UNIVERSITI PUTRA MALAYSIA

MAASS CUSP FORM ON ASYMMETRIC HYPERBOLIC TORUS

NOR SYAZANA SHAMSUDDIN

FS 2018 16

MAASS CUSP FORM ON ASYMMETRIC HYPERBOLIC TORUS

By

NOR SYAZANA BINTI SHAMSUDDIN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

November 2017

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DEDICATIONS

To all of my love;
My Husband
Ummi
My Sister
My Family

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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November 2017

Chairman : **Assoc. Prof. Hishamuddin Zainuddin, PhD**
Faculty : **Science**

The quantum system describing a free particle moving on a cusped hyperbolic surface is represented using the eigenfunction of the hyperbolic Laplace-Beltrami operator. The eigenspectra contained both continuous and discrete spectra, but the focus here is only on the discrete part. The eigenfunctions have to be computed numerically and they are known as Maass cusp form (MCF). The hyperbolic surface of interest here is the singly punctured two-torus. Past research has shown that the case of the symmetric torus has degenerate eigenvalues. The purpose of this research is to find the eigenvalues for asymmetric torus, deformed from symmetric torus by moving the vertices of its fundamental domain at the real axis, as well as to investigate the degeneracy behavior of its eigenvalues.

There are three models that are being explored, namely $\mathscr{F}_1$ with vertices at -1, $\frac{1}{2}$, 1, and ∞, $\mathscr{F}_2$ with vertices at -3, 0, 2 and ∞ and the last one $\mathscr{F}_3$ with vertices at -2, 0, 1 and ∞. Despite having different cusp widths, all models are ensured to have the same area. Since the domain of the torus in the hyperbolic plane needs an equivalent fundamental domain where the cusp is represented by the point of imaginary infinity for a convenient computation, a cusp reduction method is constructed including the equations for the generators in order to act as the side identification.

Consider that the asymmetric torus has no parity symmetry, an algorithm of MCF with exponential expansion is developed using Mathematica. The computation of MCF is an adapted algorithm of Hejhal and Then, i.e. based on implicit automorphy and finite Fourier series. There are 37 eigenvalues found for asymmetric torus $\mathscr{F}_1$

and 24 eigenvalues for asymmetric torus $\mathscr{F}_2$ between range [0, 15]. Both domains have non-degenerate eigenvalues. Remarkably, all eigenvalues of $\mathscr{F}_2$ are also eigenvalues for $\mathscr{F}_1$, suggesting that the unique MCF for $\mathscr{F}_1$ are newforms while those of $\mathscr{F}_2$ are oldforms. In the same range, the computed algorithm for asymmetric torus $\mathscr{F}_3$ gives out 36 eigenvalues and surprisingly these eigenvalues are doubly degenerate.

It is believed that the equivalent fundamental domain for $\mathscr{F}_3$ has extra symmetry compared to $\mathscr{F}_1$ and $\mathscr{F}_2$. Apparently, equivalent fundamental domain for $\mathscr{F}_3$ has symmetry at each vertices, meanwhile the other two does not have. All the candidate eigenvalues given by the algorithm went through checking procedure stated in the literature so that only authentic eigenvalues have been chosen. Those procedures are y-independent solution, automorphy condition, Hecke relation and Ramanujan-Petersson conjecture. Later, the eigenstates of selected eigenvalues from each surface are visualized using contour plot and density plot in the Mathematica.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

FUNGSI BERBENTUK JURING MAASS DIATAS PERMUKAAN TORUS HIPERBOLIK BERASIMETRI

Oleh

NOR SYAZANA BINTI SHAMSUDDIN

November 2017

Pengerusi : **Assoc. Prof. Hishamuddin Zainuddin, PhD**
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Sistem kuantum yang menghuraikan pergerakan sesuatu zarah bebas di atas permukaan hiperbolik berjuring diwakili oleh fungsi eigen operator hiperbolik Laplace-Beltrami. Spektra eigen bagi permukaan meliputi kedua-dua spektra yang selanjar dan diskrit. Fungsi eigen harus diselesaikan secara berangka dan ianya dikenali sebagai fungsi berbentuk juring Maass (MCF). Permukaan hiperbolik yang menjadi minat di sini ialah permukaan sebuah torus. Kajian lepas menunjukkan permukaan torus yang simetri mempunyai nilai eigen yang degenerat. Matlamat kajian ini adalah untuk mencari nilai eigen bagi torus asimetri, yang mana diubah bentuknya dengan memindahkan bucu torus pada paksi nyata, serta menyiasat kelakuan degenerasi nilai eigennya.

Terdapat tiga model yang dikaji, iaitu $\mathscr{F}_1$ dengan bucu di -1, $\frac{1}{2}$, 1 dan ∞, $\mathscr{F}_2$ dengan bucu di -3, 0, 2 dan ∞, dan yang terakhir $\mathscr{F}_3$ dengan bucu di -2, 0, 1 dan ∞. Walaupun mempunyai lebar juring yang berbeza, permukaan torus dipastikan supaya mempunyai luas yang sama. Memandangkan domain torus dalam satah hiperbolik ini memerlukan domain asas setara di mana juring diwakili oleh titik khayalan tak terhingga bagi memudahkan pengiraan, maka kaedah pengurangan juring dibina, termasuklah persamaan bagi penjana yang bertindak sebagai pengecaman sisi torus tersebut.

Oleh sebab torus asimetri tidak mempunyai simetri pariti, satu algoritma MCF dengan pengembangan eksponen dihasilkan menggunakan perisian Mathematica. Pen-

giraan MCF ini adalah algoritma terubahsuai Hejhal dan Then yang berlandaskan automorf tersirat dan siri Fourier terhingga. Program ini dijalankan ke atas torus simetri dan hasilnya bertepatan dengan kajian lepas. Oleh itu, dilaksanakan ke atas model torus asimetri dengan yakin. Terdapat 37 nilai eigen bagi torus asimetri $\mathscr{F}_1$ dan 24 nilai eigen bagi torus asimetri $\mathscr{F}_2$ di antara julat [0, 15]. Kedua-dua domain menunjukkan nilai eigen yang tidak degenerat. Semua nilai eigen bagi $\mathscr{F}_2$ adalah nilai eigen bagi $\mathscr{F}_1$, menunjukkan bahawa MCF unik bagi $\mathscr{F}_1$ adalah bentukkan baru manakala $\mathscr{F}_2$ adalah bentukkan lama. Dalam julat yang sama, pengiraan algoritma bagi torus asimetri $\mathscr{F}_3$ menghasilkan 32 nilai eigen dan yang mengejutkan nilai eigen ini berganda dua.

Ia dipercayai bahawa domain asas setara $\mathscr{F}_3$ mempunyai simetri tambahan jika dibandingkan dengan domain asas setara $\mathscr{F}_1$ dan $\mathscr{F}_2$. Secara jelasnya, domain asas setara $\mathscr{F}_3$ memiliki simetri pada setiap bucu, manakala tidak pada domain yang lain. Kesemua nilai eigen yang dihasilkan oleh algoritma akan melalui prosedur-prosedur pemeriksaan supaya hanya nilai eigen yang sahih dipilih. Prosedur-prosedur tersebut adalah penyelesaian y-bebas, syarat automorf, hubungan Hecke dan andaian Ramanujan-Petersson. Seterusnya, keadaan eigen bagi nilai eigen terpilih untuk setiap model divisualisasikan menggunakan plot kontur dan plot ketumpaan dalam perisian Mathematica.

ACKNOWLEDGEMENTS

In the name of Allah S.W.T. the Most Gracious and Most Merciful.

Alhamdulillah, thanks to Allah for all the past two years and a half that I survive in completing my master and grateful for the blessing in my life.

I would like to appreciate my supervisor, Assoc. Prof. Dr. Hishamuddin Zainuddin for the opportunity to be in his supervision and unlimited support from the beginning until the end of my master's study. His comment and suggestion in this field and research are very helpful to me since I was new in this field of study. Not forgetting here to appreciate and acknowledge support from member of the supervisory committee, Dr. Chan Kar Tim which also help me a lot during these years, especially in the software and the computation.

I would like to extend my thanks and appreciation to all lecturers in the Department of Physics and staff on INSPEM for their help and moral support during completing my research. Last but not least, this gratitude goes to friends in the QuEST where they are continuously given positive support through thick and thin during the journey on accomplishes this study.

Finally, I would like to thank my mom, Zainab Jusoh and my sister, Shamsunizai Shamsuddin, together with my siblings and my bestfriend for their encouragement and endless prayer. My appreciation also goes to my husband for the faith he had in me to finish the study toward the end.

I certify that a Thesis Examination Committee has met on 30 November 2017 to conduct the final examination of Nor Syazana binti Shamsuddin on her thesis entitled "Maass Cusp Form on Asymmetric Hyperbolic Torus" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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CHAPTER 1

INTRODUCTION

1.1 Briefly on Quantum Chaos

One of the uses of Schrodinger's equation in the quantum mechanics is to describe the quantum system of a particle. The probability function of the solution to the Schrodinger's equation provides the informations of the probability of finding the particle in certain region and time (Griffiths and Harris, 1995). The Schrodinger equation in Euclidean space is defined as $H\psi = E\psi$, where the Hamiltonian operator for Euclidean space is $H = -\frac{\hbar^2}{2m}\nabla^2 + V$ with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, V is the potential of the system considered and E is the energy of the particle (Robinett and Murphy, 1997; Griffiths and Harris, 1995). It can be said that Schrodinger's equation in quantum mechanics is comparatively as Newton's second law of motion in the classical mechanic, and all the information about the system are embodied in the solution function of the equation.

In this study, we are interested in particle moving freely in hyperbolic space where the metric on the surface is defined as $ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right)$ (Anderson, 2005). The quantum system of the particle moving on those surfaces is governed by time-independent Schrodinger equation $H\psi = E\psi$ with the Hamiltonin $H = -\Delta$, where Δ is the non-Euclidean Laplace operator (assume $\hbar = 2m = 1$). The surface considered here have a cusp where the particle can enter from infinitely far away or leaving the surface (Gutzwiller, 1990). Thus, the eigenstates of the particle can be corresponded to the bounded motion on the surface or unbounded motion involving points at infinity. As a result of the presence of the cusp, the spectrum of the Laplacian consists of both discrete and continuous parts, where the former are for the bound states and the latter are for the scattering states (Then, 2007). In this research, the considered quantum states are for the discrete part where they are spanned by a discrete eigenfunction namely Maass waveform, a non-holomorphic modular form originated by Hans Maass in 1949 (Terras, 1985). In general, such eigenstates are not known analytically and hence require development of complex programs

Classical mechanics on the hyperbolic surface are known to exhibit chaotic behaviour. Due to this, in 1898, Hadamard studied the free motion of a ball on a surface of negative curvature without boundary (Avelin, 2003) rather than flat billiard table. This led to Hadamard's research in 1898 which is the first proved example of the chaotic dynamics known as Hadamard dynamical system or fevered as Hadamard billiards. In this system, the motion of the particle is considered to be free (frictionless) on a compact Riemann surface of constant negative curvature (Avelin, 2007;

Gutzwiller, 1990). He then showed that the system is chaotic when the long-time behavior of the system is very insensitive to initial conditions, plus every trajectories of the particle move away from every other.

The study of modified billiard of the Hadamard billiard is introduced by Emil Artin in 1924, later known as Artin billiard, which is characterized by the point particle's free motion on a non-compact Riemann surface of constant negative curvature. The configuration space of this billiard has the topology of a sphere containing an open end (cusp) at infinity. The cusp represented as vertices that are located infinitely far away and hence the domain of non-compact Riemann surface with finite area. These kind of domains can be regarded as mathematical models for many physical situations which illustrate the point particle coming from infinitely far away outside the domain and entering the domain as in the scattering problem (Gutzwiller, 1990).

Since the investigation by Lobachevsky, Poincare and Hadamard in the 19^{th} century, the motion on these kind of dynamical system becomes attention to researchers due to mathematical connection in number theory, differential geometry and group theory. Meanwhile, in physics, it can be related to string theory and Quantum Hall effect (Pnueli, 1994). Another significant related topic is quantum chaos. The quantum energy levels are connected to the classical periodic orbits through the trace formula of Gutzwiller (Bogomolny et al., 1995), or known as Selberg's trace formula, one of the important results in mathematics. Gutzwiller (1980) was the first to indicate that the results of Selberg's trace formula is crucial for the understanding of quantum chaos. Quantum chaos is said to have application in cosmology (Then, 2007) and condensed matter (Hurt, 2000; Gubin and Santos, 2012).

Quantum chaos are not well understood and studying quantum systems on hyperbolic surfaces maybe useful. The study on quantum chaos generally presumed to constitute all complication related to the quantum mechanical behavior of classically chaotic system (Stöckmann, 2000). Quantum chaology by definition in Berry (1989) is the study of semiclassical, but nonclassical, phenomena characteristic where classical counterparts exhibit chaos. He emphasized that semiclassical treatment is meant to take the Plank constant, h, in an equation describing the system to tend to zero. One can refer Nonnenmacher (2008), Berry (1977) and Stöckmann (2000) for more details on this topic.

1.2 Problem Statement

A punctured surface (with one or more cusp) defines on constant negative curvature, such as the Artin billiard, becomes an object of study in the context of quantum chaos. Artin billiard, as mentioned before, can be described by the use of modular group, i.e. a discrete subgroup of the Projective Special Linear group, PSL(2,$\mathbb{R}$).

There are other surfaces and groups that can be considered to this kind of research such as singly punctured torus (commutator subgroup of modular group) (Gutzwiller, 1983; Antoine et al., 1990; Pnueli, 1994; Chan et al., 2013a; Siddig, 2009), triply punctured two-sphere (principal congruence subgroup of level two) (Chan et al., 2016), punctured surfaces characterized by $\sum_{g,k}$ where g denotes the genus and k is the number of cusp (Lévay, 2000), moonshine group (Jorgenson et al., 2014; Conway et al., 2004; Cummins and Gannon, 1997), Picard group (Aurich et al., 2004; Then, 2006; Then, 2007), Bianchi group (Steil, 1999) and deformation of cusp for subgroup of modular group (Avelin, 2003; Avelin, 2007; Farmer and Lemurell, 2005)

One of listed the surfaces at the beginning of this subsection, and being the focus in this research is the singly punctured torus (Chan et al., 2013b). The symmetric torus is generated by the commutator subgroup of modular group Γ', and the side identifications of the torus are made using the generators of the subgroup Γ'. The fundamental domain of Γ' is well-known to have a parity symmetry at $x = 0$. Thus the domain can also be generated by the reflection operator J, leading to possibility of two different eigenfunctions of even and odd class. Surprisingly, when both classes are considered together, their eigenvalues are doubly degenerate, i.e. having the same eigenvalue for two different eigenstates. The result by Chan et al. is of interest here namely what cause the degeneracy of the eigenvalues.

The suspected explanation of the results is that there are extra symmetries on the torus described in Chan et al. (2013b). In our study, the torus is to be deformed, in order to reduce the symmetries of the fundamental domain and the deformed torus is being named as asymmetric torus since the major radius of the torus is not equal to its minor radius. At the same time, the symmetry which gives the degeneracy is also being studied here.

1.3 Objectives

The intent of the present research is to acquire the eigenvalues of the Hamiltonian of a quantum particle moving on a asymmetric hyperbolic torus. The objectives are motivated by the possible degeneracy or nondegeneracy of eigenvalues for the hyperbolic torus in general. The objectives of this study are as follows:

1. To deform the torus in order to reduce the symmetry of the fundamental domain, forming an asymmetric torus.

2. To find the generators of a group that represents the asymmetric torus.

3. To construct a method which transforms the fundamental domain of the asymmetric torus with the cusp represent by four vertices at the boundary to the

equivalent fundamental domain with only a point at the boundary, namely at $y = \infty$.

4. To compute the eigenvalues of the Hamiltonian of a particle moving on the corresponding asymmetric torus.

1.4 Scope of study

The study will focus on determining the eigenvalues of three specific hyperbolic asymmetric tori and check whether those tori have degenerate or non-degenerate eigenvalues. The computation will be done by using a Mathematica program, developed based on the MCF algorithm with both exponential and cosine/sine expansion. Along the way, a general equation for the generator of the torus and also the general cusp reduction method will be developed. In addition, a program to verify the authenticity of the eigenvalues has been constructed based on methods mention in the literature, where each candidate eigenvalue, output from the Mathematica program, will go through four procedures before being declared as valid eigenvalues.

The proposed MCF computation can be done numerically, and each model needs to have different programs in Mathematica due to the different geometries of the asymmetric tori where it will affect the computation. Only three models of the asymmetric torus are chosen because the cusp reduction method can only be applied to some range of the major and minor radii. A bigger deformation will not result in the needed hyperbolic tori.

1.5 Outline of the Thesis

The thesis is divided into eight chapters. In Chapter One, a brief motivation of the study has been presented, as well as the problem statements and objectives of this research.

Chapter Two presents the review of Maass waveforms and the literature that is related to the study of the punctured surfaces. In addition, a description on Hejhal's algorithm and published work related to the algorithm are given in this chapter.

Chapter Three analyzes the mathematical groundwork for the hyperbolic geometry and also the discrete subgroup of $PSL(2,\mathbb{R})$. An attention is given to the surfaces of symmetric torus, including the subgroup of the $PSL(2,\mathbb{R})$ generating such torus. The theoretical framework on the Maass waveform, which consist of modified K-Bessel function and Hecke operator are also discussed.

Chapter Four introduced a construction of the general torus, covering also establishing the generators for the side identifications which can be applied on both symmetric and asymmetric tori. The chapter continues with the description of the method to reduce the vertices representing the cusp. A few examples of the cusp reduction method are given at the end of this chapter.

Chapter Five is dedicated to the computation of Maass cusp form both exponential expansion and cosine/sine expansions, where the former is for a general domain and the latter is for domains with parity symmetry. Pullback algorithm, one of the important algorithm for the needed computation is also demonstrated there. A comparison of the eigenvalues for the symmetric torus resulting from computation of both expansions is made to check the accuracy of the modified algorithm.

Chapter Six presents the computational work in Maass cusp form for the models of asymmetric torus that give no degeneracy in the eigenvalues. A Maass cusp form algorithm with the exponential expansion and the pullback algorithm on the previous chapter is deployed here. The numerical results and topoghraphies of the eigenstates are also shown here. Meanwhile, the same description as Chapter Six is applied to Chapter Seven, but for the model of the asymmetric torus with doubly degenerate eigenvalues.

The final Chapter contains the conclusion of this research. At the same time, we give some suggestions of the causes of the degeneracy of the eigenvalues. There are also some recommendations for the future work related to the study of hyperbolic asymmetric torus.

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LIST OF PUBLICATIONS

The following are the list of publications that arise from this study.

Journal articles:

Nor Syazana Shamsuddin, Hishamuddin Zainuddin, & Chan Kar Tim (2017, January). Computing Maass cusp form on general hyperbolic torus. In *AIP Conference Proceedings* (Vol. 1795, No. 1, p. 020014). AIP Publishing.

Hishamuddin Zainuddin, Chan Kar Tim, **Nor Syazana Shamsuddin**, & Nurisya Mohd Shah (2017, January). Quantum Bound States on Some Hyperbolic Surfaces. In Journal of Physics: Conference Series (Vol. 795, No. 1, p. 012002). IOP Publishing.

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