



UNIVERSITI PUTRA MALAYSIA

***DIFFERENTIAL GAMES PROBLEMS DESCRIBED BY SYSTEM OF
INFINITE DIFFERENTIAL EQUATIONS IN HILBERT SPACE***

USMAN WAZIRI

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**DIFFERENTIAL GAMES PROBLEMS DESCRIBED BY SYSTEM OF
INFINITE DIFFERENTIAL EQUATIONS IN HILBERT SPACE**

By

USMAN WAZIRI

**Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia,
in Fulfilment of the Requirements for the Degree of Doctor of Philosophy**

January 2018



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DEDICATIONS

To my parents

Alhaji Waziri Aliyu Madara (Head District of Lariski) and Maryam Hassan.

My wife

Fauziyya Hadi Adam.

My daughters

Amatullah Usman Waziri and Amatulrahman Usman Waziri.



Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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USMAN WAZIRI

January 2018

Chair: Associate Professor Ibragimov Gafurjan, PhD
Faculty: Science

This thesis deals with the solutions of differential game problems described by some infinite systems of ordinary differential equations in Hilbert space. The infinite system arises from the solution of some control and differential game of problems described by some partial differential equations. By using decomposition method, some of these problems can be reduced to the one described by some infinite system of ordinary differential equations.

Therefore, this thesis focuses on different types of infinite systems using various approaches in Hilbert space. The first system is an infinite system of first order differential equations, and the second system is an infinite system of 2-systems of first order differential equations. For all the systems, we study the existence and uniqueness, and then we consider control and differential game problems with some forms of constraints on controls of the players.

For the first system, we present solution of optimal pursuit problems with negative coefficients, where the controls of the players are subjected to integral constraints. Pursuer's goal is to force the state of the system toward the origin and the evader tries to avoid this.

Secondly, we extend the first system and introduce another state away from that of the initial state. In this game, pursuer attempts to bring the state of the system toward another the evader's purpose is opposite where we study pursuit game problems with negative coefficients.

Furthermore, the second game is improved with various constraints and the coefficients assumed to be any real numbers, the condition of completion of pursuit with geometric and integral constraints is proposed.

For the second system, we solve pursuit differential game problem of 2-system of first-order that involves a generalization of all considered games with conjugate complex, the case of integral constraints.

The main findings and contributions of this thesis is to study differential game described by infinite system of differential equations. For the first system, we propose an optimal pursuit time. For the second the third cases, we propose a new approach of completion of pursuit and for the second system, a guaranteed pursuit time is also proposed.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

**MASALAH PERMAINAN PEMBEZAAN YANG DIHURAIKAN OLEH
SISTEM PERSAMAAN PEMBEZAAN YANG TIDAK TERHINGGA
DALAM RUANG HILBERT**

Oleh

USMAN WAZIRI

Januari 2018

Pengerusi: Profesor Madya Ibragimov Gafurjan, PhD
Fakulti: Sains

Tesis ini mengendalikan penyelesaian kepada masalah permainan pembezaan yang diterangkan oleh beberapa sistem tak terhingga persamaan pembezaan biasa dalam ruang Hilbert. Sistem tak terhingga berkenaan muncul dari penyelesaian bagi beberapa kawalan dan permainan pembezaan bagi masalah yang diterangkan oleh beberapa persamaan pembezaan separa. Dengan menggunakan kaedah pemecahan, beberapa masalah ini boleh dimudahkan kepada masalah yang diterangkan oleh sistem tak terhingga bagi persamaan pembezaan biasa.

Oleh itu, tesis ini memfokuskan kepada jenis sistem tak terhingga yang berbeza dengan menggunakan berbagai pendekatan dalam ruang Hilbert. Sistem yang pertama adalah sistem tak terhingga persamaan pembezaan peringkat pertama, dan sistem kedua adalah sistem tak terhingga bagi sistem-2 bagi persamaan pembezaan peringkat pertama. Untuk semua sistem, kami mengkaji kewujudan dan keunikan, dan kemudian kami pertimbangkan kawalan dan masalah permainan pembezaan dengan beberapa bentuk kekangan terhadap kawalan pemain.

Untuk sistem pertama, kami mempersembahkan selesaian kepada masalah pengejaran optimal dengan pekali negatif, di mana kawalan pemain adalah bergantung kepada kekangan kamiran. Tujuan pengejar adalah untuk memaksa keadaan sistem kearah asalan dan pengelak cuba untuk mengelak dari ini berlaku.

Kedua, kami memperkembangkan sistem pertama dan memperkenalkan keadaan

yang lain, pada kedudukan yang lain dari keadaan permulaan. Dalam permainan ini, pengejar cuba untuk membawa keadaan sistem ke arah yang lain. Tujuan pengelak adalah bertentangan di mana kami mengkaji masalah permainan pengejaran dengan pekali negatif.

Di samping itu, permainan kedua diperbaiki dengan pelbagai kekangan dan pekali adalah dianggap sebagai sebarang nombor nyata, keadaan bagi penyempurnaan pengejaran dengan kekangan geometri dan kamiran dicadangkan.

Untuk sistem kedua, kami menyelesaikan masalah permainan pembezaan pengejaran 2-sistem peringkat pertama yang melibatkan pengitlakan semua masalah yang diper-timbangkan dengan konjugat kompleks, kes kekangan kamiran.

Hasil dan sumbangan utama tesis ini adalah untuk mengkaji permainan pembezaan yang diterangkan oleh sistem tak terhingga persamaan pembezaan. Untuk sistem yang pertama, kami mencadangkan masa pengejaran optimal. Untuk kes kedua dan ketiga, kami cadangkan pendekatan baru bagi penyempurnaan pengejaran dan untuk sistem kedua, dicadangkan juga satu masa pengejaran yang pasti.

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Special thanks to my employers Bauchi State University Gadau and Bauchi State Government for their financial support.

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Signature: _____

Name of Member of Supervisory Committee

Associate Professor. Zarina Bibi bt Ibrahim

Signature: _____

Name of Member of Supervisory Committee

Senior Lecturer. Idham Arif Alias

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LIST OF ABBREVIATIONS

$\mathbb{R}$	Set of Real Numbers
$\mathbb{R}^n$	n -dimensional real Euclidean space
ODE	Ordinary Differential Equations
IS-1st-Order	Infinite System of 1st Order
IS-ODE	Infinite System of Ordinary Differential Equations
IS-2-Systems	Infinite System of 2-Systems
PDE	Partial Differential Equations
l_r^2	$\{\alpha = (\alpha_1, \alpha_2, \dots) : \sum_{k=1}^{\infty} \lambda_k ^r \alpha_k^2 < \infty\}$
$L_2(0, T)$	The set of square integrable functions on $[0, T]$
$L_2(0, T; l_r^2)$	$\{w(t) = (w_1(t), w_2(t), \dots) : \sum_{k=1}^{\infty} \lambda_k ^r \int_0^T w_k^2(t) dt < \infty\}$ and $w_k(\cdot) \in L_2(0, T)$
$C[0, T]$	Space of continuous functions on $[0, T]$
$C(0, T; l_r^2)$	Space of continuous functions defined on $[0, T]$ in the norm of l_r^2 with value in l_r^2
$\mathbb{C}$	Set of Complex Numbers
$\rightarrow$	Converges (approaches) to
$\dot{x}$	First derivative of x
$\frac{\partial z}{\partial t}$	First partial derivatives
$x(\cdot)$	Function of $x(t)$, $0 \leq t \leq T$
$[a, b]$	$\{x = (1-t)a + tb : 0 \leq t \leq 1\}$
(a, b)	$\{x = (1-t)a + tb : 0 < t < 1\}$
$\ f\ _p = (\int_{\mathbb{R}} f ^p dx)^{1/p}$	p -norm
$\ H\ $	Norm of the matrix H
H^*	The Transpose of the matrix H

CHAPTER 1

INTRODUCTION

This chapter presents the background about differential game problems. It also contains important definitions and standard results used. Then followed by motivation of study. Objectives of the study, scope and limitation and organization of the thesis are also discussed.

1.1 Introduction

Differential game of pursuit and evasion problems are a special kind of mathematical problems related to the modeling and analysis of conflicts in the background of dynamical systems. The game usually consists of two players with different goals. The goal of the 1st-player is to capture the 2nd-player in some sense, while that of the 2nd-player tries all the possibilities to escape from this capture. For instance, the capture could be minimizing the distance between the 1st-player and 2nd-player as much as possible while that of the avoidance is opposite. Conventionally, we call the 1st-player to be a pursuer and 2nd-players to be an evader.

The game of pursuit and evasion examines conflict problems in systems which driven by a model. A model that describes the behavior of the players which is determined by the players input through their respective control functions contained in the model. The model is usually a system of differential equation, and each player tries as much as possible to control the state of the system so that the goal will be achieved.

Differential game problems that require to finding the condition for which pursuer can capture the evader is called pursuit differential problem. Similarly, a differential game problems that requires to finding the condition for which the evader can avoid the capture from pursuer is called evasion differential problem. The system of differential equation is usually the space where pursuit and evasion differential game problems played.

Pursuit and evasion differential game are differential game of kind in which a pursuer attempts to capture evader in a minimal possible time and the evader attempts to avoid that by maximizing the capture time. A very good example of pursuit and evasion differential game is the popular Lion and Man problem. The game considers a lion and a man to be the pursuer and the evader respectively, enclosed in a circular environment. The problem required to address some questions:

1. How long can the evader (that is a man) escapes from the pursuer (that is a lion) ?

2. If the evader (i.e. a man) succeeds to escape from the pursuer (i.e. a lion) all the time, what is the least distance can the evader (i.e. a man) has between the him and the pursuer (i.e. a lion) ?

Question 1 and 2 are formulated from the man's view point, while question 3 and 4 below are formulated from the lion point of view:

3. How long does it take for the pursuer (i.e. a lion) to capture the evader (i.e. a man) ?
4. If the pursuer (i.e. a lion) does not succeed in capture the evader (i.e. a man), how close to evader (i.e. a man) can pursuer (i.e. a lion) go?

The problem that involves question 1 and 2 is referred to as evasion differential game problem. Like wise the problem concerning question 3 and 4 is called pursuit differential game problem. This problem has been investigated thoroughly in works such as Croft (1964), Flynn (1973), Lewin (1986), and Bollobas et al. (2012).

The theory of differential games constitutes a group of problems that related to game theory and optimal control theory. Differential game related to optimal control theory in the sense that optimal control problem consists of a single control function in the model and a single criterion to be optimized. However, control problem can be extended to a differential game by introducing control function of the second player to the game model. Indeed, differential game problems represent a class of optimal control problem in cases where there are more than one control function or player.

In differential game problems, the control functions are usually subjected to constraints reflecting a natural phenomenon. Conventionally, the constraints could either be integral or geometric. If the players control belong to the subset of $\mathbb{R}^n$, then they are said to be subjected to geometric constraint. A constraint is referred to as integral if the resources of the player are bounded.

Nowadays, the differential game has become an active research area with many applications. It has been applied to solve practical problems related to artificial intelligence, economics, engineering, military operation and so. For example, it has been employed for an aircraft combat, missile control, and other military strategies. There are a tremendous amount of applications in the field of mathematical finance and marketing. Most recent developments include adding stochastic to differential games. Stochastic differential game models are increasingly used in various fields ranging from market development, competition policy and investment. Another potential application includes searching the building for intruders and surgical operations to name a few.

1.2 Standard Results

In the present section, we present some standard results and basic definition which will be used in the next chapters.

1.2.1 Hilbert space

Definition 1.2.1 Let X be a complex linear space, then a function $\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}$ is referred to as the inner product, if it satisfied the following axioms:

- i. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$.
- ii. $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$.
- iii. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, the complex conjugate of $\langle x, y \rangle$.
- iv. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

where $x, y, z \in X$ and λ is a complex number.

A linear space X which has an inner product $\langle \cdot, \cdot \rangle$ is called an inner product space denoted by $(X, \langle \cdot, \cdot \rangle)$.

Example 1.2.2 A finite dimensional space (Euclidean space) $\mathbb{R}^n$ with dot product

$$\langle (x_1, x_2, \dots, x_n), (y_1, y_2, \dots, y_n) \rangle = \sum_{k=1}^n x_k y_k,$$

is an inner product space.

(Pedersen (1999))

Definition 1.2.3 Let $\{x_n\}$ be a sequence in an inner product space $(X, \langle \cdot, \cdot \rangle)$. A sequence $\{x_n\}$ is said to be a Cauchy sequence if, for every positive number $\varepsilon > 0$, there is an integer $N_\varepsilon > 0$ such that $\|x_n - x_m\| < \varepsilon$ for all natural number $m, n > N_\varepsilon$.

Brian et al. (2001)

Definition 1.2.4 A Hilbert space is an inner-product space in which every Cauchy sequence in the space converges to a point in the space. In other words, complete inner-product space is called Hilbert space.

Ponnusamy (2002)

Example 1.2.5 $\mathbb{C}^n$ is a Hilbert space with an inner-product defined by

$$\langle x, y \rangle = \sum_{k=1}^n x_k y_k$$

where $x = x_1, x_2, \dots, x_n$, $y = y_1, y_2, \dots, y_n$ is finite dimensional space in $\mathbb{C}^n$.

Example 1.2.6 Let E be a measurable subset of $\mathbb{R}$. Then the space of all square integrable function denoted by $L_2(E)$ with inner product defined by

$$\langle f, g \rangle = \int_E f \bar{g} d\mu$$

is Hilbert space.

Definition 1.2.7 Let X be a linear space. A norm on X is a real-valued function with function $\|\cdot\|$ on X satisfying the following axiom:

- i. $\|x\| \geq 0 \forall x \in X$,
- ii. $\|x\| = 0$ if and only if $x = 0$
- iii. $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$, and for all scalar λ
- iv. $\|x + y\| \leq \|x\| + \|y\| \forall x, y \in X$

A linear space X with a norm $\|\cdot\|$ on it is called a normed linear space denoted by $(X, \|\cdot\|)$. The norm is also referred to as the length of the vector x . Rao (2006).

Theorem 1.1 Every inner-product space is a normed linear space with norm defined by

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

For this proof see Rao (2006)

Definition 1.2.8 The sequence space l^p ($1 \leq p \leq \infty$) for which the norm for the sequence $\{x_n\} \in l^p$ is defined by

$$\|x\|_p = \begin{cases} (\sum_{n=1}^{\infty} |x_n|^p)^{1/p} < \infty, & \text{if } 1 \leq p < \infty, \\ \sup_{1 \leq n < \infty} |x_n| < \infty, & \text{if } p = \infty, \end{cases}$$

is normed space.

The norm defined for $1 \leq n < \infty$ is called l^p norm (or simply p -norm) and for $p = \infty$ is called l^∞ norm or simply sup-norm. Ponnusamy (2002)

1.2.2 Standard Inequalities

Theorem 1.2 (Cauchy-Schwartz inequality).

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner-product space. Cauchy-Schwartz inequality states that

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \cdot \langle y, y \rangle,$$

for all vectors $x, y \in X$.

Rao (2006)

Example 1.2.9 In Euclidean space $\mathbb{R}^n$ with the standard inner product, the Cauchy-Schwartz inequality is

$$\sum_{i=1}^n x_i y_i \leq \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \left(\sum_{i=1}^n y_i^2 \right)^{1/2}$$

Example 1.2.10 For the inner product space of square-integrable complex-valued functions, Cauchy-Schwartz inequality is

$$\left| \int f(x) \overline{g(x)} dx \right|^2 \leq \int |f(x)|^2 dx \int |g(x)|^2 dx$$

where $\overline{g(x)}$ is the conjugate of $g(x)$.

Theorem 1.3 (Minkowski's inequality).

Let $p \geq 1$ and $f, g \in L^p(\mathbb{R})$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

For the proof see Pedersen (1999)

Example 1.2.11 (Minkowski's sum inequality).

If $p \geq 1$ and $x_k, y_k \in \mathbb{R}$, $k = 1, 2, \dots$, then

$$\left[\sum_{k=1}^n |x_k + y_k|^p \right]^{1/p} \leq \left[\sum_{k=1}^n |x_k|^p \right]^{1/p} + \left[\sum_{k=1}^n |y_k|^p \right]^{1/p}$$

This result is true for infinite sum and for the details, see Rao (2006).

Example 1.2.12 (Minkowski's integral inequality).

If $p \geq 1$ and $f, g \in C[a, b]$, then

$$\left[\int_a^b |f(x) + g(x)|^p \right]^{1/p} \leq \left[\int_a^b |f(x)|^p \right]^{1/p} + \left[\int_a^b |g(x)|^p \right]^{1/p}.$$

For the details see Ponnusamy (2002)

1.2.3 Measurable Functions

Definition 1.2.13 Let X be a set. A collection Σ of subsets of X is called σ -algebra on set X , if it satisfies the following properties:

i. X belongs to Σ , i.e.

$$X \in \Sigma.$$

ii. if A belong to Σ , then the complement of A belongs to Σ , i.e.

$$A^c = \{x \in X \mid x \notin A\} \in \Sigma.$$

iii. if A_k is a sequence of Σ , then the union of the all A_k belongs to Σ , i.e.

$$A_k \in \Sigma \Rightarrow \bigcup_{k=1}^{\infty} A_k \in \Sigma.$$

The members of Σ are called measurable sets.

Definition 1.2.14 Let Σ be a σ -algebra over set X . A function $\mu : \Sigma \rightarrow \mathbb{R}$ is called a measure if it satisfies the following:

i. $\mu(E) \geq 0$ for all $E \in \Sigma$,

ii. $\mu(\cup_{k \in I} E_k) = \sum_{k \in I} \mu(E_k)$, for all countable $\{E_k\}_{k \in I}$ pairwise disjoint members of Σ ,

iii. $\mu(\emptyset) = 0$.

Definition 1.2.15 Let X be a set and let Σ be a σ -algebra defined on X . A set X together with Σ is called a measurable space and is denoted as (X, Σ) .

Definition 1.2.16 Let (X, Σ_1) and (Y, Σ_2) be measurable spaces. A measurable function is a function $f : (X, \Sigma_1) \rightarrow (Y, \Sigma_2)$ such that $f^{-1}(E) \in \Sigma_1$ for all $E \in \Sigma_2$. i.e. preimage of any measurable set is measurable.

Definition 1.2.17 *In measure theory, a property holds almost everywhere, if the set of elements for which the property does not hold has a measure zero.*

For example, if $f : [a, b] \rightarrow \mathbb{R}$ is a monotonic function, then it is differentiable almost everywhere. Aliprantis and Burkinshaw (1981).

1.3 Motivation

The differential game of pursuit and evasion problem is derived from the fact that the trajectories of pursuer and evader are given as the solution of differential equations. Also, the strategies of pursuer and evader are given as control functions occurring in the differential equations. The system of differential equations describing differential game problem could be ordinary or partial differential equations. The later seems to be of broader applications from the fact that many physical system and biological process are often governed by partial differential equations, where the state of the system is a function of time.

In the early works such as Bukovskii (1961) and Wang (1965), variation technique was employed to study differential game problem described by partial differential equations. Some further and recent works on the differential game problem described by partial differential equations include that of Chernous'ko (1996) and Tukhtasinov (1995). In each of these works decomposition method is used to reduce the problem described by partial differential equations to the one described by an infinite system of ordinary differential equations.

The later can be studied in an independent framework. For example, Ibragimov and Hasim (2010) and Ibragimov (2013a), studied differential game problem described by an infinite system of differential equations with positive coefficients. This serves as a motivations factor in attracting my attention to study differential game problem described by an infinite system of ordinary differential equations, in the case of negative coefficients.

1.4 Objectives of the Thesis

The following are objectives of the thesis:

- To obtain solution of optimal pursuit differential game problem of the following IS-ODE

$$\dot{z}_k + \lambda_k z_k = -u_k + v_k, z_k(0) = z_k^0, k = 1, 2, \dots,$$

where $z_k, u_k, v_k \in \mathbb{R}^1, z^0 = (z_1^0, z_2^0, \dots) \in l_{r+1}^2, u_1, u_2, \dots$, are control parameters of

the pursuer and $v_1, v_2, \dots$, are that of the evader, $\{\lambda_1, \lambda_2, \dots\}$ are bounded sequence of negative numbers. Pursuer tries to force the state of the system toward the origin and the evader tries to avoid this, where we obtain an equation to find the optimal pursuit time and construct the optimal strategies for players.

- To obtain a solution of pursuit game for an IS-1st-Order of ordinary differential equations with integral constraints in the case of negative coefficients

$$\dot{z}_k + \lambda_k z_k = -u_k + v_k, z_k(0) = z_k^0, k = 1, 2, \dots, \quad (1.1)$$

where $z^0 = (z_1^0, z_2^0, \dots) \in l_{r+1}^2$. We denote a given state $z^1 = (z_1^1, z_2^1, \dots) \in l_{r+1}^2$, of the system (1.1) and all other parameters are stated as in the first system above. In this case, pursuer actions to bring the state of the system toward another and evader actions to stop this.

- To obtain condition of completion of pursuit with geometric and integral constraints described by IS-1st-Order of ordinary differential equations of the space l_2

$$\dot{z}_k = -\lambda_k z_k - u_k + v_k, z_k(0) = z_k^0, k = 1, 2, \dots, \quad (1.2)$$

where $z_k, u_k, v_k \in \mathbb{R}^1$, $z^0 = (z_1^0, z_2^0, \dots) \in l_2$, $\{\lambda_1, \lambda_2, \dots\}$ are assumed to be any real numbers. We denote a given state $z^1 = (z_1^1, z_2^1, \dots) \in l_2$, of the system (1.2). Pursuer tries to brings the state of the system to coincides with another state of the space l_2 .

- To obtain guaranteed pursuit time for an IS-2-Systems of 1st-Order of ordinary differential equations in Hilbert space l_2

$$\begin{aligned} \dot{x}_k &= -\beta_k x_k - \alpha_k y_k - u_{1k} + v_{1k}, \\ \dot{y}_k &= \alpha_k x_k - \beta_k y_k - u_{2k} + v_{2k}, \\ x_k(0) &= x_k^0, y_k(0) = y_k^0, k = 1, 2, \dots \end{aligned}$$

where, α_k, β_k are real numbers, $\beta_k \geq 0$ with initial state $x^0 = (x_1^0, x_2^0, \dots) \in l_2$, $y^0 = (y_1^0, y_2^0, \dots) \in l_2$. We denote $x^1 = (x_1^1, x_2^1, \dots) \in l_2$, $y^1 = (y_1^1, y_2^1, \dots) \in l_2$ be another state, a control parameters of the pursuer is given by $u = (u_{11}, u_{12}, u_{21}, u_{22}, \dots)$, and that of the evader is given by $v = (v_{11}, v_{12}, v_{21}, v_{22}, \dots)$. In the game, the goal of the pursuer is to force the state of the system toward another state and the evader actions in the opposite. Integral constraints are imposed on the control functions of the players

1.5 Scope and Limitation

In this thesis, we focus our attention on pursuit and evasion differential game problem described by infinite systems of differential equations in Hilbert space and limited only

to systems of infinite first-order ordinary differential equations.

1.6 Organization of the Thesis

This thesis is organized into eight chapters. The remaining part of the thesis is presented as follows:

In Chapter 2, we give an introduction to the chapter. We brief on a history of the emergence of differential games and review the related literature on some important works.

Chapter 3 begins with an introduction. The solution of 1st-order linear equation is presented. The proof of existence-uniqueness theorem to the considered systems is also discussed.

Chapter 4 focuses on the solution of optimal pursuit time for an IS-1st-Order ordinary differential equations with negative coefficients, where the control functions of the players are subjected to integral constraints. At first, we give an introduction to the chapter and then proceed with a solution of control problem involving infinite system of ordinary differential equations. The chapter is then concluded.

After introducing the chapter, we study a control problem described by infinite system of differential equations in Chapter 5. The condition of completion of pursuit in differential game problem described by IS-1st-Order ordinary differential equations with integral constraints are then presented. The chapter ends with a brief conclusion.

In Chapter 6, after introducing the chapter, then we focuses to the solution of pursuit game problem described by infinite system of differential equations where the control geometric and integral constraints are imposed to the control functions of the player. Furthermore, the chapter includes control problems described by infinite system of differential equations with geometric and integral constraints. The chapter ends with conclusion.

In Chapter 7, we present differential game pursuit time for an IS-2-Systems of 1st-Order in Hilbert space. Following an introduction, control problem described by an an IS-2-Systems of 1st-Order are considered. Each component of the control functions of players is subjected to integral constraints. An equation to find guaranteed pursuit time is obtained. Finally, we conclude the chapter.

Last but not the least, we give the general conclusion and some future research works

in Chapter 8.



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