervisor (chairman of the supervisory committee) Associate Prof. Dr. Gafurjan Ibragimov for his tremendous contributions and excellent supervision. His valued assistance and comments in the preparation and completion of this thesis are extremely appreciated. Special thanks to Associate Prof. Dr. Zarina Bibi Bt. Ibrahim and Dr. Idham Arif Alias members of the supervisory committee. Thank you for your constructive efforts towards this thesis. I really appreciate your contributions.

My deepest gratefulness to my parents Alhaji Waziri Aliyu Madara (Head District of Lariski) and Hajiya Maryam Hassan for their parental support and prayers toward my victory. Many thanks to my wife Hajiya Fauziyya Hadi Adam and my daughters Amatullah Usman Waziri and Amatulrahman Usman Waziri for their tolerance and sacrifice during my study.

Special thanks to my employers Bauchi State University Gadau and Bauchi State Government for their financial support.

I certify that a Thesis Examination Committee has met on 25 January 2018 to conduct the final examination of Usman Waziri on his thesis entitled "Differential Games Problems Described by System of Infinite Differential Equations in Hilbert Space" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

Members of the Thesis Examination Committee were as follows:

Mohd Rizam bin Abu Bakar, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Norazak bin Senu, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Leong Wah June, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Khalik G. Guseinov, PhD
Professor
Anadolu University
Turkey
(External Examiner)

NOR AINI AB. SHUKOR, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 28 March 2018

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

The members of the Supervisory Committee were as follows:

Ibragimov Gafurjan
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Zarina Bibi bt Ibrahim
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Idham Arif Alias
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

ROBIAH BINTI YUNUS, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

Declaration by graduate student

I hereby confirm that:

- this thesis is my original work;
- quotations, illustrations and citations have been duly referenced;
- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
- intellectual property from the thesis and copyright of thesis are fully-owned by Universiti Putra Malaysia, as according to the Universiti Putra Malaysia (Research) Rules 2012;
- written permission must be obtained from supervisor and the office of Deputy Vice-Chancellor (Research and Innovation) before thesis is published (in the form of written, printed or in electronic form) including books, journals, modules, proceedings, popular writings, seminar papers, manuscripts, posters, reports, lecture notes, learning modules or any other materials as stated in the Universiti Putra Malaysia (Research) Rules 2012;
- there is no plagiarism or data falsification/fabrication in the thesis, and scholarly integrity is upheld as according to the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) and the Universiti Putra Malaysia (Research) Rules 2012. The thesis has undergone plagiarism detection software.

Signature:____________________ Date:____________________

Name and Matric No: Usman Waziri, GS 43402

Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________________
Name of Chairman of Supervisory Committee
Associate Professor. Ibragimov Gafurjan

Signature: ______________________________
Name of Member of Supervisory Committee
Associate Professor. Zarina Bibi bt Ibrahim

Signature: ______________________________
Name of Member of Supervisory Committee
Senior Lecturer. Idham Arif Alias

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

$\mathbb{R}$	Set of Real Numbers
$\mathbb{R}^n$	n-dimentional real Euclidean space
ODE	Ordinary Differential Equations
IS-1st-Order	Infinite System of 1st Order
IS-ODE	Infinite System of Ordinary Differential Equations
IS-2-Systems	Infinite System of 2-Systems
PDE	Partial Differential Equations
l_r^2	$\{\alpha = (\alpha_1, \alpha_2, ...) : \sum_{k=1}^{\infty} \lvert\lambda_k\rvert^r \alpha_k^2 < \infty\}$
$L_2(0,T)$	The set of square integrable functions on $[0,T]$
$L_2(0,T;l_r^2)$	$\{w(t) = (w_1(t), w_2(t), ...) : \sum_{k=1}^{\infty} \lvert\lambda_k\rvert^r \int_0^T w_k^2(t)dt < \infty\}$ and $w_k(\cdot) \in L_2(0,T)$
$C[0,T]$	Space of continuous functions on $[0,T]$
$C(0,T;l_r^2)$	Space of continuous functions defined on $[0,T]$ in the norm of l_r^2 with value in l_r^2
$\mathbb{C}$	Set of Complex Numbers
$\rightarrow$	Converges (approaches) to
$\dot{x}$	First derivative of x
$\frac{\partial z}{\partial t}$	First partial derivatives
$x(\cdot)$	Function of $x(t)$, $0 \leq t \leq T$
$[a,b]$	$\{x = (1-t)a + tb : 0 \leq t \leq 1\}$
(a,b)	$\{x = (1-t)a + tb : 0 < t < 1\}$
$\lvert\lvert f\rvert\rvert_p = (\int_{\mathbb{R}} \lvert f\rvert^p dx)^{1/p}$	p-norm
$\lvert\lvert H\rvert\rvert$	Norm of the matrix H
H^*	The Transpose of the matrix H

CHAPTER 1

INTRODUCTION

This chapter presents the background about differential game problems. It also contains important definitions and standard results used. Then followed by motivation of study. Objectives of the study, scope and limitation and organization of the thesis are also discussed.

1.1 Introduction

Differential game of pursuit and evasion problems are a special kind of mathematical problems related to the modeling and analysis of conflicts in the background of dynamical systems. The game usually consists of two players with different goals. The goal of the 1st-player is to capture the 2nd-player in some sense, while that of the 2nd-player tries all the possibilities to escape from this capture. For instance, the capture could be minimizing the distance between the 1st-player and 2nd-player as much as possible while that of the avoidance is opposite. Conventionally, we call the 1st-player to be a pursuer and 2nd-players to be an evader.

The game of pursuit and evasion examines conflict problems in systems which driven by a model. A model that describes the behavior of the players which is determined by the players input through their respective control functions contained in the model. The model is usually a system of differential equation, and each player tries as much as possible to control the state of the system so that the goal will be achieved.

Differential game problems that require to finding the condition for which pursuer can capture the evader is called pursuit differential problem. Similarly, a differential game problems that requires to finding the condition for which the evader can avoid the capture from pursuer is called evasion differential problem. The system of differential equation is usually the space where pursuit and evasion differential game problems played.

Pursuit and evasion differential game are differential game of kind in which a pursuer attempts to capture evader in a minimal possible time and the evader attempts to avoid that by maximizing the capture time. A very good example of pursuit and evasion differential game is the popular Lion and Man problem. The game considers a lion and a man to be the pursuer and the evader respectively, enclosed in a circular environment. The problem required to address some questions:

1. How long can the evader (that is a man) escapes from the pursuer (that is a lion) ?

2. If the evader (i.e. a man) succeeds to escape from the pursuer (i.e. a lion) all the time, what is the least distance can the evader (i.e. a man) has between the him and the pursuer (i.e. a lion) ?

Question 1 and 2 are formulated from the man's view point, while question 3 and 4 below are formulated from the lion point of view:

3. How long does it take for the pursuer (i.e. a lion) to capture the evader (i.e. a man) ?

4. If the pursuer (i.e. a lion) does not succeed in capture the evader (i.e. a man), how close to evader (i.e. a man) can pursuer (i.e. a lion) go?

The problem that involves question 1 and 2 is referred to as evasion differential game problem. Like wise the problem concerning question 3 and 4 is called pursuit differential game problem. This problem has been investigated thoroughly in works such as Croft (1964), Flynn (1973), Lewin (1986), and Bollobas et al. (2012).

The theory of differential games constitutes a group of problems that related to game theory and optimal control theory. Differential game related to optimal control theory in the sense that optimal control problem consists of a single control function in the model and a single criterion to be optimized. However, control problem can be extended to a differential game by introducing control function of the second player to the game model. Indeed, differential game problems represent a class of optimal control problem in cases where there are more than one control function or player.

In differential game problems, the control functions are usually subjected to constraints reflecting a natural phenomenon. Conventionally, the constraints could either be integral or geometric. If the players control belong to the subset of $\mathbb{R}^n$, then they are said to be subjected to geometric constraint. A constraint is referred to as integral if the resources of the player are bounded.

Nowadays, the differential game has become an active research area with many applications. It has been applied to solve practical problems related to artificial intelligence, economics, engineering, military operation and so. For example, it has been employed for an aircraft combat, missile control, and other military strategies. There are a tremendous amount of applications in the field of mathematical finance and marketing. Most recent developments include adding stochastic to differential games. Stochastic differential game models are increasingly used in various fields ranging from market development, competition policy and investment. Another potential application includes searching the building for intruders and surgical operations to name a few.

1.2 Standard Results

In the present section, we present some standard results and basic definition which will be used in the next chapters.

1.2.1 Hilbert space

Definition 1.2.1 *Let X be a complex linear space, then a function $\langle .,. \rangle : X \times X \to \mathbb{C}$ is referred to as the inner product, if it satisfied the following axioms:*

i. $\langle x,x \rangle \geq 0$ *and* $\langle x,x \rangle = 0$ *if and only if* $x = 0$.

ii. $\langle \lambda x,y \rangle = \lambda \langle x,y \rangle$.

iii. $\langle x,y \rangle = \overline{\langle y,x \rangle}$, *the complex conjugate of* $\langle x,y \rangle$.

iv. $\langle x+y,z \rangle = \langle x,z \rangle + \langle y,z \rangle$.

where $x,y,z \in X$ and λ is a complex number.
A linear space X which has an inner product $\langle .,. \rangle$ is called an inner product space denoted by $(X, \langle .,. \rangle)$.

Example 1.2.2 *A finite dimensional space (Euclidean space) $\mathbb{R}^n$ with dot product*

$$\langle (x_1,x_2...,x_n),(y_1,y_2...,y_n) \rangle = \sum_{k=1}^{n} x_k y_k,$$

is an inner product space.

(Pedersen (1999))

Definition 1.2.3 *Let $\{x_n\}$ be a sequence in an inner product space $(X, \langle .,. \rangle)$. A sequence $\{x_n\}$ is said to be a Cauchy sequence if, for every positive number $\varepsilon > 0$, there is an integer $N_\varepsilon > 0$ such that $\|x_n - x_m\| < \varepsilon$ for all natural number $m,n > N_\varepsilon$.*

Brian et al. (2001)

Definition 1.2.4 *A Hilbert space is an inner-product space in which every Cauchy sequence in the space converges to a point in the space. In other words, complete inner-product space is called Hilbert space.*

Ponnusamy (2002)

Example 1.2.5 *$\mathbb{C}^n$ is a Hilbert space with an inner-product defined by*

$$\langle x,y\rangle = \sum_{k=1}^{n} x_k y_k$$

where $x = x_1, x_2, ..., x_n$, $y = y_1, y_2, ..., y_n$ is finite dimensional space in $\mathbb{C}^n$.

Example 1.2.6 *Let E be a measurable subset of $\mathbb{R}$. Then the space of all square integrable function denoted by $L_2(E)$ with inner product defined by*

$$\langle f,g\rangle = \int_E f\bar{g}d\mu$$

is Hilbert space.

Definition 1.2.7 *Let X be a linear space. A norm on X is a real-valued function with function $\|\cdot\|$ on X satisfying the following axiom:*

i. $\|x\| \geq 0 \; \forall x \in X$,

ii. $\|x\| = 0$ *if and only if* $x = 0$

iii. $\|\lambda x\| = |\lambda| \, \|x\|$ *for all* $x \in X$, *and for all scalar* λ

iv. $\|x+y\| \leq \|x\| + \|y\| \; \forall x,y \in X$

A linear space X with a norm $\|\cdot\|$ on it is called a normed linear space denoted by $(X, \|\cdot\|)$. The norm is also referred to as the length of the vector x. Rao (2006).

Theorem 1.1 *Every inner-product space is a normed linear space with norm defined by*

$$\|x\| = \sqrt{\langle x,x\rangle}.$$

For this proof see Rao (2006)

Definition 1.2.8 *The sequence space l^p $(1 \leq p \leq \infty)$ for which the norm for the sequence $\{x_n\} \in l^p$ is defined by*

$$\|x\|_p = \begin{cases} \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty, \; if \; 1 \leq p < \infty, \\ sup_{1 \leq n < \infty}, |x_n| < \infty, \; if \; p = \infty, \end{cases}$$

is normed space.

The norm defined for $1 \leq n < \infty$ is called l^p norm (or simply p-norm) and for $p = \infty$ is called l^∞ norm or simply sup-norm. Ponnusamy (2002)

1.2.2 Standard Inequalities

Theorem 1.2 *(Cauchy-Schwartz inequality).*
Let $(X, \langle ., . \rangle)$ be an inner-product space. Cauchy-Schwartz inequality states that

$$|\langle x,y \rangle|^2 \leq \langle x,x \rangle \cdot \langle y,y \rangle ,$$

for all vectors $x, y \in X$.

Rao (2006)

Example 1.2.9 *In Euclidean space $\mathbb{R}^n$ with the standard inner product, the Cauchy-Schwartz inequality is*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} y_i^2 \right)^{1/2}$$

Example 1.2.10 *For the inner product space of square-integrable complex-valued functions, Cauchy-Schwartz inequality is*

$$\left| \int f(x)\overline{g(x)}dx \right|^2 \leq \int |f(x)|^2 dx \int |g(x)|^2 dx$$

where $\overline{g(x)}$ is the conjugate of $g(x)$.

Theorem 1.3 *(Minkowski's inequality).*
Let $p \geq 1$ and $f, g \in L^p(\mathbb{R})$. Then

$$\|f+g\|_p \, \|f\|_p + \|g\|_p \, .$$

For the proof see Pedersen (1999)

Example 1.2.11 *(Minkowski's sum inequality).*
If $p \geq 1$ and $x_k, y_k \in \mathbb{R}$, $k = 1, 2, ...$, then

$$\left[\sum_{k=1}^{n} |x_k + y_k|^p \right]^{1/p} \leq \left[\sum_{k=1}^{n} |x_k|^p \right]^{1/p} + \left[\sum_{k=1}^{n} |y_k|^p \right]^{1/p}$$

This result is true for infinite sum and for the details, see Rao (2006).

Example 1.2.12 *(Minkowski's integral inequality).*

If $p \geq 1$ and $f, g \in C[a,b]$, then

$$\left[\int_a^b |f(x)+g(x)|^p\right]^{1/p} \leq \left[\int_a^b |f(x)|^p\right]^{1/p} + \left[\int_a^b |g(x)|^p\right]^{1/p}.$$

For the details see Ponnusamy (2002)

1.2.3 Measurable Functions

Definition 1.2.13 *Let X be a set. A collection Σ of subsets of X is called σ- algebra on set X, if it satisfies the following properties:*

i. X belongs to Σ, i.e.

$$X \in \Sigma.$$

ii. if A belong to Σ, then the complement of A belongs to Σ, i.e.

$$A^c = \{x \in X |\, x \notin A\} \in \Sigma.$$

iii. if A_k is a sequence of Σ, then the union of the all A_k belongs to Σ, i.e.

$$A_k \in \Sigma \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \Sigma.$$

The members of $\sum$ are called measurable sets.

Definition 1.2.14 *Let Σ be a σ- algebra over set X. A function $\mu : \sigma \to \mathbb{R}$ is called a measure if it satisfies the following:*

i. $\mu(E) \geq 0$ for all $E \in \Sigma$,

ii. $\mu(\cup_{k\in I} E_k) = \Sigma_{k\in I}\mu(E_k)$, for all countable $\{E_k\}_{k\in I}$ pairwise disjoint members of Σ,

iii. $\mu(\oslash) = 0$.

Definition 1.2.15 *Let X be a set and let Σ be a σ- algebra defined on X. A set X together with Σ is called a measurable space and is denoted as (X, Σ).*

Definition 1.2.16 *Let (X, Σ_1) and (Y, Σ_2) be measurable spaces. A measurable function is a function $f : (X, \Sigma_1) \to (Y, \Sigma_2)$ such that $f^{-1}(E) \in \Sigma_1$ for all $E \in \Sigma_2$. i.e. preimage of any measurable set is measurable.*

Definition 1.2.17 *In measure theory, a property holds almost everywhere, if the set of elements for which the property does not hold has a measure zero.*

For example, if $f:[a,b] \to \mathbb{R}$ is a monotonic function, then it is differentiable almost everywhere. Aliprantis and Burkinshaw (1981).

1.3 Motivation

The differential game of pursuit and evasion problem is derived from the fact that the trajectories of pursuer and evader are given as the solution of differential equations. Also, the strategies of pursuer and evader are given as control functions occurring in the differential equations. The system of differential equations describing differential game problem could be ordinary or partial differential equations. The later seems to be of broader applications from the fact that many physical system and biological process are often governed by partial differential equations, where the state of the system is a function of time.

In the early works such as Bukovskii (1961) and Wang (1965), variation technique was employed to study differential game problem described by partial differential equations. Some further and recent works on the differential game problem described by partial differential equations include that of Chernous'ko (1996) and Tukhtasinov (1995). In each of these works decomposition method is used to reduce the problem described by partial differential equations to the one described by an infinite system of ordinary differential equations.

The later can be studied in an independent framework. For example, Ibragimov and Hasim (2010) and Ibragimov (2013a), studied differential game problem described by an infinite system of differential equations with positive coefficients. This serves as a motivations factor in attracting my attention to study differential game problem described by an infinite system of ordinary differential equations, in the case of negative coefficients.

1.4 Objectives of the Thesis

The following are objectives of the thesis:

- To obtain solution of optimal pursuit differential game problem of the following IS-ODE
$$\dot{z}_k + \lambda_k z_k = -u_k + v_k, z_k(0) = z_k^0, k = 1,2,...,$$
where $z_k, u_k, v_k \in \mathbb{R}^1, z^0 = (z_1^0, z_2^0, ...) \in l_{r+1}^2, u_1, u_2, ...,$ are control parameters of

the pursuer and $v_1, v_2, ...,$ are that of the evader, $\{\lambda_1, \lambda_2, ...\}$ are bounded sequence of negative numbers. Pursuer tries to force the state of the system toward the origin and the evader tries to avoid this, where we obtain an equation to find the optimal pursuit time and construct the optimal strategies for players.

- To obtain a solution of pursuit game for an IS-1st-Order of ordinary differential equations with integral constraints in the case of negative coefficients

$$\dot{z_k} + \lambda_k z_k = -u_k + v_k, z_k(0) = z_k^0, k = 1, 2, ..., \tag{1.1}$$

  where $z^0 = (z_1^0,\, z_2^0, \ldots,) \in l_{r+1}^2$. We denote a given state $z^1 = (z_1^1,\, z_2^1, \ldots,) \in l_{r+1}^2$, of the system (1.1) and all other parameters are stated as in the first system above. In this case, pursuer actions to bring the state of the system toward another and evader actions to stop this.

- To obtain condition of completion of pursuit with geometric and integral constraints described by IS-1st-Order of ordinary differential equations of the space l_2

$$\dot{z_k} = -\lambda_k z_k - u_k + v_k, z_k(0) = z_k^0, k = 1, 2, ..., \tag{1.2}$$

  where $z_k, u_k, v_k \in \mathbb{R}^1$, $z^0 = (z_1^0,\, z_2^0, \ldots,) \in l_2$, $\{\lambda_1, \lambda_2, ...\}$ are assumed to be any real numbers. We denote a given state $z^1 = (z_1^1,\, z_2^1, \ldots,) \in l_2$, of the system (1.2). Pursuer tries to brings the state of the system to coincides with another state of the space l_2.

- To obtain guaranteed pursuit time for an IS-2-Systems of 1st-Order of ordinary differential equations in Hilbert space l_2

$$\begin{aligned} &\dot{x}_k = -\beta_k x_k - \alpha_k y_k - u_{1k} + v_{1k}, \\ &\dot{y}_k = \alpha_k x_k - \beta_k y_k - u_{2k} + v_{2k}, \\ &x_k(0) = x_k^0,\ y_k(0) = y_k^0,\ k = 1, 2, ... \end{aligned}$$

  where, α_k, β_k are real numbers, $\beta_k \geq 0$ with initial state $x^0 = (x_1^0, x_2^0, ...) \in l_2$, $y^0 = (y_1^0, y_2^0, ...) \in l_2$. We denote $x^1 = (x_1^1, x_2^1, ...) \in l_2$, $y^1 = (y_1^1, y_2^1, ...) \in l_2$ be another state, a control parameters of the pursuer is given by $u = (u_{11}, u_{12}, u_{21}, u_{22}, ...)$, and that of the evader is given by $v = (v_{11}, v_{12}, v_{21}, v_{22}, ...)$. In the game, the goal of the pursuer is to force the state of the system toward another state and the evader actions in the opposite. Integral constraints are imposed on the control functions of the players

1.5 Scope and Limitation

In this thesis, we focus our attention on pursuit and evasion differential game problem described by infinite systems of differential equations in Hilbert space and limited only to systems of infinite first-order ordinary differential equations.

1.6 Organization of the Thesis

This thesis is organized into eight chapters. The remaining part of the thesis is presented as follows:

In Chapter 2, we give an introduction to the chapter. We brief on a history of the emergence of differential games and review the related literature on some important works.

Chapter 3 begins with an introduction. The solution of 1st-order linear equation is presented. The proof of existence-uniqueness theorem to the considered systems is also discussed.

Chapter 4 focuses on the solution of optimal pursuit time for an IS-1st-Order ordinary differential equations with negative coefficients, where the control functions of the players are subjected to integral constraints. At first, we give an introduction to the chapter and then proceed with a solution of control problem involving infinite system of ordinary differential equations. The chapter is then concluded.

After introducing the chapter, we study a control problem described by infinite system of differential equations in Chapter 5. The condition of completion of pursuit in differential game problem described by IS-1st-Order ordinary differential equations with integral constraints are then presented. The chapter ends with a brief conclusion.

In Chapter 6, after introducing the chapter, then we focuses to the solution of pursuit game problem described by infinite system of differential equations where the control geometric and integral constraints are imposed to the control functions of the player. Furthermore, the chapter includes control problems described by infinite system of differential equations with geometric and integral constraints. The chapter ends with conclusion.

In Chapter 7, we present differential game pursuit time for an IS-2-Systems of 1st-Order in Hilbert space. Following an introduction, control problem described by an an IS-2-Systems of 1st-Order are considered. Each component of the control functions of players is subjected to integral constraints. An equation to find guaranteed pursuit time is obtained. Finally, we conclude the chapter.

Last but not the least, we give the general conclusion and some future research works in Chapter 8.

REFERENCES

Alias, I. A., Ibragimov, G., Kuchkarov, A., and Sotvoldiyev, A. (2016a). Differential game with many pursuers when controls are subjected to coordinate-wise integral constraints. *Malaysian Journal of Mathematical Sciences*, 10(2):195–207.

Alias, I. A., Ibragimov, G., and Rakhmanov, A. (2016b). Evasion differential game of infinitely many evaders from infinitely many pursuers in hilbert space. *Dynamic Games and Applications*, 6(2):1–13.

Alias, I. A., Salleh, Y., and Azamov, A. (2013). Evasion differential game of two evaders and one pursuer with integral constraints. *AIP Conference Proceedings*, 1557(1):140–145.

Aliprantis, C. and Burkinshaw, O. (1981). On weakly compact operators on banach lattices. *Proceedings of the American Mathematical Society*, 83(3):573–578.

Allahabi, F. and Ibragimov, G. I. (2012). Evasion from one pursuer in a hilbert space. *International Journal of Modern Physics: Conference Series*, 9:529–536.

Allahabi, F. A. A. (2011). The existence-uniqueness theorem for a system of differential equations in the spaces. *Malaysian Journal of Mathematical Sciences*, 5(1):111–123.

Avdonin, S. A. and Ivanov, S. A. (1995). *Families of Exponentials : The Method of Moments in Controllability Problems for Distributed Parameter Systems*. Cambridge University Press.

Balakrishnan, A. (1965). Optimal control problems in banach spaces. *Journal of the Society for Industrial and Applied Mathematics, Series A: Control*, 3(1):152–180.

Başar, T., Bensoussan, A., and Sethi, S. P. (2010). Differential games with mixed leadership: The open-loop solution. *Applied Mathematics and Computation*, 217(3):972–979.

Berkovitz, L. D. (1967). Necessary conditions for optimal strategies in a class of differential games and control problems. *SIAM Journal on Control*, 5(1):1–24.

Berkovitz, L. D. (1988). Characterizations of the values of differential games. *Applied mathematics & optimization*, 17(1):177–183.

Bollobas, B., Leader, I., and Walters, M. (2012). Lion and man can both win? *Israel Journal of Mathematics*, 189(1):267–286.

Brian, S. T., Judith, B. B., and Andrew, M. B. (2001). Elementary real analysis.

Butkovskiy, A. (1975). *Methods of the Distributed Parameter Systems Control.* Moscow.

Case, J. H. (1979). *Economics and the Competitive Process*. New York University Press.

Chernous' ko, F. (1976). A problem of evasion from many pursuers. *Journal of Applied Mathematics and Mechanics*, 40(1):14–24.

Chernous'ko, F. (1992). Bounded controls in distributed parameter systems. *Journal of Applied Mathematics and Mechanics*, 56(5):707–723.

Chernous'ko, F. (2004). Optimal control for a class of systems subjected to disturbances. *Journal of applied mathematics and mechanics*, 68(4):503–510.

Croft, H. T. (1964). lion and man: A postscript. *Journal of the London Mathematical Society*, 1(1):385–390.

Dockner, E. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press.

Egorov, A. (2004). *Principles of the Control Theory*. Nauka, Moscow.

Flynn, J. O. (1973). Lion and man: The boundary constraint. *SIAM journal on Control*, 11(3):397–411.

Friedman, A. (1971). *Differential Games*. Wiley-Interscience, New York.

Friedman, A. (1994). Differential games. *Handbook of game theory with economic applications*, 2:781–799.

Guseinov, K. G., Neznakhin, A., and Ushakov, V. (1999). Approximate construction of attainability sets of control systems with integral constraints on the controls. *Journal of Applied Mathematics and Mechanics*, 63(4):557–567.

Ibragimov, G. (1998). A game of optimal pursuit of one object by several. *Journal of applied mathematics and mechanics*, 62(2):187–192.

Ibragimov, G. (2002a). A game problem on a closed convex set. *Siberian advances in mathematics*, 12(3):16–31.

Ibragimov, G. (2002b). A problem of optimal pursuit in systems with distributed parameters. *Journal of applied mathematics and mechanics*, 66(5):719–724.

Ibragimov, G. (2004). Collective pursuit with integral constraints on the controls of players. *Siberian Advances in Mathematics*, 14(2):14–26.

Ibragimov, G. (2013a). Optimal pursuit time for a differential game in the hilbert space l_2. *ScienceAsia 39S*, 6(8):9.

Ibragimov, G., Akhmedov, A., Puteri, N. I., and Abdul Manaf, N. (2017). Pursuit differential game described by infinite first order 2-systems of differential equations. *Malaysian Journal of Mathematical Sciences*, 11(2):181–190.

Ibragimov, G., Allahabi, F., and Kuchkarov, A. (2014a). A pursuit problem in an infinite system of second-order differential equations. *Ukrainian Mathematical Journal*, 65(8).

Ibragimov, G., Hasim, R. M., Rakhmanov, A., and Alias, I. A. (2016). Optimal pursuit time in differential game for an infinite system of differential equations. *Malaysian Journal of Mathematical Sciences*, 10(S):267–277.

Ibragimov, G. and Hussin, N. A. (2010). A pursuit-evasion differential game with many pursuers and one evader. *Malaysian Journal of Mathematical Sciences*, 4(2):183–194.

Ibragimov, G. and Ja'afaru, A. (2011). On existence-uniqueness of solution to countable number of first order differential equations in the space l_2. *Journal of Applied Sciences Research*, 7(12):1860–1864.

Ibragimov, G. and Ja'afaru, A. (2011a). On control problem described by infinite system of first order differential equations. *Australian Journal of Basic and Applied Science*, 5(10):736–742.

Ibragimov, G., Mat Hashim, R., and Azamov, A. (2008). Existence and uniqueness of the solution for an infinite system of differential equations. *International Journal of Mathematics and Statistics*, 1(2):9–14.

Ibragimov, G., Rahmanov, A., and Alias, I. A. (2014b). Construction of strategies of pursuers in a differential game of many players with state and integral constraints. *International Conference on Mathematical Sciences and Statistics 2013*, pages 37–43.

Ibragimov, G., Rasid, N. A., Kuchkarov, A., and Ismail, F. (2015a). Multi pursuer differential game of optimal approach with integral constraints on controls of players. *Taiwanese Journal of Mathematics*, 19(3):pp–963.

Ibragimov, G., Tukhtasinov, M., Mat Hasim, R., and Alias, I. A. (2015b). A pursuit problem described by infinite system of differential equations with coordinate-wise integral constraints on control functions. *Malaysian Journal of Mathematical Sciences*, 9(1):67–76.

Ibragimov, G. I. (2001). On a multiperson pursuit problem with integral constraints on the controls of the players. *Mathematical Notes*, 70(1):181–191.

Ibragimov, G. I. (2005). Optimal pursuit with countably many pursuers and one evader. *Differential Equations*, 41(5):627–635.

Ibragimov, G. I. (2013b). The optimal pursuit problem reduced to an infinite system of differential equations. *Journal of Applied Mathematics and Mechanics*, 77(5):470–476.

Ibragimov, G. I. and Hasim, R. M. (2010). Pursuit and evasion differential games in hilbert space. *International Game Theory Review*, 12(03):239–251.

Ibragimov, G. I., Khakestari, M., and Kuchkarov, A. S. (2012). Solution of a linear pursuit-evasion differential game with closed and convex terminal set. *Journal of Mathematical and Fundamental Sciences*, 44(1):1–12.

Ibragimov, G. I. and Rikhsiev, B. B. (2006). On some sufficient conditions for optimality of the pursuit time in the differential game with multiple pursuers. *Automation and Remote Control*, 67(4):529–537.

Ibragimov, G. I. and Salimi, M. (2009). Pursuit-evasion differential game with many inertial players. *Mathematical Problems in Engineering*, Article ID 653723, 15 pages doi:10.1155/2009/653723.

Ilin, V. (2001). Boundary control of a string oscillating at one end, with the other end fixed and under the condition of the existence of finite energy. *Dokl. RAN*, 378(6):743–747.

Isaacs, R. (1965). *Differential Games, a Mathematical Theory with Applications to Optimization, Control and Warfare*. New York: John Wiley and Sons.

Isaacs, R. (30 November 1954-25 March 1955). *Differential Games:RM-1391, RM-1399, RM-1411 AND RM-1486*. Rand Reports.

Ivanov, I. B. (1980). Effect of surface mobility on the dynamic behavior of thin liquid films. *Pure and Applied Chemistry*, 52(5):1241–1262.

Ivanov, R. and Ledyaev, Y. S. (1981). Time optimality for the pursuit of several objects with simple motion in a differential game. *Trudy Matematicheskogo Instituta imeni VA Steklova*, 158:87–97.

Ja'afaru, A. B. and Ibragimov, G. (2011). Differential games described by infinite system of differential equations of second order. the case of negative coefficients. *International Journal of Pure and Applied Mathematics*, 70(7):927–938.

Ja'afaru, A. B. and Ibragimov, G. (2012). On some pursuit and evasion differential game problems for an infinite number of first-order differential equations. *Journal of Applied Mathematics*, Article ID 717124, 13 pages doi:10.1155/2012/717124.

Kirk, D. (1998). Optimal control theory. an introduction, dover publications. *San José, California*.

Krasovskii, N. N., Subbotin, A. I., and Kotz, S. (1987). *Game-Theoretical Control Problems*. Springer-Verlag New York, Inc.

Kuchkarov, A. S. (2003). The pursuit-evasion problem in the sphere. *Dokl. Adyg.(Cherkess.) Mezhd. Akad. Nauk*, 6(2):104–108.

Kuchkarov, A. S. (2007). The problem of optimal approach in locally euclidean spaces. *Automation and Remote Control*, 68(6):974–978.

Kuchkarov, A. S. and Rikhsiev, B. (2001). A pursuit problem under phase constraints. *Automation and Remote Control*, 62(8):1259–1262.

Kuhn, H. W. and Szego, G. P. (1971). *Differential Games and Related Topics*. North-Holland Pub. Co.

Lee, E. B. and Markus, L. (1967). Foundations of optimal control theory. Technical report, DTIC Document.

Leitmann, G. (1974). *Cooperative and Non-cooperative Many Players Differential Games*. Springer.

Leong, W. J. and Gafurjan, I. I. (2007). A differential game of multiperson pursuit in the hilbert space. *Communications in Mathematical Analysis*, 2(1).

Leong, W. J. and Ibragimov, G. (2008). A multiperson pursuit problem on a closed convex set in hilbert space. *Far east journal of applied mathematics*, 33(2):205–214.

Levchenkov, A. and Pashkov, A. (1990). Differential game of optimal approach of two inertial pursuers to a non-inertial evader. *Journal of Optimization Theory and Applications*, 65(3):501–518.

Lewin, J. (1986). The lion and man problem revisited. *Journal of Optimization Theory and Applications*, 49(3):411–430.

Lewin, J. (1994). Differential games: Theory and methods for solving game problem with singular surfaces.

Lions, J.-L. (1968). Contrôle optimal de systemes gouvernés par des équations aux dérivées partielles.

Mamatov, M. S. (2009a). On the theory of differential pursuit games in distributed parameter systems. *Automatic Control and Computer Sciences*, 43(1):1–8.

Mamatov, M. S. (2009b). Application of the finite difference method to the problem of pursuit in the distributed-parameter systems. *Automation and Remote Control*, 70(8):1376–1384.

Mamatov, M. S. and Tukhtasinov, M. (2009). Pursuit problem in distributed control systems. *Cybernetics and systems analysis*, 45(2):297–302.

Mehlmann, A. (2013). *Applied Differential Games*. Springer Science & Business Media.

Melikyan, A. and Ovakimyan, N. (1991). Singular trajectory in the problem of simple pursuit on a manifold. *Journal of Applied Mathematics and Mechanics*, 55(1):48–55.

Melikyan, A. and Ovakimyan, N. (1993). A differential game of simple approach in manifolds. *Journal of Applied Mathematics and Mechanics*, 57(1):47–57.

Nikolaevich, K. N. and Subbotin, A. I. (1988). *Game-theoretical Control Problems*. N.Y.,Springer, 517.

Nikolskii, M. (1969). The direct method in linear differential games with integral constraints. *Controlled Systems, IM, IK, SO AN SSSR*, (2):49–59.

Osipov, I. and Okhezin, S. (1977). Theory of positional control in hyperbolic systems. In *Akademiia Nauk SSSR, Doklady*, volume 233, pages 551–554.

Osipov, Y. S. (1975). Theory of differential games in systems with distributed parameters. *Doklady Akademii Nauk SSSR*, 223(6):1314–1317.

Paiewonsky, B. (1965). Optimal control-a review of theory and practice. *Aiaa Journal*, 3(11):1985–2006.

Pashkov, A. and Teorekhov, S. (1983). On a game of optimal pursuit of one object by two objects. *Prikl. Mat. Mekh*, 47(6):898–903.

Pedersen, M. (1999). *Functional Analysis in Applied Mathematics and Engineering*, volume 31. CRC press.

Pinch, E. R. (1995). *Optimal Control and the Calculus of Variations*. Oxford University Press.

Ponnusamy, S. (2002). Foundations of functional analysis, alpha science international publishers, uk, 2002. *Also, Narosa Publishing House, India*.

Pontryagin, L. S. (1987). *Mathematical Theory of Optimal Processes*. CRC Press.

Pshenichnyi, B. and Onopchuk, Y. N. (1968). Linear differential games with integral constraints. *Akademii Nauk SSSR, Tekhnicheskaya Kibernetika*, 1968(1):13–22.

Rao, K. C. (2006). Functional analysis, alpha science international.

Sakawa, Y. (1966). Optimal control of a certain type of linear distributed-parameter systems. *IEEE Transactions on Automatic Control*, 11(1):35–41.

Salimi, M., Ibragimov, G. I., Siegmund, S., and Sharifi, S. (2016). On a fixed duration pursuit differential game with geometric and integral constraints. *Dynamic Games and Applications*, 6(3):409–425.

Salleh, Y., Ibragimov, G., Rahmanov, A., Khaldjigitov, A., Eshkuvatov, Z. K., Kilicman, A., and June, L. W. (2013). Construction the evasion strategy in the differential game of many pursuers and one evader with integral constraints. *AIP Conference Proceedings*, 1557(1):296–299.

Samatov, B. (2013). Problems of group pursuit with integral constraints on controls of the players. i. *Cybernetics and Systems Analysis*, 49(5):756–767.

Satimov, N. and Kuchkarov, A. (2001). Many pursuers evasion on the suface. *Automation and Remote Control*, (1):51–55.

Satimov, N. and Mamatov, M. S. (1983). On the pursuit and evasion problem in differential games between groups of pursuers and evaders. *Dokl. Akad. Nauk Uzb. SSR*, (4):3–6.

Satimov, N., Rikhsiev, B. B., and Khamdamov, A. (1983). On a pursuit problem for n-person linear differential and discrete games with integral constraints. *Sbornik: Mathematics*, 46(4):459–471.

Satimov, N. Y. and Tukhtasinov, M. (2005a). On some game problems for first-order controlled evolution equations. *Differential Equations*, 41(8):1169–1177.

Satimov, N. Y. and Tukhtasinov, M. (2005b). Some game problems in distributed controlled systems. *Journal of applied mathematics and mechanics*, 69(6):885–890.

Satimov, N. Y. and Tukhtasinov, M. (2006). Game problems on a fixed interval in controlled first-order evolution equations. *Mathematical notes*, 80(3):578–589.

Satimov, N. Y., Tukhtasinov, M., and Ismatkhodzhaev, S. (2007). On an evasion problem on a semi-infinite interval for a class of controlled distributed systems. *Mathematical Notes*, 81(1):260–267.

Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. *International journal of game theory*, 4(1):25–55.

Sinitsyn, A. (1993). Construction of the cost function in the game of pursuit by several objects. *Prikl. Mat. Mekh*, 57(1):52–57.

Soravia, P. (1993). Pursuit-evasion problems and viscosity solutions of isaacs equations. *SIAM journal on control and optimization*, 31(3):604–623.

Tukhtasinov, M. (1995). Some problems in the theory of differential pursuit games in systems with distributed parameters. *Journal of Applied Mathematics and Mechanics*, 59(6):935–940.

Tukhtasinov, M. and Mamadaliyev, N. (2011). On the problems of pursuit and deviation from meeting in the class of the distributed control systems. *On certain analytic functions defined by a generalized linear operator 1 Afaf Abubaker, Maslina Darus 2 Optimal control problems for partial differential equations from popula-tion dynamics*, page 161.

Tukhtasinov, M. and Mamatov, M. S. (2008). On pursuit problems in controlled distributed systems. *Mathematical notes*, 84(1-2):256–262.

Tukhtasinov, M. and Mamatov, M. S. (2009). On transfer problems in control systems. *Differential Equations*, 45(3):439–444.

Turetsky, V. and Shinar, J. (2003). Missile guidance laws based on pursuit-evasion games formulations. *Automatica*, 39(4):607–618.

Vagin, D. and Petrov, N. (2002). A problem of group pursuit with phase constraints. *Journal of applied mathematics and mechanics*, 66(2):225–232.

Vaisburd, I. and Osipov, I. S. (1975). Differential games of encounter for distributed-parameter systems. *Journal of Applied Mathematics and Mechanics*, 39(5):772–779.

Varaiya, P. (1970). N-person nonzero sum differential games with linear dynamics. *SIAM Journal on Control*, 8(4):441–449.

Wang, P. (1965). *Control of Distributed Parameters system, Advance in Control Systems; Theory and Applications.* Academic Press, New York.

Weeren, A., Schumacher, J., and Engwerda, J. (1999). Asymptotic analysis of linear feedback nash equilibria in nonzero-sum linear-quadratic differential games. *Journal of Optimization Theory and Applications*, 101(3):693–722.