Malaysian Journal of Mathematical Sciences 12(3): 287–303 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Classification of Two-Dimensional Jordan Algebras Over $\mathbb{R}$

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Received: 7 March 2018

Accepted: 18 August 2018

ABSTRACT

In the paper we describe all (not necessarily commutative) Jordan algebra structures on a two-dimensional vector space over $\mathbb{R}$ in terms of their matrices of structure constants.

Keywords: Basis, Jordan algebra, matrix of structure constants.

1. Introduction

The classification problem of finite dimensional algebras is one of the important problems of algebra. Many authors have considered this problem for different classes of two-dimensional algebras Althoen and Hansen (1992), Althoen and Kugler (1983), Bermúdez et al. (2008) and Goze and Remm (2017). In Petersson (2000) the problem is considered for all two-dimensional algebras over any fields by a basis free (invariant) approach. Unlike the approach of Petersson (2000) the same problem is considered in Ahmed et al. (2017b) by the use of structure constants approach over algebraically closed fields. The results of Ahmed et al. (2017b) have been used in Ahmed et al. (2017a) and Ahmed et al. (2017c) to classify two-dimensional (not necessarily commutative) Jordan algebras and left (right) unital algebras over algebraically closed fields. In the present paper we represent all possible (not necessarily commutative) Jordan algebra structures on two-dimensional vector space over the field of real numbers $\mathbb{R}$. For some computations we used Computer Software Mathematica. The commutative case of two-dimensional real Jordan algebras and a complete classification of all N-dimensional non-associative Jordan algebras with $(N-3)$-dimensional, $(N-2)$-dimensional, $(N-1)$-dimensional annihilators over an algebraically closed field of characteristic $\neq 2$ have been given in Bermúdez et al. (2008), Wallace (1970) and Hegazi and Abdelwahab (2017), respectively.

2. Preliminaries

Let $\mathbb{F}$ be any field and $A \otimes B$ stand for the Kronecker product of matrices A and B over $\mathbb{F}$.

Definition 2.1. *A vector space* $\mathbb{A}$ *over* $\mathbb{F}$ *with multiplication* $\cdot : \mathbb{A} \otimes \mathbb{A} \to \mathbb{A}$ *given by* $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} \cdot \mathbf{v}$ *such that*

$$(\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) + \beta(\mathbf{v} \cdot \mathbf{w}), \quad \mathbf{w} \cdot (\alpha\mathbf{u} + \beta\mathbf{v}) = \alpha(\mathbf{w} \cdot \mathbf{u}) + \beta(\mathbf{w} \cdot \mathbf{v})$$

whenever $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{A}$ *and* $\alpha, \beta \in \mathbb{F}$*, is said to be an algebra.*

Definition 2.2. *Two algebras* $\mathbb{A}$ *and* $\mathbb{B}$ *are called isomorphic if there is an invertible linear map* $\mathbf{f} : \mathbb{A} \to \mathbb{B}$ *such that*

$$\mathbf{f}(\mathbf{u} \cdot_{\mathbb{A}} \mathbf{v}) = \mathbf{f}(\mathbf{u}) \cdot_{\mathbb{B}} \mathbf{f}(\mathbf{v})$$

whenever $\mathbf{u}, \mathbf{v} \in \mathbb{A}$.

Let $(\mathbb{A}, \cdot)$ be a m-dimensional algebra over $\mathbb{F}$ and $e = (e_1, e_2, ..., e_m)$ be its basis. Then the bilinear map $\cdot$ is represented by a matrix $A = (A_{ij}^k) \in$

$M(m \times m^2; \mathbb{F})$ as follows
$$\mathbf{u} \cdot \mathbf{v} = eA(u \otimes v),$$
for $\mathbf{u} = eu, \mathbf{v} = ev$, where $u = (u_1, u_2, ..., u_m)^T$, and $v = (v_1, v_2, ..., v_m)^T$ are column coordinate vectors of $\mathbf{u}$ and $\mathbf{v}$, respectively. The matrix $A \in M(m \times m^2; \mathbb{F})$ defined above is called the matrix of structure constants (MSC) of $\mathbb{A}$ with respect to the basis e. Further we assume that a basis e is fixed and we do not make a difference between the algebra $\mathbb{A}$ and its MSC A.

If $e' = (e'_1, e'_2, ..., e'_m)$ is another basis of $\mathbb{A}$, $e'g = e$ with $g \in G = GL(m; \mathbb{F})$, and A' is MSC of $\mathbb{A}$ with respect to e' then it is known that $A' = gA(g^{-1})^{\otimes 2}$ holds (see Ahmed et al. (2017b)). Thus, we can reformulate the concept of isomorphism of algebras as follows.

Definition 2.3. *Two m-dimensional algebras $\mathbb{A}$, $\mathbb{B}$ over $\mathbb{F}$, given by their matrices of structure constants A, B, are said to be isomorphic if there exists $g \in GL(m; \mathbb{F})$ such that $B = gA(g^{-1})^{\otimes 2}$.*

Further we consider only the case $m = 2$, for the simplicity we use $A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix}$ for MSC, where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ stand for any elements of $\mathbb{F}$ (structure constants of $\mathbb{A}$). We need the following classification result from Bekbaev (2017).

Theorem 2.1. *Any non-trivial 2-dimensional real algebra is isomorphic to only one of the following algebras given by their matrices of structure constants:*

$$A_{1,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2+1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1+1 & -\alpha_2 \end{pmatrix}, \textit{ where } \mathbf{c} = (\alpha_1, \alpha_2, \alpha_4, \beta_1) \in \mathbb{R}^4,$$

$$A_{2,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \textit{ where } \beta_1 \geq 0, \ \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3,$$

$$A_{3,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \textit{ where } \beta_1 \geq 0, \ \mathbf{c} = (\alpha_1, \beta_1, \beta_2) \in \mathbb{R}^3,$$

$$A_{4,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}, \textit{ where } \mathbf{c} = (\beta_1, \beta_2) \in \mathbb{R}^2,$$

$$A_{5,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 1-\alpha_1 & 0 \end{pmatrix}, \textit{ where } \mathbf{c} = (\alpha_1, \beta_2) \in \mathbb{R}^2,$$

$$A_{6,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 2\alpha_1-1 & 1-\alpha_1 & 0 \end{pmatrix}, \textit{ where } \mathbf{c} = \alpha_1 \in \mathbb{R},$$

$$A_{7,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \textit{where } \beta_1 \geq 0,\ \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2,$$

$$A_{8,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \textit{where } \beta_1 \geq 0,\ \mathbf{c} = (\alpha_1, \beta_1) \in \mathbb{R}^2,$$

$$A_{9,r}(\mathbf{c}) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}, \textit{ where } \mathbf{c} = \beta_1 \in \mathbb{R},$$

$$A_{10,r}(\mathbf{c}) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}, \textit{ where } \mathbf{c} = \alpha_1 \in \mathbb{R},$$

$$A_{11,r} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}, \quad A_{12,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix},$$

$$A_{13,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \end{pmatrix}, \ A_{14,r} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \ A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

3. Classification of 2-dimensional real Jordan algebras

Recall that an algebra $\mathbb{A}$ is said to be a Jordan algebra if $(\mathbf{u}\cdot\mathbf{v})\cdot\mathbf{u}^2 = \mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{u}^2)$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{A}$. In terms of its MSC A of the algebra $\mathbb{A}$ the condition above is written as follows:

$$(A(A \otimes A) - A(E \otimes A(E \otimes A)))(u \otimes e_i \otimes u \otimes u) = 0, \quad i = 1, 2, \qquad (1)$$

for $u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$

If we rewrite the condition (1) in terms of x, y and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ and use the fact that x, y are free we obtain a system of equations with respect to $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ as follows

$$
\begin{aligned}
\alpha_1\alpha_2\beta_1 - \alpha_1\alpha_3\beta_1 - \alpha_4\beta_1^2 + \alpha_2\beta_1\beta_2 &= 0\\
\alpha_2^2\beta_1 - \alpha_3^2\beta_1 + \alpha_1\alpha_4\beta_1 + \alpha_2\beta_2^2 - \alpha_1\alpha_3\beta_3 - 2\alpha_4\beta_1\beta_3 + \alpha_2\beta_2\beta_3 &= 0\\
2\alpha_2\alpha_4\beta_1 + \alpha_4\beta_2^2 - \alpha_2\alpha_3\beta_3 - \alpha_3^2\beta_3 - \alpha_4\beta_3^2 - \alpha_4\beta_1\beta_4 + \alpha_2\beta_2\beta_4 &= 0\\
\alpha_4^2\beta_1 - \alpha_3\alpha_4\beta_3 + \alpha_4\beta_2\beta_4 - \alpha_4\beta_3\beta_4 &= 0\\
\alpha_2\beta_1^2 + \beta_1\beta_2^2 - \alpha_1\beta_1\beta_3 - \beta_1^2\beta_4 &= 0\\
-\alpha_1\alpha_3\beta_1 + 2\alpha_2\beta_1\beta_2 - \alpha_1\beta_2^2 + \beta_2^3 + \alpha_1^2\beta_3 + \alpha_2\beta_1\beta_3 - \alpha_3\beta_1\beta_3 - \alpha_1\beta_2\beta_3 &\\
+\beta_2^2\beta_3 - \alpha_1\beta_3^2 + \alpha_1\beta_1\beta_4 - 2\beta_1\beta_3\beta_4 &= 0\\
-\alpha_2\alpha_3\beta_1 - \alpha_3^2\beta_1 + \alpha_4\beta_1\beta_2 - \alpha_3\beta_2^2 + \alpha_1\alpha_2\beta_3 + \alpha_1\alpha_3\beta_3 - \alpha_4\beta_1\beta_3 + \alpha_2\beta_2\beta_3 &\\
-\alpha_3\beta_2\beta_3 - \alpha_3\beta_3^2 + 2\alpha_2\beta_1\beta_4 + \alpha_3\beta_1\beta_4 - \alpha_1\beta_2\beta_4 + 2\beta_2^2\beta_4 - \beta_3^2\beta_4 - \beta_1\beta_4^2 &= 0\\
-\alpha_3\alpha_4\beta_1 + \alpha_1\alpha_4\beta_3 - \alpha_4\beta_3^2 + \alpha_4\beta_1\beta_4 - \alpha_3\beta_2\beta_4 + \alpha_2\beta_3\beta_4 + \beta_2\beta_4^2 - \beta_3\beta_4^2 &= 0\\
-\alpha_1^2\alpha_2 + \alpha_1^2\alpha_3 - \alpha_2^2\beta_1 + \alpha_1\alpha_4\beta_1 - \alpha_1\alpha_3\beta_2 - \alpha_4\beta_1\beta_2 + \alpha_1\alpha_2\beta_3 + \alpha_2\beta_1\beta_4 &= 0\\
-\alpha_1\alpha_2^2 + 2\alpha_1\alpha_3^2 - \alpha_1^2\alpha_4 - \alpha_2\alpha_4\beta_1 + \alpha_3\alpha_4\beta_1 - \alpha_2^2\beta_2 - \alpha_2\alpha_3\beta_2 - \alpha_3^2\beta_2 + &\\
\alpha_1\alpha_4\beta_2 - \alpha_4\beta_2^2 + \alpha_2\alpha_3\beta_3 + 2\alpha_1\alpha_4\beta_3 - \alpha_4\beta_2\beta_3 - \alpha_1\alpha_3\beta_4 + \alpha_2\beta_2\beta_4 + \alpha_2\beta_3\beta_4 &= 0\\
\alpha_2\alpha_3^2 + \alpha_3^3 - 2\alpha_1\alpha_2\alpha_4 - \alpha_2\alpha_4\beta_2 + \alpha_2\alpha_4\beta_3 + 2\alpha_3\alpha_4\beta_3 - \alpha_2^2\beta_4 - \alpha_2\alpha_3\beta_4 &\\
-\alpha_3^2\beta_4 + \alpha_1\alpha_4\beta_4 - \alpha_4\beta_2\beta_4 + \alpha_2\beta_4^2 &= 0\\
\alpha_3^2\alpha_4 - \alpha_1\alpha_4^2 + \alpha_4^2\beta_3 - \alpha_2\alpha_4\beta_4 &= 0.
\end{aligned} \tag{2}
$$

Therefore, the entries of MSC (structure constants) of a Jordan algebra $\mathbb{A}$ must satisfy the system of equations (2). Using (2) along with the results of Theorem 2.1 we prove the next theorem.

Theorem 3.1. *Any non-trivial Jordan algebra structure on a two-dimensional vector space over the field of real numbers is isomorphic to only one of the following pairwise non isomorphic algebras given by their matrices of structure constants:*

- $J_{1,r} = A_{2,r}\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $J_{2,r} = A_{2,r}\left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $J_{3,r} = A_{3,r}(\frac{1}{2}, 0, \frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $J_{4,r} = A_{3,r}(\frac{1}{2}, 0, -\frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$
- $J_{5,r}(\alpha_1) = A_{5,r}(\alpha_1, -1 + 2\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -1 + 2\alpha_1 & -\alpha_1 + 1 & 0 \end{pmatrix}$,

  where $\alpha_1 \in \mathbb{R}$ *and* $\alpha_1 \neq \frac{1}{10}\left(5 \pm \sqrt{5}\right)$,

- $J_{6,r}(\alpha_1) = A_{5,r}\left(\alpha_1, \sqrt{\alpha_1 - \alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1 - \alpha_1^2} & -\alpha_1 + 1 & 0 \end{pmatrix},$ *where* $0 \leq \alpha_1 \leq 1$,
- $J_{7,r}(\alpha_1) = A_{5,r}\left(\alpha_1, -\sqrt{\alpha_1 - \alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -\sqrt{\alpha_1 - \alpha_1^2} & -\alpha_1 + 1 & 0 \end{pmatrix},$ *where* $0 < \alpha_1 < 1$,
- $J_{8,r} = A_{6,r}\left(\frac{1}{10}\left(5 - \sqrt{5}\right)\right) = \begin{pmatrix} \frac{1}{10}\left(5 - \sqrt{5}\right) & 0 & 0 & 0 \\ 1 & -\frac{\sqrt{5}}{5} & \frac{1}{10}\left(5 + \sqrt{5}\right) & 0 \end{pmatrix},$
- $J_{9,r} = A_{6,r}\left(\frac{1}{10}\left(5 + \sqrt{5}\right)\right) = \begin{pmatrix} \frac{1}{10}\left(5 + \sqrt{5}\right) & 0 & 0 & 0 \\ 1 & \frac{\sqrt{5}}{5} & \frac{1}{10}\left(5 - \sqrt{5}\right) & 0 \end{pmatrix},$
- $J_{10,r} = A_{10,r}\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix},$
- $J_{11,r} = A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

Proof. To prove this theorem we solve the system of equations (2) for all MSC listed in Theorem 2.1.

For $A_{1,r}(\alpha_1, \alpha_2, \alpha_4, \beta_1) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_2 + 1 & \alpha_4 \\ \beta_1 & -\alpha_1 & -\alpha_1 + 1 & -\alpha_2 \end{pmatrix}$ the system of equations (2) becomes

$$\begin{aligned}
-\alpha_1\beta_1 - \alpha_1\alpha_2\beta_1 - \alpha_4\beta_1^2 &= 0 \\
-\alpha_1 + \alpha_1^2 - 2\alpha_1\alpha_2 + 3\alpha_1^2\alpha_2 - \beta_1 - 2\alpha_2\beta_1 - 2\alpha_4\beta_1 + 3\alpha_1\alpha_4\beta_1 &= 0 \\
-1 + \alpha_1 - 3\alpha_2 + 3\alpha_1\alpha_2 - 2\alpha_2^2 + 3\alpha_1\alpha_2^2 - \alpha_4 + 2\alpha_1\alpha_4 + 3\alpha_2\alpha_4\beta_1 &= 0 \\
-\alpha_4 + \alpha_1\alpha_4 + \alpha_1\alpha_2\alpha_4 + \alpha_4^2\beta_1 &= 0 \\
-\alpha_1\beta_1 + 2\alpha_1^2\beta_1 + 2\alpha_2\beta_1^2 &= 0 \\
-\alpha_1 + 5\alpha_1^2 - 6\alpha_1^3 - \beta_1 + 2\alpha_2\beta_1 - 6\alpha_1\alpha_2\beta_1 &= 0 \\
-1 + 4\alpha_1 - 4\alpha_1^2 + 2\alpha_1\alpha_2 - 6\alpha_1^2\alpha_2 - \beta_1 - 4\alpha_2\beta_1 - 6\alpha_2^2\beta_1 - \alpha_4\beta_1 &= 0 \\
-\alpha_1\alpha_2 - 2\alpha_2^2 - \alpha_4 + 3\alpha_1\alpha_4 - 2\alpha_1^2\alpha_4 - \alpha_4\beta_1 - 2\alpha_2\alpha_4\beta_1 &= 0 \\
2\alpha_1^2 + \alpha_1\alpha_2 - 2\alpha_2^2\beta_1 + 2\alpha_1\alpha_4\beta_1 &= 0 \\
3\alpha_1 + \alpha_2 + 7\alpha_1\alpha_2 + 6\alpha_1\alpha_2^2 + 3\alpha_1\alpha_4 - 6\alpha_1^2\alpha_4 + \alpha_4\beta_1 &= 0 \\
1 + 5\alpha_2 + 8\alpha_2^2 + 6\alpha_2^3 + 2\alpha_4 - 2\alpha_1\alpha_4 + 3\alpha_2\alpha_4 - 6\alpha_1\alpha_2\alpha_4 &= 0 \\
\alpha_4 + 2\alpha_2\alpha_4 + 2\alpha_2^2\alpha_4 + \alpha_4^2 - 2\alpha_1\alpha_4^2 &= 0.
\end{aligned} \tag{3}$$

Due to the equation 1 (of (3)) the following two cases occur:
Case 1. $\alpha_1 + \alpha_1\alpha_2 + \alpha_4\beta_1 = 0.$ Then due to the equation 4 one has $\alpha_4 = 0$, in

particular $-\alpha_1 = \alpha_1\alpha_2$, and $\alpha_1 - 2\alpha_2^2 = 0$. This implies that either $\alpha_2 = 0$ or $\alpha_2 = -1$. In the both cases the equation 11 provides a contradiction. So in Case 1 the system of equations (3) is inconsistent.
Case 2. $\beta_1 = 0$. Due to the equation 6 (of (3)) one has $\alpha_1 = 0$ or $\frac{1}{2}$ or $\frac{1}{3}$. In $\alpha_1 = 0$ case the equation 7 provides a contradiction, in $\alpha_1 = \frac{1}{2}$ the equation 2 implies $\alpha_2 = -1$ and the equation 4 implies $\alpha_4 = 0$, but the equation 3 provides a contradiction. In $\alpha_1 = \frac{1}{3}$ case the equation 2 implies $\alpha_2 = -\frac{2}{3}$ and the equation 4 implies $\alpha_4 = 0$, but the equation 7 provides a contradiction. So there is no Jordan algebra among $A_{1,r}$.

For $A_{2,r}(\alpha_1, \beta_1, \beta_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & -\alpha_1 + 1 & 0 \end{pmatrix}$ the system of equations (2) becomes

$$-\beta_1^2 = 0,\ -2\beta_1 + 3\alpha_1\beta_1 = 0,\ -1 + 2\alpha_1 - \alpha_1^2 + \beta_2^2 = 0,\ -\alpha_1\beta_1 + \alpha_1^2\beta_1 + \beta_1\beta_2^2 = 0,$$

$$-\alpha_1 + 3\alpha_1^2 - 2\alpha_1^3 - \alpha_1\beta_2 + \alpha_1^2\beta_2 + \beta_2^2 - 2\alpha_1\beta_2^2 + \beta_2^3 = 0,\ -\beta_1 + \alpha_1\beta_1 + \beta_1\beta_2 = 0,$$

$$-1 + 3\alpha_1 - 2\alpha_1^2 = 0,\ \alpha_1\beta_1 - \beta_1\beta_2 = 0,\ 2\alpha_1 - 3\alpha_1^2 - \beta_2 + 2\alpha_1\beta_2 - \beta_2^2 = 0,\ 1 - 2\alpha_1 = 0.$$

Hence, one gets

$$\beta_1 = 0, \quad \alpha_1 = \frac{1}{2} \ \text{ and } \ \beta_2 = \pm\frac{1}{2}$$

and we obtain the following Jordan algebras

$$J_{1,r} = A_{2,r}\left(\tfrac{1}{2}, 0, \tfrac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad J_{2,r} = A_{2,r}\left(\tfrac{1}{2}, 0, -\tfrac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

For $A_{3,r}(\alpha_1, \beta_1, \beta_2) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & \beta_2 & 1 - \alpha_1 & 0 \end{pmatrix}$ the system of equations (2) becomes

$$\begin{gathered} \beta_1^2 = 0, \quad 2\beta_1 - 3\alpha_1\beta_1 = 0, \quad 1 - 2\alpha_1 + \alpha_1^2 - \beta_2^2 = 0, \quad -\alpha_1\beta_1 + \alpha_1^2\beta_1 + \beta_1\beta_2^2 = 0, \\ -\alpha_1 + 3\alpha_1^2 - 2\alpha_1^3 - \alpha_1\beta_2 + \alpha_1^2\beta_2 + \beta_2^2 - 2\alpha_1\beta_2^2 + \beta_2^3 = 0, \quad \beta_1 - \alpha_1\beta_1 - \beta_1\beta_2 = 0, \\ 1 - 3\alpha_1 + 2\alpha_1^2 = 0, \quad -\alpha_1\beta_1 + \beta_1\beta_2 = 0, \quad -2\alpha_1 + 3\alpha_1^2 + \beta_2 - 2\alpha_1\beta_2 + \beta_2^2 = 0, \quad 1 - 2\alpha_1 = 0. \end{gathered} \tag{4}$$

Due to the equations 1 and 3 in the system of equations (4) we get
$\beta_1 = 0, \alpha_1 = \frac{1}{2}$, then $\beta_2 = \pm\frac{1}{2}$, hence we obtain the following Jordan algebras

$$J_{3,r} = A_{3,r}(\tfrac{1}{2}, 0, \tfrac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad J_{4,r} = A_{3,r}(\tfrac{1}{2}, 0, -\tfrac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}.$$

For $A_{4,r}(\beta_1, \beta_2) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & \beta_2 & 1 & -1 \end{pmatrix}$ the equation 11 of (2) shows the inconsistence of the system.

For $A_{5,r}(\alpha_1,\beta_2)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \beta_2 & -\alpha_1+1 & 0\end{pmatrix}$ the system of equations (2) is equivalent to

$$-\alpha_1+3\alpha_1^2-2\alpha_1^3-\alpha_1\beta_2+\alpha_1^2\beta_2+\beta_2^2-2\alpha_1\beta_2^2+\beta_2^3=0.$$

So β_2 may be equal only to $-1+2\alpha_1$ or $-\sqrt{\alpha_1-\alpha_1^2}$ or $\sqrt{\alpha_1-\alpha_1^2}$. Taking into account the fact that in $\alpha_1=0,1$ cases $\sqrt{\alpha_1-\alpha_1^2}=-\sqrt{\alpha_1-\alpha_1^2}=0$, in $\alpha_1=\frac{1}{10}\left(5\pm\sqrt{5}\right)$ cases $-1+2\alpha_1$ is equal to one of $\pm\sqrt{\alpha_1-\alpha_1^2}$ one gets the following Jordan algebras

$$J_{5,r}(\alpha_1)=A_{5,r}(\alpha_1,-1+2\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & -1+2\alpha_1 & -\alpha_1+1 & 0\end{pmatrix},$$

where $\alpha_1\in\mathbb{R}$ and $\alpha_1\neq\frac{1}{10}\left(5\pm\sqrt{5}\right)$,

$$J_{6,r}(\alpha_1)=A_{5,r}\left(\alpha_1,\sqrt{\alpha_1-\alpha_1^2}\right)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & \sqrt{\alpha_1-\alpha_1^2} & -\alpha_1+1 & 0\end{pmatrix},$$

where $0\leq\alpha_1\leq 1$,

$$J_{7,r}(\alpha_1)=A_{5,r}\left(\alpha_1,-\sqrt{\alpha_1-\alpha_1^2}\right)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 0 & -\sqrt{\alpha_1-\alpha_1^2} & -\alpha_1+1 & 0\end{pmatrix},$$

where $0<\alpha_1<1$.

For $A_{6,r}(\alpha_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 0\\ 1 & 2\alpha_1-1 & -\alpha_1+1 & 0\end{pmatrix}$ the system of equations (2) is equivalent to

$$1-5\alpha_1+5\alpha_1^2=0$$

and therefore one has the following Jordan algebras

$$J_{8,r}=A_{6,r}\left(\tfrac{1}{10}\left(5-\sqrt{5}\right)\right)=\begin{pmatrix}\frac{1}{10}\left(5-\sqrt{5}\right) & 0 & 0 & 0\\ 1 & -\frac{\sqrt{5}}{5} & \frac{1}{10}\left(5+\sqrt{5}\right) & 0\end{pmatrix},$$

$$J_{9,r}=A_{6,r}\left(\tfrac{1}{10}\left(5+\sqrt{5}\right)\right)=\begin{pmatrix}\frac{1}{10}\left(5+\sqrt{5}\right) & 0 & 0 & 0\\ 1 & \frac{\sqrt{5}}{5} & \frac{1}{10}\left(5-\sqrt{5}\right) & 0\end{pmatrix}.$$

For $A_{7,r}(\alpha_1,\beta_1)=\begin{pmatrix}\alpha_1 & 0 & 0 & 1\\ \beta_1 & -\alpha_1+1 & -\alpha_1 & 0\end{pmatrix}$ we obtain the system of equations

$$\beta_1^2=0,\quad 3\alpha_1\beta_1=0,\quad 1-2\alpha_1=0,\quad \beta_1-2\alpha_1\beta_1+2\alpha_1^2\beta_1=0,\quad 1-5\alpha_1+8\alpha_1^2-6\alpha_1^3=0,$$
$$-2\alpha_1^2=0,\qquad -\beta_1+2\alpha_1\beta_1=0,\qquad -1+4\alpha_1-6\alpha_1^2=0.$$

and evidently this system is inconsistent.

It is easy to see that the structure constants of $A_{8,r}(\alpha_1, \beta_1)) = \begin{pmatrix} \alpha_1 & 0 & 0 & -1 \\ \beta_1 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$ and $A_{9,r}(\beta_1) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ \beta_1 & 1 & 0 & -1 \end{pmatrix}$ from Theorem 2.1 do not satisfy the system of equations (2).

For $A_{10,r}(\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1-\alpha_1 & -\alpha_1 & 0 \end{pmatrix}$ the system of equations (2) is equivalent to

$$1 - 5\alpha_1 + 8\alpha_1^2 - 6\alpha_1^3 = 0,$$

and therefore $\alpha_1 = \frac{1}{3}$, and one gets Jordan algebra

$$J_{10,r} = A_{10,r}\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}.$$

It is easy to see that the structure constants of the algebras $A_{11,r},\ A_{12,r},\ A_{13,r},\ A_{14,r}$ from the list of Theorem 2.1 also do not satisfy the system of equations (2) hence they also are not Jordan algebras. The algebra $A_{15,r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ is a Jordan algebra. $\square$

Note that the derivation algebras and the automorphism groups the above presented algebras have been given in Ahmed et al. (2018). Let us specify unital algebras among the algebras listed in Theorem 3.1.

Definition 3.1. *An element $\mathbf{1}_L$ ($\mathbf{1}_R$) of an algebra $\mathbb{A}$ is called a left (respectively, right) unit if $\mathbf{1}_L \cdot \mathbf{u} = \mathbf{u}$ (respectively, $\mathbf{u} \cdot \mathbf{1}_R = \mathbf{u}$) for all $\mathbf{u} \in \mathbb{A}$. An algebra with the left(right) unit element is said to be left(right) unital algebra, respectively.*

Now we can describe, up to isomorphism, all two-dimensional left(right) unital real Jordan algebras. To do this it is enough to compare the list of algebras from Theorem 3.1 with that of Ahmed et al. (2017c). The final result we give as a table as follows.

Algebra	$\mathbf{1}_L$	Algebra	$\mathbf{1}_R$
$J_{1,r}=A_{2,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$	$J_{1,r}=A_{2,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$
$J_{3,r}=A_{3,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$	$J_{2,r}=A_{2,r}(\frac{1}{2},0,-\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$
$J_{5,r}(1)=A_{5,r}(1,1)$	$\begin{pmatrix}1\\t\end{pmatrix}$, where $t\in\mathbb{R}$	$J_{3,r}=A_{3,r}(\frac{1}{2},0,\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$
$J_{6,r}(\frac{1}{2})=A_{5,r}(\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$	$J_{4,r}=A_{3,r}(\frac{1}{2},0,-\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$
		$J_{5,r}(\frac{1}{2})=A_{5,r}(\frac{1}{2},0)$	$\begin{pmatrix}2\\t\end{pmatrix}$, where $t\in\mathbb{R}$
		$J_{6,r}(\frac{1}{2})=A_{5,r}(\frac{1}{2},\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$
		$J_{7,r}(\frac{1}{2})=A_{5,r}(\frac{1}{2},-\frac{1}{2})$	$\begin{pmatrix}2\\0\end{pmatrix}$

A Jordan algebra $\mathbb{A}$ is said to be commutative if $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}$ whenever $\mathbf{u},\mathbf{v}\in\mathbb{A}$. From the result presented above one can easily derive the following classification result on commutative Jordan algebras.

Corollary 3.1. *Any non-trivial commutative Jordan algebra structure on a two-dimensional real vector space is isomorphic to only one of the following pairwise non isomorphic commutative Jordan algebras:*

$$J_{1c,r}=\begin{pmatrix}\frac{1}{2}&0&0&1\\0&\frac{1}{2}&\frac{1}{2}&0\end{pmatrix},\ J_{2c,r}=\begin{pmatrix}\frac{1}{2}&0&0&-1\\0&\frac{1}{2}&\frac{1}{2}&0\end{pmatrix},\ J_{3c,r}=\begin{pmatrix}\frac{2}{3}&0&0&0\\0&\frac{1}{3}&\frac{1}{3}&0\end{pmatrix},$$

$$J_{4c,r}=\begin{pmatrix}1&0&0&0\\0&0&0&0\end{pmatrix},\ J_{5c,r}=\begin{pmatrix}\frac{1}{2}&0&0&0\\0&\frac{1}{2}&\frac{1}{2}&0\end{pmatrix},\ J_{6c,r}=\begin{pmatrix}0&0&0&0\\1&0&0&0\end{pmatrix}.$$

Remark 3.1. *One of the nearest class of algebras to the class of Jordan algebras is the class of power-associative algebras. The power-associativity of an algebra $\mathbb{A}$ means that any subalgebra of $\mathbb{A}$ generated by a single element is associative. The description of all two-dimensional power associative real Jordan algebras has been given in Wallace (1970) as follows: $A_1,A_2,A_3,A(\sigma),B_1,B_2,B_3,B_4,B_5$, where $\sigma\in\mathbb{R}$ (note that the algebra A_2 in this list is extra since it can be given as $A_2=A(0)$). They can be found in the list of Theorem 3.1 as follows:*

$$A_1 \textit{ is the trivial algebra, } A_3=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}\simeq J_{5,r}(\tfrac{1}{2}),$$

$$A(\sigma)=\begin{pmatrix}1+\sigma&0&0&0\\0&1&\sigma&0\end{pmatrix}\simeq J_{5,r}(\tfrac{1+\sigma}{1+2\sigma}),\ \textit{where } \sigma\neq-\tfrac{1}{2},\tfrac{-1\pm\sqrt{5}}{2},$$

$A(\frac{-1}{2}) = J_{10,r},\ A(\frac{-1+\sqrt{5}}{2}) = J_{6,r}(\frac{5+\sqrt{5}}{10}),$

$A(\frac{-1-\sqrt{5}}{2}) = J_{7,r}(\frac{5-\sqrt{5}}{10}),\ B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \simeq J_{11,r},$

$B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \simeq J_{6,r}(1),\ B_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \simeq J_{1,r},$

$B_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \simeq J_{3,r},\ B_5 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq J_{6,r}(\frac{1}{2}).$

Note that the algebras $A(1), B_1, B_2, B_3, B_4, B_5$ from the list above are commutative and they are identified with the algebras

$$J_{3c,r}, J_{6c,r}, J_{4c,r}, J_{1c,r}, J_{2c,r}, J_{5c,r}$$

from Corollary 3.1.

Comparing the results of the present paper and that of Wallace (1970) we conclude that

$$J_{2,r},\ J_{4,r},\ J_{6,r}(\alpha_1),\ \text{where } 0 \leq \alpha_1 < 1 \text{ and } \alpha_1 \neq \frac{1}{2}, \frac{5+\sqrt{5}}{10},$$
$$J_{7,r}(\alpha_1),\ \text{where } 0 < \alpha_1 < 1 \text{ and } \alpha_1 \neq \frac{5-\sqrt{5}}{10},\ J_{8,r},\ J_{9,r}.$$

is a list of representatives of two-dimensional non power-associative Jordan algebras.

Note that the algebra with MSC $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is an example of two-dimensional Jordan algebra which is not power associative over any field $\mathbb{F}$.

Remark 3.2. *The description of two-dimensional (nontrivial) commutative Jordan algebras has been given in Bermúdez et al. (2008) as follows $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5$. There are the following isomorphisms between the algebras listed in Corollary 3.1 and that of Bermúdez et al. (2008).*

$$J_{1c,r} \simeq \psi_0,\ J_{2c,r} \simeq \psi_5,\ J_{3c,r} \simeq \psi_4,\ J_{4c,r} \simeq \psi_2,\ J_{5c,r} \simeq \psi_1,\ J_{6c,r} \simeq \psi_3.$$

3.1 Appendix

For the sake of completeness we present here, without proof, the description of two-dimensional Jordan algebras over any algebraically closed fields. The detailed proof has been given in Ahmed et al. (2017a).

Theorem 3.2. *Any nontrivial Jordan algebra structure on a two-dimensional vector space over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F}) \neq 2,3$ *and* $5)$ *is isomorphic to only one of the following pairwise non isomorphic Jordan algebras listed by their matrices of structure constants, where* $i \in \mathbb{F}$ *such that* $i^2 = -1$:

- $J_1 = A_2\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $J_2 = A_2\left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$,
- $J_3(\alpha_1) = A_4(\alpha_1, -1+2\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -1+2\alpha_1 & -\alpha_1+1 & 0 \end{pmatrix}$,
where $\alpha_1 \in \mathbb{F}$ *and* $\alpha_1 \neq \frac{1}{10}\left(5 \pm \sqrt{5}\right)$,
- $J_4(\alpha_1) = A_4\left(\alpha_1, \sqrt{\alpha_1 - \alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1 - \alpha_1^2} & -\alpha_1+1 & 0 \end{pmatrix}$,
where $\alpha_1 \in \mathbb{F}$,
- $J_5(\alpha_1) = A_4\left(\alpha_1, -\sqrt{\alpha_1 - \alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -\sqrt{\alpha_1 - \alpha_1^2} & -\alpha_1+1 & 0 \end{pmatrix}$,
where $\alpha_1 \in \mathbb{F}$ *and* $\alpha_1 \neq 0, 1$,
- $J_6 = A_5\left(\frac{1}{10}\left(5-\sqrt{5}\right)\right) = \begin{pmatrix} \frac{1}{10}\left(5-\sqrt{5}\right) & 0 & 0 & 0 \\ 1 & -\frac{\sqrt{5}}{5} & \frac{1}{10}\left(5+\sqrt{5}\right) & 0 \end{pmatrix}$,
- $J_7 = A_5\left(\frac{1}{10}\left(5+\sqrt{5}\right)\right) = \begin{pmatrix} \frac{1}{10}\left(5+\sqrt{5}\right) & 0 & 0 & 0 \\ 1 & \frac{\sqrt{5}}{5} & \frac{1}{10}\left(5-\sqrt{5}\right) & 0 \end{pmatrix}$,
- $J_8 = A_8\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$,
- $J_9 = A_8\left(\frac{1}{2} - \frac{i}{2}\right) = \begin{pmatrix} \frac{1}{2} - \frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} & 0 \end{pmatrix}$,
- $J_{10} = A_8\left(\frac{1}{2} + \frac{i}{2}\right) = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} - \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} & 0 \end{pmatrix}$,
- $J_{11} = A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Accordingly, for the cases of $\mathbb{F}$ with $Char(\mathbb{F}) = 2,3$ and 5 one has the following results.

Theorem 3.3. *Any nontrivial Jordan algebra structure on a two-dimensional vector space over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F}) = 2)$ *is isomorphic to only one of the following pairwise non isomorphic Jordan algebras listed by their matrices of structure constants:*

- $J_{1,2} = A_{3,2}(0,0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, *where* $\beta_1 \in \mathbb{F}$,
- $J_{2,2}(\alpha_1) = A_{4,2}(\alpha_1, 1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & 1 & \alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$,
- $J_{3,2}(\alpha_1) = A_{4,2}\left(\alpha_1, \sqrt{\alpha_1 + \alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1 + \alpha_1^2} & \alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$ *and* $\alpha_1^2 + \alpha_1 + 1 \neq 0$,
- $J_{4,2} = A_{5,2}(\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 1 & 1 & \alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$ *and* $\alpha_1^2 + \alpha_1 + 1 = 0$,
- $J_{5,2} = A_{8,2}(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$,
- $J_{6,2} = A_{10,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,
- $J_{7,2} = A_{12,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

Theorem 3.4. *Any nontrivial Jordan algebra structure on a two-dimensional vector space over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F}) = 3)$ *is isomorphic to only one of the following pairwise non isomorphic Jordan algebras listed by their matrices of structure constants, where* $i \in \mathbb{F}$ *such that* $i^2 = -1$:

- $J_{1,3} = A_{2,3}(-1,0,1) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$,
- $J_{2,3} = A_{2,3}(-1,0,-1) = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$,
- $J_{3,3} = A_{4,3}(\alpha_1, -1-\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -1-\alpha_1 & -\alpha_1 + 1 & 0 \end{pmatrix}$, *where* $\alpha_1 \in \mathbb{F}$ *and* $\alpha_1 \neq -1 \pm i$,
- $J_{4,3} = A_{4,3}\left(\alpha_1, \sqrt{\alpha_1 - \alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1 - \alpha_1^2} & -\alpha_1 + 1 & 0 \end{pmatrix}$,

- $J_{5,3} = A_{4,3}\left(\alpha_1, -\sqrt{\alpha_1-\alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -\sqrt{\alpha_1-\alpha_1^2} & -\alpha_1+1 & 0 \end{pmatrix},$
 where $\alpha_1 \in \mathbb{F}$ *and* $\alpha_1 \neq 0, 1,$
- $J_{6,3} = A_{5,3}(-1+i) = \begin{pmatrix} -1+i & 0 & 0 & 0 \\ 1 & -i & -1-i & 0 \end{pmatrix},$
- $J_{7,3} = A_{5,3}(-1-i) = \begin{pmatrix} -1-i & 0 & 0 & 0 \\ 1 & i & -1+i & 0 \end{pmatrix},$
- $J_{8,3} = A_{8,3}(-1+i) = \begin{pmatrix} -1+i & 0 & 0 & 0 \\ 0 & -1-i & 1-i & 0 \end{pmatrix},$
- $J_{9,3} = A_{8,3}(-1-i) = \begin{pmatrix} -1-i & 0 & 0 & 0 \\ 0 & -1+i & 1+i & 0 \end{pmatrix},$
- $J_{10,3} = A_{10,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$
- $J_{11,3} = A_{12,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

Theorem 3.5. *Any nontrivial Jordan algebra structure on a two-dimensional vector space over an algebraically closed field* $\mathbb{F}$ $(Char(\mathbb{F}) = 5)$ *is isomorphic to only one of the following pairwise non isomorphic Jordan algebras listed by their matrices of structure constants, where* $i \in \mathbb{F}$ *such that* $i^2 = -1$ *:*

- $J_{1,5} = A_2\left(\frac{1}{2}, 0, \frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$
- $J_{2,5} = A_2\left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$
- $J_{3,5} = A_4(\alpha_1, -1+2\alpha_1) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -1+2\alpha_1 & -\alpha_1+1 & 0 \end{pmatrix},$
- $J_{4,5} = A_4\left(\alpha_1, \sqrt{\alpha_1-\alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \sqrt{\alpha_1-\alpha_1^2} & -\alpha_1+1 & 0 \end{pmatrix},$
- $J_{5,5} = A_4\left(\alpha_1, -\sqrt{\alpha_1-\alpha_1^2}\right) = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & -\sqrt{\alpha_1-\alpha_1^2} & -\alpha_1+1 & 0 \end{pmatrix},$
 where $\alpha_1 \neq 0, 1,$
- $J_{6,5} = A_8\left(\frac{1}{3}\right) = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix},$

- $J_{7,5} = A_8\left(\frac{1}{2} - \frac{i}{2}\right) = \begin{pmatrix} \frac{1}{2} - \frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} + \frac{i}{2} & -\frac{1}{2} + \frac{i}{2} & 0 \end{pmatrix},$
- $J_{8,5} = A_8\left(\frac{1}{2} + \frac{i}{2}\right) = \begin{pmatrix} \frac{1}{2} + \frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} - \frac{i}{2} & -\frac{1}{2} - \frac{i}{2} & 0 \end{pmatrix},$
- $J_{9,5} = A_9 = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 \\ 1 & \frac{2}{3} & \frac{-1}{3} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 1 & -1 & \frac{1}{2} & 0 \end{pmatrix},$
- $J_{10,5} = A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$

From the results presented above one can easily derive the following classification results on commutative Jordan algebras.

Corollary 3.2. *Any nontrivial commutative Jordan algebra structure on a two-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic not 2 and 3 is isomorphic to only one of the following pairwise non isomorphic Jordan algebras listed by MSC:*

$$J_{1c} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},\ J_{2c} = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix},\ J_{3c} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$J_{4c} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},\ J_{5c} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Corollary 3.3. *Any nontrivial commutative Jordan algebra structure on a two-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic 2 is isomorphic to only one of the following pairwise non isomorphic commutative Jordan algebras listed by MSC:*

$$J_{1c,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ J_{2c,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix},\ J_{3c,2} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$J_{4c,2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Corollary 3.4. *Any nontrivial commutative Jordan algebra structure on a two-dimensional vector space over an algebraically closed field $\mathbb{F}$ of characteristic 3 is isomorphic to only one of the following pairwise non isomorphic commutative Jordan algebras listed by MSC:*

$$J_{1c,3} = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix},\ J_{2c,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ J_{3c,3} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix},$$
$$J_{4c,3} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ J_{5c,3} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Acknowledgement

The authors acknowledge the referees for their comments.

References

Ahmed, H., Bekbaev, U., and Rakhimov, I. (2017a). Classification of two-dimensional Jordan algebras. *AIP Conference Proceedings*, 1905:030003–1–030003–8.

Ahmed, H., Bekbaev, U., and Rakhimov, I. (2017b). Complete classification of two-dimensional algebras. *AIP Conference Proceedings*, 1830:070016–1 – 070016–11.

Ahmed, H., Bekbaev, U., and Rakhimov, I. (2017c). Two-dimensional left (right) unital algebras over algebraically closed fields and $\mathbb{R}$. *arXive: 1712.06731*.

Ahmed, H., Bekbaev, U., and Rakhimov, I. (2018). The authmorphism groups and derivation algebras of two-dimensional algebras. *Journal of Generalized Lie Theory and Application*, 1(12):1–9.

Althoen, S. and Hansen, K. (1992). Two-dimensional real algebras with zero divisors. *Acta Sci.Math.(Sceged)*, 56:23–42.

Althoen, S. and Kugler, L. (1983). When is $\mathbb{R}^2$ a division algebra? *Amer. Math. Monthly*, 90:625–635.

Bekbaev, U. (2017). Complete classification of two-dimensional general, commutative, commutative Jordan, division and evolution algebras. *arXiv: 1705.01237*.

Bermúdez, J., Campoamor-Stursberg, R., Garcìa Vergnolle, L., and Hernández, J. (2008). Contractions d'algèbres de Jordan en dimension 2. *Journal of Algebra*, 319(6):2395–2409.

Goze, M. and Remm, E. (2017). 2-dimensional algebras application to Jordan, g-associative and hom-associative algebras. *Journal of Generalized Lie Theory and Applications*, 11(2):1 – 11.

Hegazi, A. S. and Abdelwahab, H. (2017). The classification of n-dimensional non- associative Jordan algebras with $(n-3)$-dimensional annihilator. *Communications in Algebra*, 46(2):629–643.

Petersson, H. (2000). The classification of two-dimensional nonassicative algebras. *Result. Math.*, 3:120 − 154.

Wallace, E. (1970). Two-dimensional power associative algebras. *Mathematics Magazine*, 43(3):158–162.