UNIVERSITI PUTRA MALAYSIA

EXISTENCES AND UNIQUENESS OF SOLUTIONS FOR SOME CLASSES OF ITERATIVE FRACTIONAL FUNCTIONAL INTEGRAL AND DIFFERENTIAL EQUATIONS

FATEN HASAN MOHAMMED DAMAG

FS 2018 8

EXISTENCES AND UNIQUENESS OF SOLUTIONS FOR SOME CLASSES OF ITERATIVE FRACTIONAL FUNCTIONAL INTEGRAL AND DIFFERENTIAL EQUATIONS

By

FATEN HASAN MOHAMMED DAMAG

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

November 2017

DEDICATIONS

To my father who is not here to see my success: Dad your daughter has realized your dream so now you will be happy.
"YEMEN".

&

To my mother Saaedah, who has helped and supported me and given me the strength and determination to live my dream. Mom I love you very much.
"YEMEN".

&

To my brothers, Adul Rahman, Mobrok, Mousa, Haroun, Bader, my sisters, Amira, Hanan, Abeer, Wahibah, thank you all for everything that you have been to me. I love you all.
"YEMEN".

&

To my family and all whom I love.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

EXISTENCES AND UNIQUENESS OF SOLUTIONS FOR SOME CLASSES OF ITERATIVE FRACTIONAL FUNCTIONAL INTEGRAL AND DIFFERENTIAL EQUATIONS

By

FATEN HASAN MOHAMMED DAMAG

November 2017

Chairman: Professor Adem Kilicman, PhD
Faculty: Science

The iterative functional equations are important classes and deal with fractional differential and integral equations which involve the composition of unknown. The related theory for some classes of iterative fractional integral and differential equations are not established yet and there are still many open problems in this field.

In this thesis, we focus on some classes of initial value problems in the iterative fractional integral and differential equations. The existence and uniqueness of the solutions for such equations, including their qualitative behavior are investigated.

Fixed Point Theorems, Schauder Fixed Point Theorem, Banach Fixed Point Theorem, Browder-Ghode-Kirk Fixed Point Theorem, Schaefer Fixed Point Theorem, Burton fixed point theorem, Weakly Picard Operator, Power Series, Non Expansive Operators and function g-Non Expansive Operators are used to prove the existences and uniqueness of solutions for particular classes of iterative fractional integral and differential equations.

We establish the sufficient conditions for the existence of solutions for three types of iterative fractional equations i.e iterative fractional differential equations, iterative fractional integro-differential equations and iterative fractional integral equations. Further, the uniqueness and existence of solutions are proved. The convergence of solutions for special type of iterative fractional differential equations is also studied.

The existences and uniqueness of solutions for generalized iterative fractional differential equations, iterative fractional differential equations with state dependent, and system of iterative fractional differential equations, and the existences and uniqueness of solutions for generalized classes of iterative fractional integral equations and system of iterative fractional integral equations are proved.

We also examine the existences and uniqueness of solutions for nonlinear iterative fractional differential equations by testing the convergence of solutions. The sufficient conditions for nonlinear quadratic iterative integral equations are established. A new type of stability based on the Burton fixed point theorem for the general iterative fractional differential equations is investigated. In order to verify the theorems, several examples are provided.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KEWUJUDAN DAN KEUNIKAN PENYELESAIAN bAGI BEBERAPA KELAS PERSAMAAN KAMIRAN DAN PEMBEZAAN FUNGSIAN PECAHAN BERLELAR

Oleh

FATEN HASAN MOHAMMED DAMAG

November 2017

Pengerusi: Adem Kilicman, PhD
Fakulti: Sains

Persamaan fungsi berlelar adalah kelas penting dan berkaitan dengan persamaan pembezaan dan kamiran pecahan yang melibatkan komposisi yang tak diketahui. Teori yang berkaitan untuk beberapa kelas persamaan kamiran dan pembezaan pecahan berlelar belum ditubuhkan dan masih terdapat banyak masalah terbuka dalam bidang ini.

Dalam tesis ini, kami memberi tumpuan kepada beberapa kelas masalah nilai awal dalam persamaan kamiran dan pembezaan pecahan berlelar. Kewujudan dan keunikan penyelesaian untuk persamaan tersebut, termasuk kelakuan kualitatif mereka disiasat.

Teorem Titik Tetap, Teorem Titik Tetap Schauder, Teorea Titik Tetap Banach, Teorem Titik Tetap Browder-Ghode-Kirk, Teorem Titik Tetap Schaefer, Teorem Titik Tetap Burton, Operator Weakly Picard, Siri Kuasa, Operator Tak Mengembang dan fungsi g -Operator Tak Mengembang digunakan untuk membuktikan kewujudan dan keunikan penyelesaian untuk kelas tertentu bagi persamaan kamiran dan pembezaan pecahan berlelar.

Kami menubuhkan syarat kecukupan untuk kewujudan penyelesaian untuk tiga jenis persamaan pecahan berlelar iaitu persamaan pembezaan pecahan berlelar, persamaan kamiran-pembezaan pecahan berlelar dan persamaan kamiran pecahan berlelar. Seterusnya, keunikan dan kewujudan penyelesaian dibuktikan. Penumpuan penyelesaian untuk persamaan pembezaan pecahan jenis khas juga dikaji.

Kewujudan dan keunikan penyelesaian bagi persamaan pembezaan pecahan berlelar teritlak, persamaan pembezaan pecahan berlelar dengan keadaan bersandar, dan sistem persamaan pembezaan pecahan berlelar, dan kewujudan dan keunikan penyelesaian untuk kelas teritlak bagi persamaan kamiran pecahan berlelar dan sistem persamaan kamiran pecahan berlelar persamaan dibuktikan.

Kami juga meneliti kewujudan dan keunikan penyelesaian bagi persamaan pembezaan pecahan berlelar tak linear dengan menguji penumpuan penyelesaian. Syarat kecukupan untuk persamaan kamiran berlelar kuadratik tak linear ditubuhkan. Jenis kestabilan baharu berdasarkan Teorem Titik Tetap Burton untuk persamaan pembezaan pecahan berlelar umum diselidiki. Untuk mengesahkan teorem tersebut, beberapa contoh disediakan

ACKNOWLEDGEMENTS

"My gratitude, first of all, is due to Allah, for giving me inspiration and ability to achieve this work".

I would like to extend my gratitude and thanks to my supervisor Prof. Dr. Adem Kilicman for his support and patience and for opening the " mathematics door" for me; for teaching me modern mathematics and guiding me through my PhD. program,

I would like to put on record my thanks to my supervisory committee, Dr.Rabah W. Ibrahim, Dr. Fadzilah Bt Md Ali, and Assoc.Dr. Siti Hasana Bt Sapar, I am grateful for your support and patience in teaching me so many useful things. My thanks are also due to the staff members of the Mathematics Department for your help and encouragement.

Also, I owe much to my family: my late father, who fired my ambition with his faith in me; my loving who has always given me her love and support; and my brothers and sisters from whom I have drawn so much strength and support I love you all.

I would also like to a big thank you to all my friends and my staff of Univeresiti Taiz Yemen such as my supervisor in Master program Prof. Dr. Adel Ahmed and my friend Dr. Saad and to the staff of Universiti Putra Malaysia for their assistance in ways big and small. God bless you all.

Finally, I would like to thank everybody who was important to the successful realization of this thesis, as well as I express my apology that I could not mention you all personally.

I certify that a Thesis Examination Committee has met on 30 November 2017 to conduct the final examination of Faten Hasan Mohammed Damag on his thesis entitled "Existence and Uniqueness of Solutions for Some Classes of Iterative Fractional Functional Integral and Differential Equations" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

Members of the Thesis Examination Committee were as follows:

Fudziah binti Ismail, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Leong Wah June, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Nik Mohd Asri bin Nik Long, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Chaudry Masood Khalique, PhD
Professor
North-West University
South Africa
(External Examiner)

NOR AINI AB. SHUKOR, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 27 February 2018

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy.

The members of the Supervisory Committee were as follows:

Adem Kilicman, PhD
Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Siti Hasana Bt Sapar,PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Fadzilah Bt Md Ali, PhD
Senior lecturer
Faculty of Science
Universiti Putra Malaysia
(Member)

Rabah W. Ibrahim, PhD
Senior lecturer
Institute of Mathematical Science
University of Malaysia
(Member)

ROBIAH BINTI YUNUS, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

Declaration by graduate student

I hereby confirm that:

- this thesis is my original work;
- quotations, illustrations and citations have been duly referenced;
- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
- intellectual property from the thesis and copyright of thesis are fully-owned by Universiti Putra Malaysia, as according to the Universiti Putra Malaysia (Research) Rules 2012;
- written permission must be obtained from supervisor and the office of Deputy Vice-Chancellor (Research and Innovation) before thesis is published (in the form of written, printed or in electronic form) including books, journals, modules, proceedings, popular writings, seminar papers, manuscripts, posters, reports, lecture notes, learning modules or any other materials as stated in the Universiti Putra Malaysia (Research) Rules 2012;
- there is no plagiarism or data falsification/fabrication in the thesis, and scholarly integrity is upheld as according to the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) and the Universiti Putra Malaysia (Research) Rules 2012. The thesis has undergone plagiarism detection software.

Signature: ______________________ Date: ______________________

Name and Matric No.: ______________________________________

Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ____________________________
Name of Chairman of Supervisory Committee: ____________________________

Signature: ____________________________
Name of Member of Supervisory Committee: ____________________________

Signature: ____________________________
Name of Member of Supervisory Committee: ____________________________

Signature: ____________________________
Name of Member of Supervisory Committee: ____________________________

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

Γ	Gamma function
FC	Fractional calculus
ODEs	Ordinary differential equations
PDEs	Partial differential equations
E_{β}	Mittag-Leffler function of one-parameter function
FDEs	Fractional differential equations
FIEs	Fractional integrals equations
erfc	Complementary error function
$E_{\beta,\gamma}$	Mittag-Leffler function of two-parameter function
IFIEs	Iterative fractional integral equations
IFDEs	Iterative fractional differential equations
IVP	Initial value problem
BGK	Browder-Ghode-Kirk fixed point theorem
PO	Picard operator
WPO	Weakly Picard operator
R-L	Riemann-Liouvilles
IIEs	Iterative integral equations
IDEs	Iterative differential equations
VIEs	Volterra Integral equations
FVIIEs	Fractional Volterra iterative integral equations
BVP	Boundary value problem

CHAPTER 1

INTRODUCTION

1.1 Fractional Calculus

Fractional Calculus (FC) is a branch of mathematical that has been developed in the traditional definitions of calculation of integral operators and derivatives. The FC is also referred to by several other names, among them generalized integral and differential calculus and calculus of arbitrary order. The name "FC "has been maintained from the time when it meant calculus of rational order (Podlubny, 1998; Malinowska et al., 2015).

Fractional Calculus was started three centuries ago (Sabatier et al., 2007; Malinowska et al., 2015). FC can be considered an old subject and at the same time it is a novel one. It was back in the 17^{th} century from a few speculations of Leibniz and Euler, and gradually advanced until the present (Machado et al., 2011). The most common notations for $\beta - th$ order derivative of a function $v(s)$ defined in interval (a, b) are $v^{(\beta)}(s)$ or $D^{\beta}v(s)$. Negative values of β correspond to fractional integrals (Podlubny, 1998; Loverro, 2004).

The past few decades have seen the increasing application of FC in pure mathematics and other scientific fields. However, it would be wrong to classify FC as a newcomer to science for in actual fact its origin is almost as old as that of classical calculus itself (Oldham and Spanier, 1974; Loverro, 2004). Over the past decade, the FC has been acknowledged as one of the greatest tools for describing long memory processes. The fractional models are useful in engineering, chemistry, mechanics, electric power, biology and physics, as well as in pure mathematics (Hilfer, 2000; Loverro, 2004). The most significant of these models are defined by differential equations with derivatives of fractional order. Their development has greater complexity compared with the classical integer-order case and studying the correspondence theory is a highly demanding task (Loverro, 2004; Machado et al., 2011). Despite the fact that some outcomes of qualitative analysis of fractional differential equations are similarly obtainable, numerous classical methods face difficulty in enforcing directly to the fractional differential equation. Therefore, there is a need to develop specifically new theories and methods to address the problem of the difficulty encountered in research. Compared to the classical theory of differential equations, work on the theory of fractional differential equations is still in its infancy (Zhou, 2014).

1.1.1 Historical Preface of Fractional Calculus

The concept of derivatives of fractional order first appeared in a notable communication between L'Hospital and Leibniz in 1695. So, the birthday of FC was on September 30, 1695. On that day, L'Hopital asked Leibniz, "What would the result be if $n = \frac{1}{2}$?"Leibniz's replied: "An apparent paradox, from which one day useful consequences will be drawn."It was thus that, FC came into existence. Following L'Hospital's and Leibniz's first enquiry, fractional calculus was exclusively a field only for the best minds in mathematics. The question raised by Leibniz for some derivatives of fractional order was a topical issue for over 300 years (Ross, 1975; Kiryakova, 1993; Loverro, 2004; Malinowska et al., 2015).

FC is an old topic since, starting from some speculations of Leibniz in 1695 and Euler in 1730, it has been developed up to nowadays. A list of scientists had interested for provided important contributions up in this field, includes: Euler, Lagrange, Laplace, Fourier, Abel, Liouville, Riemann, Greer, Holmgren, Grunwald, Letnikov, Sonin, Laurent, Nekrassov, Krug, Hadamard, Heaviside, Pincherle, Hardy and Littlewood, Weyl, Lèvy, Marchaud, Davis, Zygmund, Love, Erdèlyi , Kober, Widder, Riesz and Feller (see Podlubny (1998); Zhou (2014); Malinowska et al. (2015)). Nevertheless, studying the derivatives with non-integer order was in the literature until 1819, when Lacroix (1800) presented an introduction to fractional derivative on the basis of the conventional expression for the $n-th$ derivative of the power function (Kiryakova, 1993). Over the years, FC has become a very attractive subject for mathematicians, and several varying forms of fractional (i.e. non integer) differential operators have been introduced: the Grunwald-Letnikov, Riemann-Liouville Hadamard, Caputo, Riesz (see Oldham and Spanier (1974); Kiryakova (1993); Podlubny (1998); Hilfer (2000)), and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas et al. (2004) or variable order fractional operators introduced by Samko and Ross (1993).

In 2010, an interesting perspective to the subject, unifying all mentioned notions of fractional derivatives and integrals, was introduced in Agrawal (2010) and later studied in Bourdin et al. (2014), Klimek and Lupa (2013), Odzijewicz et al. (2012b,a, 2013).

The first application of semi-derivatives (derivatives on the order of half) was made by Abel in 1823 (see Miller and Ross (1993)). These applications of FC are in relation to the integral equation for solving the problem tautochronous, related to the determination of the shape of the curve so that the time of descent of frictionless point mass sliding down along the curve as a result of gravity is not influenced by the starting point. Recent decades have shown that derivatives of arbitrary order and integral are suitable for the description of the real materials properties, such as in the case of polymers. New fractional order models are more satisfactory than the former integrated command. The derivatives of fractional order are great tools for describing the properties of memory and several processes and hereditary material, while with the integrated control models such effects are neglected (Samko et al., 1993; Podlubny, 1998; Gorenflo and Mainardi, 2000; Malinowska et al., 2015).

In the second half of the $20th$ century, much research in FC was published in the engineering literature. In fact, recent progress of FC issues dominates modern examples of applications in differential and integral equations, physics, signal processing, fluid mechanics, viscoelasticity, mathematical biology and electrochemistry (Debnath, 2004; Odzijewicz et al., 2012a). The majority of the mathematical theories that can be applied when examining FC already existed before the advent of the last millennium. However, the last century was the time when the most significant developments occurred by way of application in engineering and various scientific fields. Mathematics has sometimes had to be modified to meet what was demanded by physical reality (Debnath, 2003; Loverro, 2004; Gorenflo and Mainardi, 2008; Dalir and Bashour, 2010).

1.1.2 Different Definitions of Fractional Calculus

FC has some different definitions (see (Oldham and Spanier, 1974; Ross, 1975; Miller and Ross, 1993; Mainardi, 1997; Podlubny, 1998; Debnath, 2003; Sabatier et al., 2007)) as:

Definition 1. *The Abel integral representation is defined by*

$$\int_0^s \frac{v'(\tau)}{(s-\tau)^\beta} d\beta = \Psi(s)$$

for arbitrary β and then

$$v(s) = \frac{1}{\Gamma(1-\beta)} \cdot \frac{d^{-\beta}\Psi(s)}{ds^{-\beta}}. \tag{1.1}$$

Definition 2. *The Riemann fractional integral is defined by*

$$D^{-\beta}v(s) = \frac{1}{\Gamma(\beta)} \int_c^s \frac{G(\mu)}{(s-\mu)^{1-\beta}} d\mu. \tag{1.2}$$

Definition 3. *The Cauchy integral formula are defined by*

$$G^{(m)}(s) = \frac{m!}{2i\pi} \int_c^{s^+} \frac{G(\mu)}{(s-\mu)^{m+1}} d\mu, \tag{1.3}$$

and substituted m by β to get

$$D^\beta v(s) = \frac{\Gamma(\beta+1)}{2i\pi} \int_c^{s^+} \frac{G(\mu)}{(s-\mu)^{1+\beta}} d\mu. \tag{1.4}$$

The proper definition of fractional integral operator is Riemann-Liouvilles

Definition 4. *The Riemann-Liouvilles ($R-L$) derivatives of fractional order is defined by*

$$D_a^{\beta} G(s) = \frac{1}{\Gamma(m-\beta)} [\frac{d}{du}]^m \int_a^s \frac{G(\mu)}{(s-\mu)^{\beta-m+1}} d\mu, \ \ (m-1) \leq \beta < m, \tag{1.5}$$

in which m is an integer and β is a real number.

Definition 5. *The Grunwald-Letnikove derivatives of fractional order is defined by*

$$D_a^{\beta} G(s) = \lim_{h \to 0} \frac{1}{h^{\beta}} \sum_{i=0}^{\frac{s-\beta}{h}} (-1)^i \left(\begin{array}{c} \beta \\ i \end{array} \right) G(t-ih). \tag{1.6}$$

Definition 6. *The Caputo derivatives of fractional order is defined by*

$$C_a^{D^{\beta}} G(\tau) = \frac{1}{\Gamma(m-\beta)} \int_a^{\tau} \frac{G^{(m)}(\mu)}{(\tau-\mu)^{\gamma-m+1}} d\mu, \ \ (m-1) \leq \beta < m, \tag{1.7}$$

where m is an integer and β is a real number.

Comparison between Riemann-Liouvilles ($R-L$) and Caputo derivatives.
The definitions of $R-L$ and Caputo are the popular definitions of fractional derivative, and so most of the studies in this area used them. On the other hand, there are differences between them such as:-

(1) If the function is positive then the Caputo sense and Riemann fractional directives are overlapping

(2) If the function is not positive then computation of fractional derivatives are different, in particular at zero.

(3) Both of them are not commutative.

$${}_a^C D_t^{\beta} ({}_a^C D_t^n f(t)) ={}_a^C \ D_t^{\beta+n} f(t).(n = 1, 2, \ldots; m-1 < \beta < m. \tag{1.8}$$

$${}_a D_t^n ({}_a D_t^{\beta} f(t)) ={}_a \ D_t^{\beta+n} f(t).(n = 1, 2, \ldots; m-1 < \beta < m. \tag{1.9}$$

The interchange of the differentiation operators in formulas (1.8) and (1.9) is allowed under different conditions

$${}_a^C D_t^{\beta} ({}_a^C D_t^n f(t)) ={}_a^C \ D_t^n ({}_a^C D_t^{\beta} f(t)) ={}_a^C \ D_t^{\beta+n} f(t). \tag{1.10}$$

$$f^{(n)} = 0, \ \ s = m, (m+1) + \cdots, n. \ n = 1, 2, \ldots; m-1 < \beta < m.$$

$${}_a D_t^n ({}_a D_t^{\beta} f(t)) ={}_a \ D_t^{\beta} ({}_a D_t^n f(t)) ={}_a \ D_t^{\beta+n} f(t). \tag{1.11}$$

$$f^{(n)} = 0, \ \ s = m, (m+1) + \cdots, n. \ n = 1, 2, \ldots; m-1 < \beta < m.$$

(4) The $R-L$ fractional derivative and the Caputo fractional derivative are also particular cases of the following sequential derivative

$$ {}_aD_t^{\beta} f(t) = \frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt}\,{}_aD_t^{(n-\beta)} f(t),\ (n-1) \leq \beta < n \tag{1.12}$$

$$ {}_a^CD_t^{\beta} f(t) = {}_a^CD_t^{(n-\beta)}\frac{d}{dt}\frac{d}{dt}\cdots\frac{d}{dt} f(t),\ (n-1) < \beta \leq n \tag{1.13}$$

The properties of the R-L derivatives and the Caputo derivatives of the same cumulative order β are different due to the different sequence of differential operators $\frac{d}{dt}$ and ${}_aD_t^{(n-\beta)}$.

(5) The derivative of Caputo at a constant c is zero, whereas $R-L$ fractional derivative to a constant c is nonzero but is equal to

$$D_s^{\beta} c = \frac{c(s-a)^{-\beta}}{\Gamma(1-\beta)}.$$

(6) The Laplace transform technique in $R-L$ requires the knowledge of the (bounded) initial values of the fractional integral $J^{n-\beta}$ and of its integer derivatives of order $k = 1, 2, \ldots, n-1$. In Caputo requires the knowledge of the (bounded) initial values of the function and of its integer derivatives of order $k = 1, 2, \ldots, n-1$ in analogy with the case when $\beta = n$.

(7) The Table 1.1 explains some differences between the properties of $R-L$ and Caputo definitions.

Property	Riemann-Liouvill	Caputo
$g(t) = c = const$	$\mathrm{D}^{\beta} c = \dfrac{c.s^{-\beta}}{\Gamma(1-\beta)}, c = const$	$\mathrm{D}^{\beta} c = 0, c = const$
Representation	$\mathrm{D}_+^{\beta} g(s) = D^n J^{n-\beta} g(s)$	$D_+^{\beta} g(s) = J^{n-\beta} D^n g(s)$
Interpolation	$\lim_{\beta\to n} D_+^{\beta} g(s) = g^{(n)}(s)$ $\lim_{\beta\to n-1} D_+^{\beta} g(s) = g^{(n-1)}(s)$	$\lim_{\beta\to n} D_+^{\beta} g(s) = g^{(n)}(s)$ $\lim_{\beta\to n-1} D_+^{\beta} g(s) = g^{(n-1)}(s)$ $-g^{(n-1)}(0)$
Linearity	$D^{\beta}(\lambda g(s) + h(s)) = \lambda D^{\beta} g(s)$ $+D^{\beta} h(s)$	$D_+^{\beta}(\lambda g(s) + h(s)) = \lambda D_+^{\beta} g(s)$ $+D_+^{\beta} h(s)$
Non-commutation	$D^m D^{\beta} g(s) = D^{\beta+m}$ $\neq D^{\beta} D^m g(s)$	$D_+^{\beta} D^m g(s) = D_+^{\beta+m}$ $\neq D^m D^{\beta} g(s)$
Laplace transform	$L\left\{D^{\beta} g(s); u\right\} = u^{\beta} G(u) -$ $\sum_{i=0}^{m-1} u^i [D^{\beta-i-1} g(s)]_{s=0}$	$L\left\{D_+^{\beta} g(s); u\right\} = u^{\beta} G(u) -$ $\sum_{i=0}^{m-1} u^{\beta-i-1} g^{(i)}(0)$
Leibniz	$D^{\beta}(g(s)h(s)) =$ $\sum_{i=0}^{\infty} \binom{\beta}{i} (D^{(\beta-i)} g(s)) h^{(i)}(s)$	$D_+^{\beta}(g(s)h(s)) =$ $\sum_{i=0}^{\infty} \binom{\beta}{i} (D^{(\beta-i)} g(s)) h^{(i)}(s)$ $-\sum_{i=0}^{n-1} \dfrac{s^{i-\beta}}{\Gamma(i+1-\beta)} \left((g(s)h(t))^{(i)}(0)\right)$

Table 1.1: Comparison between Riemann-Liouvilles and M. Caputo

In this thesis, we will use the $R-L$ and Caputo definitions in establising the theorem and solutions. On the other hand, the $R-L$ fractional derivative could hardly interpret the physical of the initial conditions required for the initial value problems that embody fractional differential equations. Furthermore, that operator has the advantages of rapid convergence, greater stability, and is more accurate in developing various types of numerical algorithms (Kilbas et al., 2006).

Remark 1. *From definitions 4 and 6, we can easily get*

$$
\begin{aligned}
D^{\beta}s^{\mu} &= \frac{s^{\mu-\beta}}{\Gamma(1-\beta+\mu)}, \\
D^{-\beta}s^{\mu} &= \frac{s^{\mu+\beta}}{\Gamma(1+\beta+\mu)}, \\
D^{\beta}A &= A\frac{s^{-\beta}}{\Gamma(1-\beta)}, \\
D^{\beta}A &= 0, \\
D^{-\beta}A &= A\frac{s^{\beta}}{\Gamma(1+\beta)}.
\end{aligned}
$$

1.1.3 Fractional Integral

The integral equation is an equation in which an unknown function appears under an integral sign. In the other hand, the fractional integral equation $(FIEs)$ is an equation in which an unknown function appears under fractional integral sign (Davis (1962); Miller and Ross (1993); Podlubny (1998); Prüss (2013)). In our scoring Cauchy formula reads

$$J_a^n g(s) = g_n(s) = \int_a^s \frac{(t-\tau)^{n-1}}{(n-1)!} g(\tau)d\tau, \;\; s>0, \;\; n\in N, \tag{1.14}$$

where N is the set of positive integers. Based on this definition it should be noted that $g_n(s)$ vanishes at $s=0$ with its derivatives of order $1,2,...,n-1$. For convention, it is required that $g(s)$ and from now on $g_n(s)$ be a causal function, (i.e. identically vanishing for $s<0$).

Naturally, it is necessary to extend the above formula from the positive integer values of the index for positive real values using the Γ function, and in fact, observing that $(n-1)! = \Gamma(n)$. By the introduction of arbitrary positive real numbers β, one can introduce the fractional integral of order $\beta > 0$:

$$J_a^{\beta} g(s) = \int_a^s \frac{(s-\tau)^{\beta-1}}{\Gamma(\beta)} g(\tau)d\tau, \;\; s>0, \;\; \beta\in R^{+}, \tag{1.15}$$

where R^{+} is the set of non negative real numbers. For complementation we introduce $G^0 := J^0 = I$ (Identity operator), (i.e. we mean $G^0 g(s) = g(s)$). Moreover, and $J^{\beta}g(0^{+})$ means the limit (if there is) of $J^{\beta}g(s)$ for $s\to 0^{+}$; this limit may be infinite.

The formulation of the integrated fractional involves some major properties, which in turn demonstrate the importance in the solution of equations including integrals and derivatives of fractional order.
First, setting $\beta = 0$ for an operator identity, we have

$$J^0 g(s) = g(s). \tag{1.16}$$

Moreover, repeated Cauchy's integral equation can be seen as

$$J^i J^j = J^{i+j} = J^j J^i,\ i, j \in N^+, \tag{1.17}$$

which can be generalized as

$$J^\beta J^\gamma = J^{\beta+\gamma} = J^\gamma J^\beta,\ \beta, \gamma \in R^+. \tag{1.18}$$

The one pre-supposed condition placed upon a function $g(s)$ that needs to be fulfilled for these and other similar properties to remain true, is that $g(s)$ be a causal function, i.e. that it is disappearing for $s \leq 0$. Another feature of $R - L$ integral emerges after the function $\Psi_\beta(s)$ is introduced by

$$\Psi_\beta(s) = \frac{s^{\beta-1}}{\Gamma(\beta)} \Rightarrow \Psi_\beta(s) * g(s) = \int_0^s \frac{(s-\tau)_+^{\beta-1}}{\Gamma(\beta)} g(\tau) d\tau, \tag{1.19}$$

where s_+ denotes the function vanishes for $s \leq 0$ and therefore Eq.(1.18) is a causal function (Oldham and Spanier (1974); Loverro (2004); Gutiérrez et al. (2010)).

1.1.4 Fractional Derivative

The differential equations can be traced to the mid 17^{th} century, when the calculation was discovered independently by Newton and Leibniz (Robinson, 2004). The differential equation is any equation that possesses derivatives, either ordinary derivatives or partial derivatives. $ODEs$ is a differential equation for a function of a single variable,(example, $(w(s))$), while a partial differential equation $(PDEs)$ is a differential equation for a function of several variables, (example, (w, z, v, s)) (Chasnov (2009); Perko (2013); Hale and Lunel (2013)). On the other hand, the fractional differential equation $(FDEs)$ an offshoot of mathematics studying the properties of derivatives of non integer orders (i.e, it is including $\iota < \beta < \iota + 1$ order) (Miller and Ross (1993); Podlubny (1998); Hilfer (2000); Zhou (2014)).

After defining the fractional integral, it is easy to define the fractional differential of any positive real power by combining the standard differential operator with a fractional integral order between 0 and 1. Simply select the operator that will be applied first. For instance, one can deduce the derivative of order $\frac{3}{2}$ of a function $g(s)$ as follows:

$$D^{\frac{3}{2}}g(s) = D^2 J^{\frac{1}{2}} g(s),$$
$$D^{\frac{3}{2}}g(s) = J^{\frac{1}{2}} D^2 g(s).$$

Both approaches provide the basis of two different definitions of fractional derivatives. The first definition in the integrated fractional before differentiation is applied, is called the fractional derivative of $R-L$ (see the definition 4). The second, which applies the integrated fractional, subsequently called the derivative of Caputo (see the definition 6)(Ishteva et al. (2005); Vance (2014)).

The fractional derivatives provide an excellent instrument for describing memory and inborn properties of different materials and processes. In particular, the issue of $FDEs$ has been gaining much prominence and attracted attention. Remarkable case studies offer the main theoretical tools for quality analysis of $FDEs$, and simultaneously show where the interconnection of the disparity between the models of integer differential and models of fractional differential, are. A great deal of work on the existence, periodicity of solutions, stability, and optimal solutions for all types of $FDEs$ has been reported (Agarwal et al. (2010); Ahmad and Nieto (2011); Ahmad et al. (2011)).

The fractional derivatives have many applications in our lives and in many scientific fields such as engineering, physics, biology and economics, etc.(Miller and Ross (1993); Podlubny (1998); Robinson (2004); Agarwal and O'Regan (2008); Coddington (2012); Hale and Lunel (2013)).

1.1.5 Concepts Related to the Fractional Calculus

An understanding of the definitions and use of FC will become clearer when discussing some quick comparatively simple definitions, that arise in studying these notions such as the Gamma function, Complementary Error Function, and the Mittag-Leffler function which are processed in the following: -

1.1.5.1 Gamma Function (Γ)

The Gamma function is essentially linked to FC by definition (see the fractional integral), where if can be clearly explained as a simple generalization factor for all real numbers defined defined as follows:

$$\Gamma(y) = \int_0^{\infty} e^z z^{y-1} dz, \; y \in R. \tag{1.20}$$

The beauty of the Γ function can be found in its properties. The Eq.(1.21) is

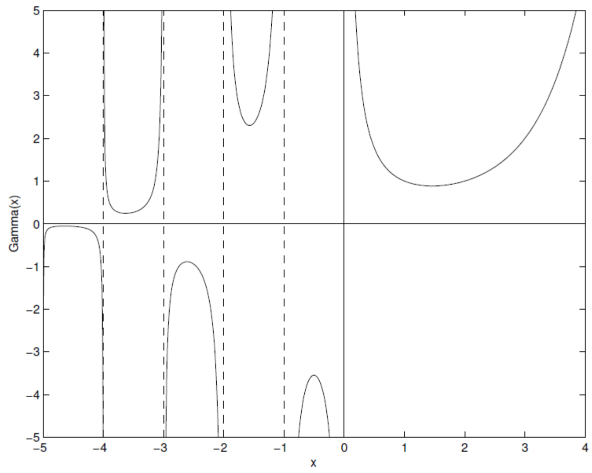

Figure 1.1: Gamma Function of the real argument

unique in which the value for any quantity is, consequence of the form of the integral, equivalent to that quantity $y-1$ times the Γ of the quantum minus one.

$$\Gamma(y+1) = y\Gamma(y);\ \texttt{for}\ y \in N^{+};\ \Gamma(y) = (y+1)!. \tag{1.21}$$

This can be demonstrated by integrating in parts. The result of this relation for integer values of y is the definition for factorial. Figure 1.1 shows the Γ function at and around zero. Note that for non positive integer values, the Γ function goes to infinity, by yet is defined at non-integer values. From the Γ function we can define the function $\Psi(s)$, which was helpful for showing alternate forms of the integral of fractional order. $\Psi(s)$ is obtained by Eq.(1.19) (Loverro, 2004).

$$\Psi_{\beta}(s) = \frac{s^{\beta-1}}{\Gamma(\beta)}. \tag{1.22}$$

1.1.5.2 Complementary Error Function (erfc)

The complementary error function (see Podlubny (1998); Ishteva (2005)) is an entire function, introduced as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-t^2} dt. \tag{1.23}$$

The Figure 1.2 explains the erfc function. Special values of the complementary error functions are

(1) $erfc(\infty) = 0,$ (2) $erfc(-\infty) = 2,$ (3) $erfc(0) = 1.$

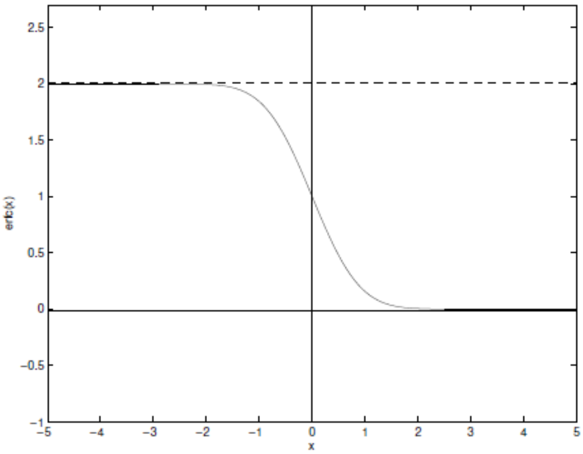

Figure 1.2: The Complementary Error Function

The following relationships are worth mentioning

(1) $erfc(-x) = 2 - erfc(x),$ (2) $\int_0^\infty erfc(x)dx = \frac{1}{\sqrt{\pi}},$

(3) $\int_0^\infty erfc^2(x)dx = \frac{2-\sqrt{2}}{\sqrt{\pi}}.$

1.1.5.3 Mittag-Leffler Function

The Mittag-Leffler function is a powerful function, which is widely used in the field of FC. It is a natural exponential from solution of differential equations of an integer order. The Mittag-Leffler function has a similar role in solving differential equations of non-integer order. Actually, the exponential function itself is specifically, one of an infinite set, of this seemingly ubiquitous function (Debnath (2003)), while the Γ function generalizes the factorial function, and the Mittag-Leffler Function generalizes the exponential function where $\beta = 1$. First, introduced as a one-parameter function by the series

$$E_\beta(v) = \sum_{i=0}^{\infty} \frac{v^i}{\Gamma(i\beta + 1)}, \ \beta > 0. \tag{1.24}$$

The exponential function corresponds to $\beta = 1$. Figure 1.3 shows the Mittag-Leffler function for different values of β. Also, it is common for Mittag-Leffler function to be represented as the function of two arguments, β and γ, by taking the following formula:-

$$E_{\beta,\gamma}(v) = \sum_{i=0}^{\infty} \frac{v^i}{\Gamma(i\beta + \gamma)}, \ \beta > 0, \ \gamma > 0. \tag{1.25}$$

Most widespread form of the function, but it is not always needed when coupled

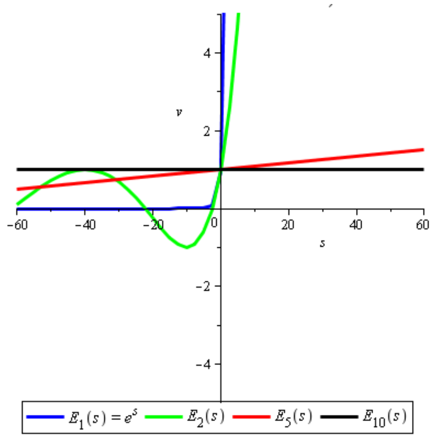

Figure 1.3: The MittagLeffler Function for different $\beta, \gamma = 1$.

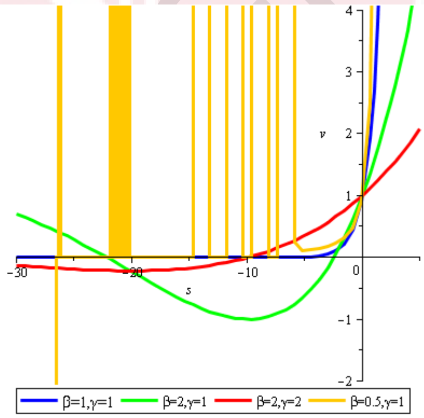

Figure 1.4: Examples of the two parameters function of Mittag-Leffler type.

with fractional differential equations. Some of its interesting properties can be found in Miller and Ross (1993); Podlubny (1998); Loverro (2004)

Example 1. *We show some examples types of Mittag-Leffler function*

$$(1)\ E_{1,1}(x) = e^x, \qquad (2)\ E_{2,1}(x^2) = \cosh(x),$$

$$(3)\ E_{2,2}(x^2) = \frac{\sinh(x)}{x}, \qquad (4)\ E_{\frac{1}{2},1}(x) = e^{x^2} erfc(-x),$$

where erfc(x) is the complementary error function and Figure 1.4 illustrates the Mittag-Leffler type functions give in Example 1 (Ishteva (2005)).

1.2 Iterative Functional Equations

The term “iterative functional equations”adopted here means “functional equations in one variable”. With the development of nonlinear dynamic systems, focus is on not only the movement behavior but also the process that touches on the problem of the inverse of the iterative operation. For the iterative functional equation generally it means identical equation consisting of unknown functions, and composite operations. The iterative functional equation has turned into a modern branch of mathematics that deals with the differential equation, integral equations and different dynamic systems. Many mathematical models in the form of the equation are the data iterative functions of the economy, biology, gemology, astronomy, geology and other disciplines (Liu and Mai (2002); Ibrahim (2012); Podisuk (2013a)).

At a given $g(u)$ of the $j-th$ iteration is the function which is composed with itself j times

$$g^0(u) = u,\ g^1(u) = g(u),\ g^2(u) = g(g^1(u)),\ \ldots,\ g^j(u) = g(g^{j-1}(u)),$$

and denoted by $g^j(u)$. A iterative functional equation refers to the equation, where analytically from a function $g(u)$ is not known, but its composition with itself is known.

Despite their prevalence, they are very hard to solve, and there are some mathematical tools for analysis. Generally, resolving mathematical equations implies deep vision and experimentation with different reformulations and substitutions. In the literature, there is in-depth research on iteration and functional equations, classic books (Kuczma et al. (1990); Nechepurenko (1997); Aczél (2014)), and Latest polls and studies (Kobza (2000); Liu and Mai (2002); Brown et al. (2003); Liu (2011a); Kruchinin and Kruchinin (2013)).

In this thesis, to focus of the mathematical model of biological experiments, so the iterative equations for all types are the best methods to study the mathematical model of biological experiments. In this studies, is interested about two types of iterative functional equation as iterative fractional integral equation $IFIEs$ and iterative fractional differential equation $IFDEs$.

1.2.1 Iterative Integral Equations of Order Fractional

The $IFIEs$ is no less important than integral iterative equations in terms of their applications and use in our daily lives but are more generalized, and accurate and applied (see the definition 1.15) (Muresan (2003); El-Sayed et al. (2009); Lauran (2011b, 2013a)).

The theory of integral equations describes numerous applications in many events and real world issues. As such, the Iterative integral equations ($IIEs$) in order fractional are usually applied in engineering, mathematics, physics, economics and biology. Bear in mind that the integral equations of fractional order are to create an interesting and important branch of the theory of integral equations. The theory of these integral equations has been developed significantly in recent years with the theory of differential equations of fractional order (Podlubny (1998); Babakhani and Daftardar-Gejji (2003)). The $IFIEs$ are open problems as they have not been studied until now so that we the first work to study this problem.

1.2.2 Iterative Differential Equations of order Fractional

Amongst the fractional differential equations ($FDEs$), there is a major class involving iteration, so the researcher will address previous studies of a special class of $FDEs$, called $IFDEs$. The $IFDEs$ are no less important than iterative differential equations ($IDEs$) in terms of their applications and use in our daily lives but are more generalized and accurate and applied because they use very little differentiation (μ) and are trapped between ι and $\iota + 1$ (Norkin et al. (1973); Ross (1975); Podlubny (1998)).

Over the past 30 years, there has been a lot of work done in the field of $IDEs$. On the other hand, the study of $IFDEs$ began no more than 10 years ago and is therefore very modern. The $IFDEs$ has emerged in a wide range of science and technical applications, including modeling of problems of natural and social sciences, such as physics, biology and economics (Stephan, 1969; Podisuk, 2013b,a).

The $IFDEs$ such as certain kinds of delay differential equations depending on the state, have distinctive features (Egri and Rus, 2007; Liu and Tunç, 2015), and the arguments of deviation depending on both the state variables w and time τ, it is of importance in the theory and practice. Furthermore, the $IFDEs$ providing an effective means for finding approximate solutions have been studied for their practical applications for a long time (Wang et al. (2013)).

1.3 Existence and Uniqueness

The primary objective of this section of the which is to introduce a unified area for researching the theories of the existence and uniqueness of a variety of $IDEs$ and $IIEs$ in fractional order. For all we know, many complex processes in nature and technology are described by functional differential equations that dominate today due to the functional components in the equations used to consider prehistory or after-effect impact. Different classes of functional differential equations and integral equations are of fundamental importance in many problems in the bionomics, epidemiology, electronics, theory of neural networks,biological ,automatic control, etc.(Norkin et al. (1973); Oregan (1995); Delbosco and Rodino (1996); Zhou (2014)).

The theorems of the existence and uniqueness of most differential equations and integral equations are generally produced by the technique of fixed point, for example, by the fixed point theorem of Schauder or by the principle of mapping contractions, etc. (Li et al. (2001); Li and Cheng (2009); Lauran (2011a); Granas and Dugundji (2013)).

1.4 Statement of the Problems

In order to find a good mathematical model for biological experiments researchers to study approximation to solutions. It was proved that fractional modeling is more appropriate and efficient method. Thus the statement of the problem in the present study are as follow:
(a) Extended some classes iterative functional in classical calculus to fractional order and use the same technique to determine the existence and uniqueness solutions as:

(1) $$w'(s) = g(s, w^{[1]}(s), w^{[2]}(s), \dots, w^{[j]}(s)). \tag{1.26}$$

(2) $$w'(s) = w^{i}(s). \tag{1.27}$$

(3) $$w'(s) = g\Big(s, w\Big(f(s) + h(w(s))\Big)\Big). \tag{1.28}$$

(4) $$\begin{cases} w_i'(s) = f_i\Big(s, w_1(s), w_2(s), w_1(w_1(s-\mu_1)), w_2(w_2(s-\mu_2))\Big), \ s \in [c,d], \\ w_i(s) = \theta_i(s), \ i = 1,2, \ s \in [s_0 - \mu_i, s_0]. \end{cases} \tag{1.29}$$

(5) $$, \quad \begin{gathered} w'(s) = g\Big(s, w(w(s)), \int_{s_0}^{s} K(s,r) w(w(r)) dr\Big), \\ w(s_0) = v_0, \end{gathered} \tag{1.30}$$

also

$$\begin{aligned} & w'(s) = g\Big(s, w(w(s)), w(w'(s)), \int_{s_0}^{s} K(s,r) w(w(r)) dr\Big), \\ & w(s_0) = w_0. \end{aligned} \tag{1.31}$$

(6) $$w(s) = h(s) + \int_{s_0}^{s} \frac{(s-u)^{\beta}}{\Gamma(\beta+1)} K(u, w(w(u))) du, \tag{1.32}$$

$$w(s) = h(s) + \int_{s_0}^{s} \frac{(s-u)^{\beta}}{\Gamma(\beta+1)} K(u, w(u), w(w(u))) du, \tag{1.33}$$

$$w(s) = h(s) + \int_{s_0}^{s} \frac{(s-u)^{\beta}}{\Gamma(\beta+1)} K(u, w(w(w(u)))) du, \tag{1.34}$$

$$w(s) = h(s) + \int_{s_0}^{s} \frac{(s-u)^{\beta}}{\Gamma(\beta+1)} K(u, w(w(u)), w(w(w(u))))du, \tag{1.35}$$

and

$$w(s) = h(s) + \int_{s_0}^{s} \frac{(s-u)^{\beta}}{\Gamma(\beta+1)} K(u, w, w(w(u), w(w(w(u))))du. \tag{1.36}$$

(7)
$$\begin{cases} w_1(s) = h_1(s) + \displaystyle\int_0^s \frac{(s-r)^{\beta}}{\Gamma(\beta+1)} k_1(r, w_1(r), w_2(r), w_1(w_1(r)))dr \\ w_2(s) = h_2(s) + \displaystyle\int_0^s \frac{(s-r)^{\beta}}{\Gamma(\beta+1)} k_2(r, w_1(r), w_2(r), w_2(w_2(r)))dr. \end{cases} \tag{1.37}$$

(b) The fractional integral and differential equations with argument depend on state variable which has been studied within the lasts 10 years. Therefore, some classes of IFDEs and IFIEs are not established yet and there are still many open peoblems to be resolved such as:

(1)
$$\frac{d^{\beta}w(s)}{ds^{\beta}} = \aleph(s, w(s), w(w(s))). \tag{1.38}$$

(2)
$$\begin{aligned} D^{\beta}w(s) &= h(s, w(s), w(w(s)), w'(s)), \\ w(s_0) &= w_0, \ w'(s_0) - w_0', \end{aligned} \tag{1.39}$$

$$\begin{aligned} D^{\beta}w(s) &= h(s, w(w'(s))), \\ w(s_0) &= w_0, \ w'(s_0) - w_0', \end{aligned} \tag{1.40}$$

$$\begin{aligned} D^{\beta}w(s) &= h(s, w(s), w(w(s)), w'(s), w''(s)), \\ w(s_0) &= w_0, \ w'(s_0) = w_0', \ w''(s_0) = w_0'', \end{aligned} \tag{1.41}$$

and

$$\begin{aligned} D^{\beta}w(s) &= h(s, w(s), w(wv''(s))), \\ w(s_0) &= w_0. \end{aligned} \tag{1.42}$$

(3)
$$\begin{cases} D^{\beta_1}w(s) = \phi_1(s)g_1\Big(s, w(s), w(w(s)), z(s), z(z(s))\Big) \\ +\displaystyle\int_0^s \frac{(s-r)^{\alpha_1-1}}{\Gamma(\alpha_1)} g_1\Big(r, w(r), z(r), w(w(r)), z(z(r))\Big)dr \\ D^{\beta_2}z(s) = \phi_2(s)g_2\Big(s, w(s), w(w(s)), z(s), z(z(s))\Big) \\ +\displaystyle\int_0^s \frac{(s-r)^{\alpha_2-1}}{\Gamma(\alpha_2)} g_2\Big(r, w(r), z(r), w(w(r)), z(z(r))\Big)dr \\ w(0) = a, z(0) = b, s \in [0,1], \end{cases} \tag{1.43}$$

(4)
$$w(s) = h(s, w(s)) + g(s, w(s)) \int_0^s \frac{(s-u)^{\beta}}{\Gamma(\beta+1)} K(u, w(u), w(w(u)))du. \tag{1.44}$$

1.5 Objectives of the Study

1. To extend the solutions of the Eqs.(1.26) and (1.27) to fractional order and prove the existence and uniqueness solutions by using Schauder fixed point theorem, Banach fixed point theorem and power series, and investigate a new type of stability method based on the Burton fixed point theorem of fractional differential equation of Caputo type.

2. To find the approximate and convergent solution by using power series of the equation (1.38).

3. To extend the solutions of the equations (1.28) and the system (1.29) to fractional order and prove the existence and uniqueness of the solutions by using Schauder fixed point theorem and Weakly Picard Operator technique.

4. To find the existence and uniqueness solutions of the equations (1.39) − (1.42) by using Browder-Ghode-Kirk fixed point theorem and Banach fixed-point theorem.

5. To extend the solutions of the equations (1.30) and (1.31) to fractional order and prove the existence and uniqueness solutions by using non-expansive operators and fixed point theorem of the equation.

6. To examine the existence and uniqueness solution of the system (1.43) by using the Banach fixed point theorem and Schaefer fixed point theorem.

7. To extend the solutions of the equations (1.32) − (1.36) and the system (1.37) to fractional order and prove the existence and uniqueness solutions by using function g non-expansive operators technique, non expansive operator technique and Schauder's fixed point theorem.

8. To propose a new method for the existing solution of equation (1.44) by using the principle of contraction and Schaefer fixed point theorem and establish conditions sufficient for existing solutions.

1.6 Outline of the Thesis

This study is deals with the qualitative aspects of the solution and not the numerical solutions, so it will focus on the existence of the solution, and uniqueness (i.e. no necessity to find a solution) because this study is theoretical and not numerical.

In this thesis, the focus is on the iterative fractional differential equations and the iterative fractional integral equations. Such equations are significant models in studying infection and are associated with studying the motion of the charged particles in delayed interaction. Also, these equations are of particular interest to mathematical models in biology, which have a large and fundamental influence on our lives such as the growth of bacteria. Furthermore, the $IFIEs$ and the $IFDEs$ provide an effective means of finding approximate solutions that have been studied for their practical applications for a long time. This thesis is organized in six self-contained chapters.

Chapter 1 provides a short history of FC and its applications. The fractional order differentiation and integration is almost as old as the classical calculation itself; however, there seems to be a striking lack of knowledge in this area among most mathematicians. This chapter also offers the iterative equations and their applications and explains why we have chosen this to be the subject of this study. Furthermore, this chapter presents the research objectives which will be discussed in the chapters that follow.

Chapter 2 reviews previous studies that focused on this area, which insists of $IDEs$ or $IFDEs$ and $IIEs$, or $IFIEs$.

Chapter 3 focuses on the achievement of the first objectives. In this chapter the focus is on the theoretical framework used to study the existence and uniqueness of the generalized $IFDEs$ and Burton stability. There is also a study of the Local existence and unique results of simple $IFDEs$ to get approximate solutions for non-linear $IFDEs$ and convergence of this equation by power series. Also, in this chapter, studies the solutions of the Cauchy problem derived from the $IFDEs$ with state-dependent by using fixed point theorem of Schauder. Finally, also presented are solutions of the system of $IFDEs$ achieved by using $R-L$ differential operator and the technique of the weakly Picard operator, and some applications of these theorems are provided.

Chapter 4 is concerned with the achievements of the second set of objectives, including the study of some classes of $IFDEs$ involving the first order and second order derivatives by using $R-L$ differential operator and fixed point theorem of Browder-Ghode-Kirk and Banach fixed-point theorem. It also involves the integro-differential equations by using R.L. differential operator and non-expansive operators and fixed point theorem. Then,the chapter proceeds to study the solutions of $IFDEs$ involving the derivatives and integral by using $R-L$ differential operator and non-expansive operators and fixed point theorem. Finally, , there is the investigation of the existence and uniqueness of the solution of system iterative integro-differential equations using Banach's theorem of fixed-point and Schaefer's fixed-point theorem with some examples of each type provided.

Chapter 5 discusses the achievement of the third objective, which studies the existing results for some $IFIEs$, fractional quadratic integral iterative equations and system of $IFIEs$ by using R.L. differential operator and non-expansive operators and fixed point theorem and the principle of contraction and Schaefer's theorem of fixed point and Schauder's fixed point theorem and some examples of each type are given.

Chapter 6 concludes the thesis and makes recommendations for future research.

REFERENCES

Aczél, J. (2014). *On applications and theory of functional equations*. Academic Press.

Agarwal, R. P., Benchohra, M., and Hamani, S. (2010). A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. *Acta Applicandae Mathematicae*, 109(3):973–1033.

Agarwal, R. P. and O'Regan, D. (2008). *An introduction to ordinary differential equations*. Springer Science & Business Media.

Agrawal, O. P. (2010). Generalized variational problems and euler–lagrange equations. *Computers and Mathematics with Applications*, 59(5):1852–1864.

Ahmad, B. and Nieto, J. J. (2011). Anti-periodic fractional boundary value problems. *Computers & Mathematics with Applications*, 62(3):1150–1156.

Ahmad, B., Ntouyas, S. K., and Alsaedi, A. (2011). New existence results for nonlinear fractional differential equations with three-point integral boundary conditions. *Advances in Difference Equations*, 2011(1):1–11.

Babakhani, A. and Daftardar-Gejji, V. (2003). Existence of positive solutions of nonlinear fractional differential equations. *Journal of Mathematical Analysis and Applications*, 278(2):434–442.

Bacotiu, C. (2004). Data dependence of the fixed points set of weakly picard operators in generalized metric spaces, studia univ. *Babes-Bolyai, Math*, 49:15–17.

Bacotiu, C. (2008). Volterra-fredholm nonlinear systems with modified argument via weakly picard operators theory. *Carpathian J. Math*, 24(2):1–9.

Balachandran, K., Park, J., and Julie, M. D. (2010). On local attractivity of solutions of a functional integral equation of fractional order with deviating arguments. *Communications in Nonlinear Science and Numerical Simulation*, 15(10):2809 – 2817.

Banaś, J. and O'Regan, D. (2008). On existence and local attractivity of solutions of a quadratic volterra integral equation of fractional order. *Journal of Mathematical Analysis and Applications*, 345(1):573–582.

Banaś, J. and Rzepka, B. (2007). Monotonic solutions of a quadratic integral equation of fractional order. *Journal of Mathematical Analysis and Applications*, 332(2):1371–1379.

Benchohra, M., Henderson, J., Ntouyas, S., and Ouahab, A. (2008). Existence results for fractional order functional differential equations with infinite delay. *Journal of Mathematical Analysis and Applications*, 338(2):1340–1350.

Berinde, V. (2007). *Iterative approximation of fixed points*, volume 1912. Springer.

Berinde, V. (2010). Existence and approximation of solutions of some first order iterative differential equations. *Miskolc Math. Notes*, 11(1):13–26.

Bourdin, L., Odzijewicz, T., Torres, D. F., et al. (2014). Existence of minimizers for generalized lagrangian functionals and a necessary optimality condition-application to fractional variational problems. *Differential and Integral Equations*, 27(7/8):743–766.

Brown, B. A., Brown, A. R., and Shlesinger, M. F. (2003). Solutions of doubly and higher order iterated equations. *Journal of Statistical Physics*, 110(3-6):1087–1097.

Buica, A. (1995). Existence and continuous dependence of solutions of some functional–differential equations. In *Babes-Bolyai Univ., Cluj–Napoca, Seminar on fixed point theory*, volume 3, pp. 1–13.

Burton, T. and Zhang, B. (2014). A Schauder-type fixed point theorem. *Journal of Mathematical Analysis and Applications*, 417(2):552–558.

Chasnov, J. R. (2009). Introduction to differential equations. *The Hong Kong University of Science and Technology, Department of Mathematics*, 2012.

Cheng, S., Si, I.-G., and Xin-Ping, W. (2002). An existence theorem for iterative functional differential equations. *Acta Mathematica Hungarica*, 94(1-2):1–17.

Chidume, C. (2009). *Geometric properties of Banach spaces and nonlinear iterations*, volume 1965. Springer.

Coddington, E. A. (2012). *An introduction to ordinary differential equations*. Courier Corporation.

Cooke, K. (1967). Functional differential systems: some models and pertubation problems. In *J. Hale and J. LaSalle (eds.) International Symposium on Differential Equations and Dynamical Systems, held in Puerto Rico*. Academic Press, N.

Corduneanu, C. (1991). *Integral equations and applications*, volume 148. Cambridge University Press, Cambridge.

Cresson, J. (2007). Fractional embedding of differential operators and lagrangian systems. *Journal of Mathematical Physics*, 48(3):033504.

Dalir, M. and Bashour, M. (2010). Applications of fractional calculus. *Appl. Math. Sci*, 4(21):1021–1032.

Darwish, M. A. (2005). On quadratic integral equation of fractional orders. *Journal of Mathematical Analysis and Applications*, 311(1):112 – 119.

Darwish, M. A. (2008). On existence and asymptotic behaviour of solutions of a fractional integral equation with linear modification of the argument. *arXiv preprint arXiv:0805.1422*.

Davis, H. T. (1962). *Introduction to nonlinear differential and integral equations*. Courier Corporation.

Debnath, L. (2003). Recent applications of fractional calculus to science and engineering. *International Journal of Mathematics and Mathematical Sciences*, 2003(54):3413–3442.

Debnath, L. (2004). A brief historical introduction to fractional calculus. *International Journal of Mathematical Education in Science and Technology*, 35(4):487–501.

Delbosco, D. and Rodino, L. (1996). Existence and uniqueness for a nonlinear fractional differential equation. *Journal of Mathematical Analysis and Applications*, 204(2):609–625.

Deng, J. and Wang, J. (2013). Existence and approximation of solutions of fractional order iterative differential equations. *Central European Journal of Physics*, 11(10):1377–1386.

Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer.

Dobritoiu, M. (2004). On an integral equation with modified argument. *Studia Univ. Babes-Bolyai*, pp. 27–33.

Eder, E. (1984). The functional differential equation $x(t) = x(x(t))$. *Journal of Differential Equations*, 54(3):390–400.

Egri, E. and Rus, I. A. (2007). First order iterative functional-differential equation with parameter. *Stud. Univ. Babes-Bolyai Math*, 52(4):67–80.

Egri, E. and Rus, I. A. (2009). First order functional differential equations with state-dependent modified argument. *Studia Universitatis Babes-Bolyai, Mathematica*, (3).

El-Sayed, A. and Hashem, H. (2013). Existence results for nonlinear quadratic integral equations of fractional order in banach algebra. *Fractional Calculus and Applied Analysis*, 16(4):816–826.

El-Sayed, A., Rida, S., and Arafa, A. (2009). On the solutions of time-fractional bacterial chemotaxis in a diffusion gradient chamber. *International Journal of Nonlinear Science*, 7(4):485–492.

Fečkan, M. (1993). On a certain type of functional differential equations. *Mathematica Slovaca*, 43(1):39–43.

Ge, W. and Mo, Y. (1997). Existence of solutions to differential-iterative equation. *Journal of Beijing Institute of Technology*, 6(3):192–200.

Gorenflo, R. and Mainardi, F. (2000). Essentials of fractional calculus. *Springer Citeseer*.

Gorenflo, R. and Mainardi, F. (2008). Fractional calculus: integral and differential equations of fractional order. *arXiv preprint arXiv:0805.3823*.

Górniewicz, L. and Rozpłoch-Nowakowska, D. (1996). On the schauder fixed point theorem. *Banach Center Publications*, 35(1):207–219.

Granas, A. and Dugundji, J. (2013). *Fixed point theory*. Springer Science & Business Media.

Grande, Z. (2011). On some equations $y'(x) = f(x, y(h(x) + g(y(x))))$. *Discussiones Mathematicae, Differential Inclusions, Control and Optimization*, 31(2):173–182.

Gutiérrez, R. E., Rosário, J. M., and Tenreiro Machado, J. (2010). Fractional order calculus: basic concepts and engineering applications. *Mathematical Problems in Engineering*, 2010(ID 375858):1–19.

Hale, J. K. and Lunel, S. M. V. (2013). *Introduction to functional differential equations*, volume 99. Springer Science Business Media.

Haloi, R., Pandey, D. N., and Bahuguna, D. (2011). Existence of solutions to a non-autonomous abstract neutral differential equation with deviated argument. *J. Nonl. Evol. Equ. Appl*, 5:75–90.

Hartung, F. and Turi, J. (1997). On differentiability of solutions with respect to parameters in state-dependent delay equations. *Journal of Differential Equations*, 135(2):192–237.

Hilfer, R. (2000). *Applications of fractional calculus in physics*. World Scientific.

Ibrahim, R. W. (2012). Existence of deviating fractional differential equation. *Cubo (Temuco)*, 14(3):129–142.

Ibrahim, R. W. (2013). Existence of iterative cauchy fractional differential equation. *Journal of Mathematics*, 2013.

Ibrahim, R. W. and Darus, M. (2014). Infective disease processes based on fractional differential equation. In *Proceedings of the 3rd International Conference on Mathematical Sciences*, volume 1602, pp. 696–703. AIP Publishing.

Ibrahim, R. W., Kilicman, A., and Damag, F. H. (2015). Mixed solutions of monotone iterative technique for hybrid fractional differential equations. *arXiv preprint arXiv:1509.08238*.

Ibrahim, R. W., Kılıçman, A., and Damag, F. H. (2016). Extremal solutions by monotone iterative technique for hybrid fractional differential equations. *Turkish Journal of Analysis and Number Theory*, 4(3):60–66.

Ishikawa, S. (1976). Fixed points and iteration of a nonexpansive mapping in a banach space. *Proceedings of the American Mathematical Society*, 59(1):65–71.

Ishteva, M. (2005). Properties and applications of the caputo fractional operator. *Department of Mathematics, University of Karlsruhe, Karlsruhe*.

Ishteva, M., Boyadjiev, L., and Scherer, R. (2005). On the caputo operator of fractional calculus and c-laguerre functions. *Mathematical Sciences Research Journal*, 9(6):161.

Katugampola, U. N. (2011). New approach to a generalized fractional integral. *Applied Mathematics and Computation*, 218(3):860–865.

Kendre, S., Kharat, V., and Narute, R. (2015). On existence of solution for iterative integrodifferential equations. *Nonlinear Analysis and Differential Equations*, 3(3):123–131.

Kilbas, A. and Marzan, S. (2005). Nonlinear differential equations with the caputo fractional derivative in the space of continuously differentiable functions. *Differential Equations*, 41(1):84–89.

Kilbas, A. A., Saigo, M., and Saxena, R. (2004). Generalized mittag-leffler function and generalized fractional calculus operators. *Integral Transforms and Special Functions*, 15(1):31–49.

Kilbas, A. A. A., Srivastava, H. M., and Trujillo, J. J. (2006). *Theory and applications of fractional differential equations*, volume 204. Elsevier Science Limited.

Kiryakova, V. S. (1993). *Generalized fractional calculus and applications*. CRC Press.

Klafter, J., Lim, S., and Metzler, R. (2012). *Fractional dynamics: recent advances*. World Scientific.

Klimek, M. (2005). Lagrangian fractional mechanicsa noncommutative approach. *Czechoslovak Journal of Physics*, 55(11):1447–1453.

Klimek, M. and Lupa, M. (2013). Reflection symmetric formulation of generalized fractional variational calculus. *Fractional Calculus and Applied Analysis*, 16(1):243–261.

Kobza, J. (2000). Iterative functional equation $x(x(t)) = f(t)$ with $f(t)$ piecewise linear. *Journal of Computational and Applied Mathematics*, 115(1):331–347.

Kruchinin, D. and Kruchinin, V. (2013). Method for solving an iterative functional equation $a^{2^n}(x) = f(x)$. *arXiv preprint arXiv:1302.1986*.

Kuczma, M., Choczewski, B., and Ger, R. (1990). *Iterative functional equations*. Number 32. Cambridge University Press.

Lacroix, S. (1800). Traité du calcul différentiel et du calcul intégral (tome 3 traité des différences et des séries)[tratado de cálculo diferencial y de cálculo integral (tomo 3 tratado de diferencias y de series)]. *París, Francia: Libraire pour les Mathématiques.(Versión digital obtenida de http://gallica. bnf. fr/el 23 de agosto de 2005).*

Lakshmikantham, V., Leela, S., and Devi, J. V. (2009). Theory of fractional dynamic systems. CSP.

Lakshmikantham, V. and Vatsala, A. (2008). Basic theory of fractional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 69(8):2677–2682.

Lauran, M. (2011a). Existence results for some differential equations with deviating argument. *Filomat*, 25(2):21–31.

Lauran, M. (2011b). Existence results for some integral equation with modified argument. *General Mathematics*, 19(3):85–92.

Lauran, M. (2012). Existence results for some nonlinear integral equations. *Miskole Mathem. Notes*, 13(1):67–74.

Lauran, M. (2013a). Nonexpansive operators associated to a system of integral equations with deviating argument. *Scientific Studies and Research*, 23(2).

Lauran, M. (2013b). Solution of first iterative differential equations. *Annals of the University of Craiova-Mathematics and Computer Science Series*, 40(1):45–51.

Lepik, . (2009). Solving fractional integral equations by the haar wavelet method. *Applied Mathematics and Computation*, 214(2):468 – 478.

Li, W.-r. and Cheng, S. S. (2009). A picard theorem for iterative differential equations. *Demonstratio Mathematica*, 42(2):373–382.

Li, W. R., Cheng, S. S., and Lu, T. T. (2001). Closed form solutions of iterative functional differential equations. *Appl. Math. E-Notes*, 1(1):4.

Li, W.-T. and Zhang, S. (2002). Classifications and existence of positive solutions of higher order nonlinear iterative functional differential equations. *Journal of Computational and Applied Mathematics*, 139(2):351–367.

Liu, B. and Tunç, C. (2015). Pseudo almost periodic solutions for a class of first order differential iterative equations. *Applied Mathematics Letters*, 40:29–34.

Liu, H. and Li, W. (2008). The exact analytic solutions of a nonlinear differential iterative equation. *Nonlinear Analysis: Theory, Methods & Applications*, 69(8):2466–2478.

Liu, J. (2017). Local analytic solutions of an iterative functional differential equation with deviating arguments depending on the state derivative. *International Journal of Emerging Engineering Research and Technology*, 5(2):21–31.

Liu, L. (2011a). Existence of analytic solutions of an iterative functional equation. *Applied Mathematics and Computation*, 217(17):7245–7257.

Liu, L. X. (2011b). Analytic solutions of an iterative functional differential equation. In *Key Engineering Materials*, volume 474, pp. 2208–2212. Trans Tech Publ.

Liu, X. and Mai, J. (2002). Analytic solutions of iterative functional equations. *Journal of Mathematical Analysis and Applications*, 270(1):200–209.

Lizama, C. and Vergara, V. (2004). Uniform stability of resolvent families. *Proceedings of the American Mathematical Society*, 132(1):175–181.

Loverro, A. (2004). Fractional calculus: history, definitions and applications for the engineer. *Rapport technique, Univeristy of Notre Dame: Department of Aerospace and Mechanical Engineering*, pp. 1–28.

Machado, J. T., Kiryakova, V., and Mainardi, F. (2011). Recent history of fractional calculus. *Communications in Nonlinear Science and Numerical Simulation*, 16(3):1140–1153.

Mainardi, F. (1997). *Fractional calculus: Fractals and fractional calculus in Continuum Mechanics*. 291–348: Springer.

Malinowska, A. B., Odzijewicc, T., and Torres, D. F. (2015). *Advanced methods in the fractional calculus of variations*. Springer.

Marian, O. I. (2006). Functional-differential equations of mixed type, via weakly picard operators. *Stud. Univ. Babes-Bolyai, Math. Li (2)*, pp. 83–95.

Miller, K. S. and Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. ISBN-10: 0471588849: Wiley-Interscience.

Mishra, L. N. and Sen, M. (2016). On the concept of existence and local attractivity of solutions for some quadratic volterra integral equation of fractional order. *Applied Mathematics and Computation*, 285:174–183.

Mollapourasl, R. and Ostadi, A. (2015). On solution of functional integral equation of fractional order. *Applied Mathematics and Computation*, 270:631–643.

Muresan, V. (2003). Volterra integral equations with iterations of linear modification of the argument. *Novi Sad J. Math*, 33(2):1–10.

Muslim, M. and Bahuguna, D. (2008). Existence of solutions to neutral differential equations with deviated argument. *Electron. J. Qual. Theory Differ. Equ*, 27:1–12.

Nechepurenko, M. (1997). Iterations of real functions and functional equations. *Institute of Computational Mathematics and Mathematical Geophysics, Novosibirsk*.

Norkin, S. et al. (1973). *Introduction to the theory and application of differential equations with deviating arguments*, volume 105. Academic Press.

Odzijewicz, T., Malinowska, A. B., and Torres, D. F. (2012a). Fractional calculus of variations in terms of a generalized fractional integral with applications to physics. In *Abstract and Applied Analysis*, volume 2012. Hindawi Publishing Corporation.

Odzijewicz, T., Malinowska, A. B., and Torres, D. F. (2012b). Generalized fractional calculus with applications to the calculus of variations. *Computers and Mathematics with Applications*, 64(10):3351–3366.

Odzijewicz, T., Malinowska, A. B., and Torres, D. F. (2013). A generalized fractional calculus of variations. *arXiv preprint arXiv:1304.5282*.

Oldham, K. and Spanier, J. (1974). *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, volume 111. Elsevier.

Olver, P. J. (2008). Numerical analysis, lecture notes. *http://www-users.math.umn.edu/ olver/num*$_/$*lno.pdf*$., 15(5) : 152 - -187. Most.$

Oprea, N. (2009). Numerical solutions of first order iterative functional-differential equations by spline functions of even degree. *Scientific Bulletin of the" Petru Maior" University of Targu Mures*, 6:34.

Oregan, D. (1995). Existence results for nonlinear integral equations. *Journal of Mathematical Analysis and Applications*, 192(3):705–726.

Otrocol, D. (2006). Iterative functional-differential system with retarded argument. *Revue d'Analyse Numerique et de Theorie de l'Approximation*, 35(2):147–160.

Pata, V. (2014). Fixed point theorems and applications. Technical report, Mimeo.

Perko, L. (2013). *Differential equations and dynamical systems*, volume 7. Springer Science Business Media.

Podisuk, M. (2002). On simple iterative ordinary differential equations. *Science Asia*, 28(2):191–198.

Podisuk, M. (2010). Numerical solution of iterative ordinary differential equation by integration method. In *International Conference on Mathematical and Computational Methods in Science and Engineering-Proceedings, Almeria, Spain*, pp. 57–62. Advances in Mathematical and Computional Methods.

Podisuk, M. (2013a). Application of simple iterative ordinary differential equations. *Procedia-Social and Behavioral Sciences*, 88:179–186.

Podisuk, M. (2013b). More on simple iterative ordinary differential equation. *Procedia-Social and Behavioral Sciences*, 88:187–195.

Podisuk, M. and Udomchalermpat, S. (2013). Numerical solutions of second order iterative ordinary differential equations. *Procedia-Social and Behavioral Sciences*, 88:196–204.

Podlubny, I. (1998). *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, volume 198. Academic press.

Prüss, J. (2013). *Evolutionary integral equations and applications*, volume 87. Birkhäuser.

Robinson, J. C. (2004). *An introduction to ordinary differential equations*. Cambridge University Press.

Ross, B. (1975). A brief history and exposition of the fundamental theory of fractional calculus. In *Fractional calculus and its applications*, pp. 1–36. Springer.

Rus, I. A. (1993). Weakly picard mappings. *Comment. Math. Univ. Carolinae*, 34(4):769–773.

Rus, I. A. and Egri, E. (2006). Boundary value problems for iterative functional-differential equations. *Studia Univ. Babes-Bolyai Math*, 51:109–126.

Rus, I. A. and Serban, M.-A. (2014). Some existence results for a system of operatorial equations. *Bulletin Mathematique De La Societe Des Sciences Mathematique De Roumanie*, 57(1):101–108.

Sabatier, J., Agrawal, O. P., and Machado, J. T. (2007). *Advances in fractional calculus*, volume 4. Springer.

Samko, S. G., Kilbas, A. A., and Marichev, O. I. (1993). Fractional integrals and derivatives. *Theory and Applications, Gordon and Breach, Yverdon*, 1993.

Samko, S. G. and Ross, B. (1993). Integration and differentiation to a variable fractional order. *Integral Transforms and Special Functions*, 1(4):277–300.

Scudo, F. M. (1971). Vito volterra and theoretical ecology. *Theoretical Population Biology*, 2(1):1–23.

Si, J. and Zhang, W. (2004). Analytic solutions of a class of iterative functional differential equations. *Journal of Computational and Applied Mathematics*, 162(2):467–481.

Si, J., Zhang, W., and Kim, G.-H. (2004). Analytic solutions of an iterative functional differential equation. *Applied Mathematics and Computation*, 150(3):647–659.

Si, J.-G., Li, W.-R., and Cheng, S. S. (1997). Analytic solutions of an iterative functional differential equation. *Computers & Mathematics with Applications*, 33(6):47–51.

Si, J.-g. and Wang, X.-p. (1998). Smooth solutions of a nonhomogeneous iterative functional differential equation with variable coefficients. *Journal of Mathematical Analysis and Applications*, 226(2):377–392.

Si, J.-G., Wang, X.-P., and Cheng, S. S. (2000). Nondecreasing and convex c2-solutions of an iterative functional-differential equation. *Aequationes Mathematicae*, 60(1-2):38–56.

Song, Y. and Kim, H. (2014). The solution of volterra integral equation of the second kind by using the elzaki transform. *Appl. Math. Sci*, 8(11):525–530.

Srivastava, H. M. and Owa, S. (1989). *Univalent functions, fractional calculus, and their applications*. Ellis Horwood; New York; Toronto: Halsted Press.

Stanek, S. (1995). On global properties of solutions of functional differential equation $x'(t) = x(x(t)) + x(t)$. *Dynamic Sys*, 4:263–278.

Stanek, S. (2004). Global properties of decreasing solutions of the equation $x'(t) = x(x(t)) + x(t)$. *Functional Differential Equations*, 4(1-2):p–191.

Stephan, B. H. (1969). On the existence of periodic solutions of z(t) = az(t r + k(t, z(t))) + f(t). *Journal of Differential Equations*, 6(3):408 – 419.

Talwong, S., Laohakosol, V., and Cheng, S. S. (2004). Power function solutions of iterative functional differential equations. *Applied Mathematics E-Notes*, 4:160–163.

TeBeest, K. G. (1997). Classroom note: Numerical and analytical solutions of volterra's population model. *SIAM Review*, 39(3):484–493.

Vance, D. (2014). Fractional derivatives and fractional mechanics. https://sites.math.washington.edu/ morrow/$336_14/papers/danny.pdf$.

Wang, J., Deng, J., and Wei, W. (2016). Fractional iterative functional differential equations with impulses. *Fixed Point Theory*, 17(1):189–200.

Wang, J., Fec, M., Zhou, Y., et al. (2013). Fractional order iterative functional differential equations with parameter. *Applied Mathematical Modelling*, 37(8):6055–6067.

Wang, J., Feckan, M., and Zhou, Y. (2014). Weakly picard operators method for modifed fractional iterative functional differential equations. *Fixed Point Theory*, 15:297–310.

Wang, J., Zhou, Y., and Medved, M. (2012). Picard and weakly picard operators technique for nonlinear differential equations in banach spaces. *Journal of Mathematical Analysis and Applications*, 389(1):261–274.

Wang, K. (1990). On the equation $x'(t) = f(x(x(t)))$. *Applied Mathematics Letters*, 33(3):405–425.

Wang, W. (2015). Positive pseudo almost periodic solutions for a class of differential iterative equations with biological background. *Applied Mathematics Letters*, 46:106–110.

Wang, X. and Si, J. (2001a). Differentiable solutions of an iterative functional equation. *Aequationes Mathematicae*, 61(1-2):79–96.

Wang, X.-P. and Si, J.-G. (2001b). Analytic solutions of an iterative functional differential equation. *Journal of Mathematical Analysis and Applications*, 262(2):490–498.

Wazwaz, A. (2011). *Linear and nonlinear integral equations*, volume 639. Springer.

Wazwaz, A.-M. (2010). *Partial differential equations and solitary waves theory*. Number 310400928. Springer Science and Business Media.

Yang, D. and Zhang, W. (2004). Solutions of equivariance for iterative differential equations. *Applied Mathematics Letters*, 17(7):759–765.

Yu, C. and Gao, G. (2005). Existence of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 310(1):26–29.

Zhang, P. (2012). Analytic solutions for iterative functional differential equations. *Electronic Journal of Differential Equations*, 2012(180):1–7.

Zhang, P. and Gong, X. (2014). Existence of solutions for iterative differential equations. *Electronic Journal of Differential Equations*, 2014(07):1–10.

Zhang, P. and Mi, L. (2009). Analytic solutions of a second order iterative functional differential equation. *Applied Mathematics and Computation*, 210(2):277–283.

Zhang, W. (1990). Discussion on the differentiable solutions of the iterated equation $i = 1n\lambda_i f^i(x) = f(x)$. *Nonlinear Analysis: Theory, Methods and Applications*, 15(4):387–398.

Zhou, Y. (2014). *Basic theory of fractional differential equations*, volume 6. World Scientific.