UNIVERSITI PUTRA MALAYSIA

CANONICAL GROUP QUANTISATION ON ONE-DIMENSIONAL COMPLEX PROJECTIVE SPACE

AHMAD HAZAZI AHAMAD SUMADI

FS 2015 49

CANONICAL GROUP QUANTISATION ON ONE-DIMENSIONAL COMPLEX PROJECTIVE SPACE

By

AHMAD HAZAZI AHAMAD SUMADI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

November 2015

DEDICATIONS

Dedicated in Humble Gratitude to my parents;
Ahamad Sumadi Hj. Marzuki and Hasnah Hj. Omar,
parent-in law; Abdul Rahman Jaffar and Hasnah Hasan,
and especially
to my beloved other half and son; Rahmah and Luqman 'Atif,
who inspires me to seek knowledge.

"Ramai orang datang bertamu
Di bawah pohon rimbun tertutup
Bersungguh-sungguh menuntut ilmu
Moga jadi pedoman hidup"

"Buah pedada batang keladi
Kembang berseri bunga senduduk
Marilah menurut resminya padi
Semakin berisi semakin menunduk"

"Pergi ke pasar membeli kangkung
Kangkung dimasak bersama tenggiri
Janganlah diturut resminya jagung
Semakin berisi semakin meninggi"

-Malay Pantun

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

CANONICAL GROUP QUANTISATION ON ONE-DIMENSIONAL COMPLEX PROJECTIVE SPACE

By

AHMAD HAZAZI AHAMAD SUMADI

November 2015

Chairman: Associate Professor Hishamuddin Zainuddin, PhD
Faculty: Science

In this thesis we study the idea of quantisation approach to study the mathematical formalism of quantum theory with the intent to relate it with the idea of geometry of quantum states, particularly, Isham's group-theoretic quantisation technique to quantise compact manifold. The core of the discussions is based upon the Isham's quantisation programme and the compact classical phase space S^2 and CP^1. In Chapter 2, we review some of the literature that give some motivations to our investigation and also of those closely related to our present work.

In Chapter 3, we emphasize on reviewing several mathematical ingredients needed and also the idea of Isham's group-theoretic quantisation method and discussed some insights to further the investigation in the subsequent chapter.

Chapter 4 consists of the author's original contributions to the thesis. In this chapter, by using the aforementioned technique proposed in Chapter 3, we quantise the systems on one-dimensional complex projective space which is topologically homeomorphic to two-dimensional sphere. These two topological spaces are regarded as the underlying compact phase spaces for which there is no longer a cotangent bundle structure. These spaces have natural symplectic structure that allows one to use them for quantisation. The crucial part is to identify canonical group that acts on the phase space. The first phase is completed by finding all the algebras related to the groups.

With the canonical groups $SO(3)$ and $SU(2)$ found, we complete the quantisation process by finding representations of the canonical groups for CP^1. It is also discussed that Isham's group-theoretic quantisation can be used for quantising complex projective spaces in general and study the complex projective space from group theoretical aspects for infinite-dimensional Hilbert space. Finally, Chapter 5 is a con-

clusion, in this chapter we summarise all our work and suggest some idea for future research.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Sarjana Sains

PENGQUANTUMAN KUMPULAN BERKANUN KE ATAS RUANG UNJURAN KOMPLEKS BERMATRA SATU

Oleh

AHMAD HAZAZI AHAMAD SUMADI

November 2015

Pengerusi: Profesor Madya Hishamuddin Zainuddin, PhD
Fakulti: Sains

Dalam tesis ini kami mengkaji idea pengquantuman bagi menyelidiki formulasi matematik bagi teori quantum dengan bermatlamat untuk mengaitkannya dengan idea geometri keadaan quantum, terutamanya teknik pengquantuman berteori-kumpulan Isham untuk mengquantumkan manifold padat.Teras perbincangan adalah berdasarkan program pengquantuman Isham pada ruang fasa klasik padat S^2 dan CP^1. Dalam Bab 2, kami melakukan tinjauan susastera yang berkait rapat dan memberikan motivasi kepada kajian kami.

Dalam Bab 3, kami memberi ulasan kepada beberapa topik matematik yang diperlukan bagi memahami keseluruhan kerja-kerja dan program Isham ini serta kami juga mengulas secara terperinci kaedah pengquantuman berteori-kumpulan Isham dan membincangkan beberapa pandangan untuk melanjutkan siasatan dalam bab berikutnya.

Bab 4 mengandungi karya asli penulis tesis. Dalam bab ini, dengan menggunakan teknik yang dicadangkan dalam Bab 3, kami mengquantumkan sistem pada ruang unjuran kompleks bermatra satu yang secara topologinya berhomeomorfik dengan sfera bermatra dua. Kedua-dua ruang topologi ini dianggap sebagai ruang fasa padat yang tidak lagi mengambil kira struktur berkas kotangen. Ruang topologi ini mempunyai struktur simplektik tabii yang boleh digunakannya untuk tujuan pengquantuman. Bahagian yang penting adalah untuk mengenal pasti kumpulan berkanun yang bertindak pada ruang fasa. Fasa pertama selesai dengan mencari semua aljabar yang berkaitan dengan kumpulan.

Setelah kumpulan-kumpulan berkanun $SO(3)$ dan $SU(2)$ dijumpai, kami melengkapkan proses pengquantuman dengan mencari semua perwakilan tak setara bagi

kumpulan berkanun. Kami juga membincangkan bahawa pengquantuman berteori-kumpulan Isham ini boleh digunakan untuk mengquantumkan ruang unjuran kompleks secara umum dan mengkaji ruang unjuran kompleks dari sifat teori kumpulan bagi ruang Hilbert bermatra ketakterhingga. Akhir sekali Bab 5 adalah kesimpulan dengan kami merumuskan semua kerja-kerja kami dan mencadangkan beberapa pandangan untuk kajian akan datang.

ACKNOWLEDGEMENTS

In The Name of Allah Most Merciful, Most Compassionate. Praises be to Allah alone, and may Allah bless Prophet Muhammad (*pbuh*), his family and companions and grant them mercy.

First and foremost, I would like to pay tribute to my teachers and mentors, a long line of illustrious people includes my generous supervisor *Assoc. Prof. Dr. Hishamuddin Zainuddin* who exposed me to the works of *Chris J. Isham* and the area of *Quantization, Quantum Foundations* and *Quantum Information Theory*. Also, for his valuable guidances on doing theoretical and mathematical physics research, and for teaching me how to realise ideas into concrete work. His insights and passion in both quantum theory and mathematical physics become an inspiration for young physicist like me.

Many thanks to *Dr. Nurisya Mohd Shah* for her enlightening discussions on *measure theory* and *group representation theory*, and to *Dr. Mohammad Alinor Abdul Kadir* for allowing me to borrow many of his mathematics books from his personal library and to whom I was introduced to *mathematical logic* and *algebraic topology*. Also, to my friend, *Umair*, for valuable consultations on LaTeX programming; to my mathematician friend, *Taufik*, for interesting discussions on pure mathematics, and to my fellow researchers in UPM Theoretical and Computational Physics Group for stimulating discussions every week!

I also would like to thank my Committee Examiners, *Assoc. Prof. Dr. Zuriati Ahmad Zukarnain*, *Assoc. Prof. Dr. Jesni Shamsul Shaari* and *Dr. Md Mahmudur Rahman* for their constructive remarks on previous version of this thesis.

Thanks also to the people who directly and indirectly involved in the process of preparing this thesis especially INSPEM staff, also many thanks are due to Universiti Putra Malaysia for providing me with a Graduate Research Fellowship scheme and also to INSPEM for some financial support.

Last but not least, I would like to thank my family members and especially my heartfelt appreciation to my beloved other half, *Rahmah Abdul Rahman*, for her love, patience and understanding. Without her life itself would be bereft of joy and happiness.

I certify that a Thesis Examination Committee has met on 26 November 2015 to conduct the final examination of Ahmad Hazazi Ahamad Sumadi on his thesis entitled "Canonical Group Quantisation on One-Dimensional Complex Projective Space" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

Members of the Thesis Examination Committee were as follows:

Zuriati Ahmad Zukarnain, PhD
Associate Professor
Faculty of Computer Science and Information Technology
Universiti Putra Malaysia
(Chairperson)

Md. Mahmudur Rahman, PhD
Senior Lecturer
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Jesni Shamsul Shaari, PhD
Associate Professor
Kuliyyah of Science
International Islamic University Malaysia
(External Examiner)

ZULKARNAIN ZAINAL, PhD
Professor and Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date: 24 MARCH 2016

This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Master of Science.

Members of the Supervisory Committee were as follows:

Hishamuddin Zainuddin, PhD
Assosiate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairperson)

Jumiah Hasan, PhD
Assosiate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

BUJANG KIM HUAT, PhD
Professor and Dean
School of Graduate Studies
Universiti Putra Malaysia

Date:

DECLARATION

Declaration by graduate student

I hereby confirm that:
- this thesis is my original work;
- quotations, illustrations and citations have been duly referenced;
- this thesis has not been submitted previously or concurrently for any other degree at any other institutions;
- intellectual property from the thesis and copyright of thesis are fully-owned by Universiti Putra Malaysia, as according to the Universiti Putra Malaysia (Research) Rules 2012;
- written permission must be obtained from supervisor and the office of Deputy Vice-Chancellor (Research and Innovation) before thesis is published (in the form of written, printed or in electronic form) including books, journals, modules, proceedings, popular writings, seminar papers, manuscripts, posters, reports, lecture notes, learning modules or any other materials as stated in the Universiti Putra Malaysia (Research) Rules 2012;
- there is no plagiarism or data falsification/fabrication in the thesis, and scholarly integrity is upheld as according to the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) and the Universiti Putra Malaysia (Research) Rules 2012. The thesis has undergone plagiarism detection software.

Signature:________________________ Date:________________________

Name and Matric No: Ahmad Hazazi Ahamad Sumadi, GS26675

Declaration by Members of Supervisory Committee

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

Signature: ______________________________
Name of
Chairman of
Supervisory
Committee: Hishamuddin Zainuddin

Signature: ______________________________
Name of
Member of
Supervisory
Committee: Jumiah Hasan

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

CCR	Canonical commutation relation
C-R	Cauchy- Riemann relation
T_xM	Tangent space of manifold M at point x
$\mathscr{C}$	Classical phase space Q
T^*Q	Set of cotangent bundle of configurations space Q
$T(\mathscr{C})$	Holomorphic tangent bundle
$N(\mathscr{C})$	Complex line bundle
CP^n	Set of n-dimensional complex projective space
$\mathbb{R}^n$	Set of n-dimensional real number
$\mathbb{R}^+$	Set of positive real line
$\mathbb{C}^n$	Set of n-dimensional complex number
$\mathbb{C}^2$	Set of 2-dimensional complex vector space
C	Set of complex line
$\Omega^2(M)$	The space of differential 2-forms on M
$\mathbb{C}^*$	Complex plane
$\mathbb{P}\mathscr{H}$	Projective Hilbert space
$\mathscr{M}$	Kähler manifold
$\mathbb{Z}$	Set of integers
$Pic(\mathscr{C})$	Picard group
pr_1	Projection mapping of the bundle
$T(M)$	Set of tangent bundle M
$L(M)$	Set of frame bundle M
$GL(n,\mathbb{C})$	General linear group with $n \times n$ complex matrix
$GL(n,\mathbb{R})$	General linear group with $n \times n$ real matrix
$SU(2)$	Special unitary group with 2×2 complex matrix with determinant 1
$SO(3)$	Special orthogonal group with 3×3 real matrix with determinant 1
$(R^3)^* \rtimes SU(2)$	Semi-direct product group of a dual vector space $(R^3)^*$ with $SU(2)$
$\mathfrak{su}(2)$	Set of Lie algebra of $SU(2)$
Diff*(M)*	A diffeomorphisms group of symplectic transformation of the symplectic manifold M
FLT	Fractional-Linear Transformation
$\mathscr{L}(\mathscr{G})$	Lie algebra of canonical group $\mathscr{G}$
$End(TM)$	Endomorphism of TM
LHS	Left hand side
RHS	Right hand side
$\hbar$	Planck's constant
WKB	Wentzel-Kramers-Brillouin for WKB approximation

CHAPTER 1

INTRODUCTION

1.1 Preamble

Historically, in twentieth century physics, the two major theories that had been a subject of discourse among physicists are Einstein's theory of relativity and quantum theory. The former caused a major reformulation of the concepts of space and time or space-time (Einstein and Lawson, 1920); the latter is the theory that revolutionised physics through the discovery of the "discreteness" energy of the hypothetical black body radiation by a German-born physicist Max Planck (Jammer, 1966). It has been used widely to understand the nature of microscopic world especially in Planck's scale. There are many applications of quantum theory in various fields of physics such as solid state physics, nuclear physics, atomic and molecular physics, condensed matter physics, etc.

Recently, the twenty-first century physics has brought us to open a new horizon of research in which there are group of computer scientists, mathematicians and physicists working together to discover a new ideas in relatively fresh research area viz. *quantum information theory*. The advent of this field has found a renewal of interest into basic quantum theory, asking new kinds of questions and making more development on the theory, and at the same time also reawakening interest in the foundational issues of quantum theory itself. For example, attempts are currently being made to understand quantum entanglement from the information-theoretic point of view (Bengtsson and Zyczkowski, 2007). In this thesis we use quantisation approach in order to study the mathematical formalism of quantum theory with the intent to relate it with the idea of geometry of quantum states.

In general, the word "quantisation" often means the discretisation of particle's energy from the ground state energy of atom in microscopic world, but in our spectrum of discussion it is different and can be understood as an appropriate procedure to construct the quantum analogue of a given classical system with a specific phase space $\mathscr{S}$. Despite of being a century old, an effort of investigating the precise mathematical formalisms are of interest for both physicists and mathematicians ever since its birth. Indeed, there are different ways to quantise a classical theory such as Feynman-path integral quantisation, Weyl-Wigner quantisation, C^*-algebra quantisation, Moyal quantisation, stochastic quantisation, quantisation by *-product, geometric quantisation, Schwinger's quantisation etc (Ali and Englis, 2005; Shaharir, 2005; Feynman and Hibbs, 1965). From all the quantisation programmes mentioned, they differ in the fundamental structures assumed on the phase space. There is no one *unique* quantisation prescription that converts a classical theory to the quantum one, producing a "well-defined" quantum formalism.

In our research, we are using a particular quantisation programme initiated by *Chris J. Isham* called the group-theoretic quantisation[1] (Isham, 1984). Isham's first attempt was to apply this programme to quantise gravity based on 3-metrics (Isham and Kakas, 1984a,b). The programme has been generalised by others for different cases (Jung, 2012; Benevides and Reyes, 2010; Bouketir, 2000; Zainuddin, 1990, 1989). Group-theoretic quantisation, mathematically, shares the same mathematical language with geometric quantisation, but it emphasises the group-theoretical aspects. The idea is to focus on the construction of a canonical group describing the symmetries of the phase space of the system under study. The canonical groups play a pivotal role corresponding to a global analogue of the canonical commutation relations (CCR) in Dirac canonical quantisation given by

$$[q^i, p_j] = i\hbar\delta^i_j; \;\; [q^i, q^j] = 0; \; [p_i, p_j] = 0 \tag{1.1}$$

where the $\hbar$ is a Planck's constant, q^i are position variables of the configuration space of the system studied and their corresponding conjugate momenta p_j. Note that in general, many physicists claimed that the starting point would seem to be the imposition of this CCR. For the other schemes, they usually have the CCR built in as an outcome at a later stage of the procedure. However, the CCR may be inappropriate as a basis for quantising classical systems on non-linear configuration spaces. For this reason one has to look for another guiding procedure to serve as a basis for quantisation. A natural ingredient would be the consideration of symmetries of the system to be quantised. Thus, one of the advantage of this programme is described by the nature of geometrical notions that allows one to understand the topological and global aspects of quantum theory in a group-theoretical context.

1.2 Organisation

In Chapter 2, we will review in general some literature that are related to our work and see how this quantisation scheme is used in a particular system and its generalisation.

In Chapter 3 we present the discussions on theory and methodology that we will use in our research. Here we will further the discussions by reviewing the preliminaries of the mathematical ingredients that are used to understand this quantisation scheme such as symplectic manifolds (the underlying mathematical structure of classical mechanics), complex and Kähler manifolds, and fibre bundle theory. Note that these geometrical tools are used in geometric quantisation programme (Brian, 2013; Woodhouse, 1997; Weinstein and Bates, 1997). In addition, group-

1. Jung (2012); Benevides and Reyes (2010) and some authors named this scheme as *"Canonical Group Quantisation"*, and we adopted this term as our main title of this thesis.

theoretical features based on Lie group actions on manifold and representation theory are extensively used in this quantisation programme and hence, we will briefly introduce an overview of, more or less, the fundamentals axioms of quantum theory known among physicists and followed by introducing the mathematical framework of Isham's group-theoretic quantisation programme with the detailed elaborations. The discussions on comparison between Isham's group-theoretic and geometric quantisation are further discussed.

In Chapter 4 is the author's contribution where this technique is applied to quantise a simple system on a compact phase spaces. The phase space chosen here is one-dimensional complex projective space CP^1 that is topologically homeomorphic to the two-sphere S^2. The case considered is slightly different compared to those of Isham and others, since the phase space $\mathscr{S}$ is no longer a cotangent bundle. Albeit, to this particular case we proceed to the next step by finding the appropriate canonical groups and its relevant unitary irreducible representations.

The final chapter is to summarise all the author's research findings and proposed some possible generalisations for future work. We suggest that, in terms of geometry of quantum states, one can study the idea of describing multiple qubit or qudit states that arise geometrically from this quantisation framework and hence to understand the idea of quantum entanglement (Bengtsson and Zyczkowski, 2007).

1.3 Problem Statements and Objectives

Based on Isham's approach, the quantisation procedure utilise geometry of phase spaces in the form of cotangent bundle T^*Q. From there, one has to find its appropriate canonical groups and followed by the unitary irreducible representation of the groups. This were done by others in several cases, for instance this programme is applied on a system of a particle on a two-torus T^2 with a background field (Zainuddin, 1989), system on a homogeneous space $SU(2)/U(1)$ (Benevides and Reyes, 2010) and system of a particle on $\mathbb{R}^+$(positive real line) with boundary conditions (Jung, 2012) etc. Furthermore, in comparison, the case of quantisation on compact phase space is well-known in geometric quantisation school (Woodhouse, 1997; Hurt, 1983; Sniatycki, 1980; Woodhouse and Simms, 1976).

Notwithstanding, motivated from the geometric quantisation school, the question arises whether it is possible or not that Isham's method be applied to the case of non-cotangent bundle structure. From here we proposed our premise of argument that we want to generalise Isham's method for the case of compact manifold as an underlying phase space.

Furthermore, complex projective spaces has been used recently in studying the geometrical feature of quantum states in the area of geometric quantum information. In this study we also want to understand the structure of this space, from the group-theoretical technique, realised as infinite-dimensional Hilbert space.

Therefore, the objectives and motivations for this thesis are as follows;

- To quantise a classical system described by simplest compact manifold. In this study they are the two-dimensional sphere S^2 and one-dimensional complex projective space CP^1.
- To find the appropriate canonical groups of both topological spaces S^2 and CP^1.
- To find inequivalent quantisations through inequivalent representations of the canonical group for CP^1.

These three objectives form the bases of our premise of arguments in Chapter 4 later on.

REFERENCES

Abraham, R. and Marsden, J. (1987). *Foundations of Mechanics*. Addison-Wesley Publishing Company, Inc., Redwood City, CA, second edition.

Abramsky, S. and Coecke, B. (2009). Categorical Quantum Mechanics. In Engesser, K., Gabbay, D. M., and Lehmann, D., editors, *Handbook of Quantum Logic and Quantum Structures*, pages 261–323, Oxford UK. Elsevier B.V.

Aharonov, Y. and Bohm, D. (1959). Significance of electromagnetic potentials in quantum theory. *Physical Review*, 115:485–491.

Ali, S. T. and Englis, M. (2005). Quantization Methods: A Guide for Physicists and Analysts. *, Rev. Math. Phys.*, 17:391.

Anandan, J. and Aharonov, Y. (1990). Geometry of quantum evolution. *Phys. Rev. Lett*, 65:1697.

Arnold, V. I. (1978). *Mathematical Methods of Classical Mechanics*. Graduate Texts in Mathematics. Springer-Verlag, New York.

Ashtekar, A. and Schilling, T. A. (1995). Geometry of quantum mechanics. In *AIP Conference Proceedings*, volume 342, pages 471–478.

Balachandran, A. P. (1994). Topology in Physics–A Perspective. *Foundations of Physics*, 24(4).

Barut, A. and Raczka, R. (1980). *Theory of Group Representations and Applications*. Polish Scientific Publishers, Warsazawa, 2nd edition.

Benevides, C. and Reyes, A. (2010). Canonical Group Quantization, Rotation Generators and Quantum Indistinguishability. In *Geometric and Topological Methods for Quantum Field Theory*, pages 344–367. Cambridge University Press.

Bengtsson, I., Brannlund, J., and Zyczkowski, K. (2002). CP^n or Entanglement Illustrated. *International Journal of Modern Physics A*, 17:4675–4695.

Bengtsson, I. and Zyczkowski, K. (2007). *Geometry of Quantum States: An Introduction to Quantum Entanglement*. Cambridge University Press.

Berndt, R. (2001). *An Introduction to Symplectic Geometry*, volume 26 of *Graduate Studies in Mathematics*. American Mathematical Society, 1st edition.

Berndt, R. (2007). *Representations of Linear Groups An Introduction Based on Examples from Physics and Number Theory*. Mathematics Subject Classification. Friedr Vieweg and Sohn Verlag, 1st edition.

Bouketir, A. (2000). *Group-Theoretic Quantisation on Spheres and Quantum Hall Effect*. PhD thesis, Universiti Putra Malaysia, Malaysia.

Brian, C. H. (2013). *Quantum Theory for Mathematicians Graduate Texts in Mathematics*. Springer, New York, 1st edition.

Brody, D. C. and Hughston, L. P. (2001). Geometric quantum mechanics. *Journal of Geometry and Physics*, 38(1):19–53.

Chang, D. C., Markina, I., and Vasilev, A. (2011). Hopf fibration: Geodesics and Distances. *Jour. Geom. Phys.*, 61:986–1000.

Chernoff, P. R. (1981). Mathematical obstructions to quantization. *Hadronic Journal; (United States)*, 4(3):879–898.

Chruscinski, D. (2006). Geometric Aspects of Quantum Mechanics and Quantum Entanglement. *Journal of Physics: Conference Series*, 30:9–16.

Corichi, A. and Ryan, P. J. (1997). Quantization of non-standard Hamiltonian systems. *J. Phys. A: Maths Gen.*, 30:3553–3572.

da Silva, A. C. (2001). *Lectures on Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, 1st edition.

David, F. (2014). *The Formalisms of Quantum Mechanics: An Introduction*, volume 893 of *Lecture Notes in Physics*. Springer International Publishing, Switzerland.

Dirac, P. A. M. (1958). *The Principles of Quantum Mechanics*. International series of monographs on physics. Clarendon Press, Oxford, 4th edition.

Dirac, P. A. M. (2003). *Lectures on Quantum Mechanics*. Dover Publications Inc.

Echeverria-Enriquez, A., Munoz-Lecanda, M. C., Roman-Roy, N., and Victoria-Monge, C. (1998). Mathematical Foundations of Geometric Quantization. *Extracta Mathematicae*, 13:135–238.

Einstein, A. and Lawson, R. W. (1920). *Relativity: The Special and General Theory*. Henry Holt, 1920 edition.

Feynman, R. and Hibbs, A. (1965). *Quantum Mechanics and Path Integrals*. Dover Publication, emended edition.

Gotay, M. J. et al. (1996a). A Groenewold-Van Hove Theorem for S^2. *American Mathematical Society*, 348.

Gotay, M. J. et al. (1996b). Obstruction Results in Quantization Theory. *J. Nonlinear Sci.*, 6:469–498.

Griffiths, P. and Harris, J. (1978). *Principles of Algebraic Geometry*. Wiley, New York.

Groenewold, H. J. (1946). On The Principles of Elementary Quantum Mechanics. *Physica*, 12:405–460.

Gurarie, D. (1992). *Symmetries and Laplacians Introduction to Harmonic Analysis, Group Representations and Applications*, volume 174 of *North- Holland Mathematics Studies*. Elsevier Science Publisher B.V., North-Holland.

Hassani, S. (1998). *Mathematical Physics: A Modern Introduction to its Foundations*. Springer-Verlag, New York.

Hou, B. Y. and Hou, B. Y. (1997). *Differential Geometry for Physicists*. World Scientific, Singapore.

Hove, L. V. (2000). Sur certaines representations unitaires dun groupe infini de transformations. In Giovannini, A., editor, *The Legacy of Léon Van Hove*, pages 5–10, Singapore. World Scientific.

Hurt, N. E. (1983). *Geometric Quantization in Action*. D. Reidel Publishing Company, Holland.

Isham, C. J. (1984). Topological and Global Aspects of Quantum Theory. In *Relativity, Groups and Topology II*, volume 40, pages 1061–1290. Elsevier Science Publisher B.V., North-Holland,Amsterdam.

Isham, C. J. (1989). *Lectures on Groups and Vector Spaces for Physicists*, volume 31 of *Lecture Notes in Physics*. World Scientific, Singapore.

Isham, C. J. (1995). *Lectures on Quantum Theory Mathematical and Structural Foundations*. Imperial College Press, London.

Isham, C. J. (1999). *Modern Differential Geometry for Physicists*, volume 61 of *Lecture Notes in Physics*. World Scientific, Singapore, 2 edition.

Isham, C. J. and Kakas, A. C. (1984a). A Group Theoretical Approach to the Canonical Quantisation of Gravity. I. Construction of the Canonical Group. *Class. Quantum Grav.*, 1(6).

Isham, C. J. and Kakas, A. C. (1984b). A Group Theoretical Approach to the Canonical Quantisation of Gravity. II. Unitary Representations of the Canonical Group. *Class. Quantum Grav.*, 1(6).

Isham, C. J. and Linden, N. (1988). A Group Theoretic Quantisation of Strings on Tori. *Class. Quantum Grav.*, 5(6):71.

Isidro, J. M. (2004). Quantum States from Tangent Vectors. *Modern Physics Letters A*, 19:2339–2352.

Jammer, M. (1966). *The Conceptual Development of Quantum Mechanics*. McGraw-Hill Book Company.

Jung, F. (2012). *Canonical Group Quantization and Boundary Conditions*. PhD thesis, Johannes Gutenberg-Universitat Mainz.

Kibble, T. W. B. (1979). Geometrization of quantum mechanics. *Comm. Math. Phys.*, 65:189–201.

Kirillov, A. A. (2001). Geometric Quantization. In Arnol'd, V. and Novikov, S. P., editors, *Dynamical Systems IV*, volume 4, pages 139–176, New York. Springer Berlin Heidelberg.

Kostant, B. (1970). Quantization and Unitary Representations. In *Lectures in modern analysis and applications III*, number 170, pages 87–208. Springer Verlag.

Kostant, B. (1975). On the definition of quantization. In *Géométrie Symplectique et Physique Mathématique, Colloques Intern. CNRS*, volume 237, pages 187–210, Paris. Centre National de la Recherche Scientifique.

Kus, M. and Zyczkowski, K. (2001). Geometry of Entangled States. *Phys. Rev. A*, 63(032307).

Levay, P. (2004). The Geometry of Entanglement: Metrics, Connections and the Geometric Phase. *Journal of Physics A: Math. Gen.*, 37:1821–1841.

Mackey, G. W. (2004). *Mathematical Foundations of Quantum Mechanics*. Dover Publications Inc.

McDuff, D. and Salamon, D. (1998). *Introduction to Symplectic Topology*. Clarendon Press.

Mosseri, R. and Dandoloff, R. (2001). Geometry of entangled states, Bloch sphere and Hopf fibrations. *J. Phys. A: Math. Gen.*, 34:10243–10252.

Nakahara, M. (2003). *Geometry, Topology and Physics*. IoP Publishing, Bristol Philadelphia.

Nash, C. and Sen, S. (2011). *Topology and Geometry for Physicists*. Dover Books on Mathematics. Dover Publications Inc.

Rosenberg, J. M. (2004). A Selective History of the Stone-von Neumann Theorem. In Doran, R. S. and Kadison, R. V., editors, *Operator Algebras, Quantization, and Noncommutative Geometry*, volume 365, pages 123–158, USA. American Mathematical Society.

Shaharir, B. M. Z. (2005). *Tabii Zarah Atom: Penjana Sains Matematik Baru dan Terserlahnya Alam Metafizik*, volume 8 of *Siri Catatan Kuliah PELANDOK*. School of Mathematical Sciences, FST, National University of Malaysia, Bangi, Malaysia.

Sniatycki, J. (1980). *Geometric Quantization and Quantum Mechanics*, volume 30. Springer, New York.

Souriau, J.-M. (1997). *Structure of Dynamical Systems : A Symplectic View of Physics*, volume 149 of *Progress in Mathematics*. Birkhauser, Boston, 1st edition.

Steenrod, N. (1951). *The Topology of Fibre Bundles*. Princeton University Press, Princeton,New Jersey.

Strocchi, F. (2005). *An Introduction to the Mathematical Structure of Quantum Mechanics*, volume 27 of *Advanced Series in Mathematical Physics*. World Scientific, Singapore.

Sumadi, A. H. A. and Zainuddin, H. (2014). Canonical Groups for Quantization on the Two-Dimensional Sphere and One-Dimensional Complex Projective Space. *Journal of Physics: Conference Series*, 553(012005).

Urbantke, H. (1991). Two-level quantum systems: States, phases, and holonomy. *Am. J. Phys.*, 59(6):503–506.

Urbantke, H. (2003). The Hopf fibration-seven times in physics. *J. of Geometry and Physics*, 46:125–150.

von Neumann, J. (1955). *Mathematical Foundations of Quantum Mechanics*, volume 2. Princeton University Press, Princeton, N.J.

Way, R. (2009). *Dynamics in the Hopf bundle, the geometric phase and implications for dynamical systems*. PhD thesis, University of Surrey, UK.

Weinstein, A. and Bates, S. (1997). *Lecture on the Geometry of Quantization*, volume 8. American Mathematical Society.

Woodhouse, N. M. J. (1997). *Geometric Quantization*. Clarendon Press, Oxford, 2nd edition.

Woodhouse, N. M. J. and Simms, D. (1976). *Lecture on Geometric Quantization*. Springer-Verlag, New York.

Wu, T. T. and Yang, C. N. (1975). Concept of non-integrable phase factors and global formulation of gauge fields. *Phys. Rev. D*, 12:3845.

Wu, T. T. and Yang, C. N. (1976). Dirac's monopole without strings: Classical Lagrangian theory. *Phys. Rev. D*, 14:437.

Zainuddin, H. (1989). Group-theoretic quantization of a particle on a torus in a constant magnetic field. *Phy. Rev. D*, 40:636.

Zainuddin, H. (1990). *Group-Theoretic Quantisation and Central Extension*. PhD thesis, University of Durham, UK.

PUBLICATIONS

Sumadi A.H.A. and Zainuddin H. (2014) Canonical Groups for Quantization on the Two-Dimensional Sphere and One-Dimensional Complex Projective Space. *Journal of Physics: Conference Series* (553) 012005

UNIVERSITI PUTRA MALAYSIA
STATUS CONFIRMATION FOR THESIS/PROJECT REPORT AND COPYRIGHT
ACADEMIC SESSION: 2015/2016

TITLE OF THE THESIS/PROJECT REPORT:
CANONICAL GROUP QUANTISATION ON ONE-DIMENSIONAL COMPLEX PROJECTIVE SPACE

NAME OF STUDENT: Ahmad Hazazi Ahamad Sumadi

I acknowledge that the copyright and other intellectual property in the thesis/project report belonged to Universiti Putra Malaysia and I agree to allow this thesis/project report to be placed at the library under the following terms:

1. This thesis/project report is the property of Universiti Putra Malaysia.
2. The library of Universiti Putra Malaysia has the right to make copies for educational purposes only.
3. The library of Universiti Putra Malaysia is allowed to make copies of this thesis for academic exchange.

I declare that this thesis is classified as:

*Please tick(✓)

◯	CONFIDENTIAL	(contain confidential information under Official Secret Act 1972).
◯	RESTRICTED	(Contains restricted information as specified by the organization/institution where research was done).
◯	OPEN ACCESS	I agree that my thesis/project report to be published as hard copy or online open acces.

This thesis is submitted for:

◯	PATENT	Embargo from ___________until ___________. (date) (date)

Approved by:

____
(Signature of Student)
New IC No/Passport No.:870217-23-5425

Date:

____
(Signature of Chairman of Supervisory Committee)
Name: **Hishamuddin Zainuddin, PhD**

Date:

[Note: If the thesis is CONFIDENTIAL or RESTRICTED, please attach with the letter from the organization/institution with period and reasons for confidentially or restricted.]