UNIVERSITI PUTRA MALAYSIA

GENERAL RELATION BETWEEN SUMS OF FIGURATE NUMBERS

MOHAMAT AIDIL MOHAMAT JOHARI

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By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

April 2013

DEDICATIONS

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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April 2013

Chair: Professor Dato' Kamel Ariffin Bin Mohd Atan, PhD

Institute: Institute For Mathematical Research

In this study, we seek to find relations between the number of representations of a nonnegative integer n as a sum of figurate numbers of different types.

Firstly, we give a relation between the number of representations, $c_k(n)$, of n as the sum k cubes and the number of representations, $p_k(n)$, of n as the sum of k triangular pyramidal numbers, namely under certain conditions

$$p_k(n) = c_k^{odd}(v),$$

where c_k^{odd} denotes the number of representations as a sum of k odd cubes and the integer v is derived from n. Then we extend this problem by considering sums of s-th powers with $s > 3$ and the associated polytopic numbers of order s.

Next, we discuss the relation between $\Phi_{(2,k)}(n)$, the number of representations of n as a sum of k fourth powers, and $\Psi_{(2,k)}(n)$, the number of representations of n

as a sum of k terms of the form $8\gamma^2 + 2\gamma$ where γ is a triangular number. When $1 \le k \le 7$, the relation is

$$\Phi_{(2,k)}(8n + k) = 2^k \Psi_{(2,k)}(n).$$

We extend this result by considering the relation between the number of representations of n as a sum of k $2m$-th powers and the number of representations of n as a sum of k terms determined by an associated polynomial of degree m evaluated at a triangular number.

Thirdly, we consider the relation between $s_k(n)$, the number of representations of n as a sum of k squares, and $e_k(n)$, the number of representations of n as a sum of k centred pentagonal numbers. When $1 \le k \le 7$, this relation is

$$\alpha_k e_k(n) = s_k\left(\frac{8n - 3k}{5}\right), \text{where } \alpha_k = 2^k + 2^{k-1}\binom{k}{4}.$$

We extend the analysis to the number of representations induced by a partition λ of k into m parts. If corresponding number of representations of n are respectively $s_\lambda(n)$ and $e_\lambda(n)$, then

$$\beta_\lambda e_\lambda(n) = s_\lambda\left(\frac{8n - 3k}{5}\right)$$

where

$$\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$$

and i_j denotes the number of parts of λ which are equal to j.

We end this thesis with a short discussion and proposal of various open problems for further research.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

HUBUNGAN UMUM ANTARA HASIL TAMBAH NOMBOR FIGURA

Oleh

MOHAMAT AIDIL BIN MOHAMAT JOHARI

April 2013

Pengerusi: Profesor Dato' Kamel Ariffin Bin Mohd Atan, PhD

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Dalam penyelidikan ini, kami mencari hubungan antara bilangan paparan bagi satu integer bukan negatif n sebagai hasil tambah nombor figura berlainan jenis.

Pertamanya kami memberi hubungan antara bilangan paparan $c_k(n)$, dengan n sebagai hasil tambah k nombor kubik, dan bilangan paparan $p_k(n)$ dengan n sebagai hasil tambah k nombor piramid bertapak segi tiga. Hubungannya adalah seperti berikut

$$p_k(m) = c_k^{odd}(v)$$

dengan $c_k^{odd}(v)$ adalah bilangan paparan bagi hasil tambah k nombor kubik ganjil dan integer v diperolehi daripada n. Kemudian kami lanjutkan masalah ini dengan mempertimbangkan hasiltambah nombor berkuasa s dengan $s > 3$ dan nombor politopik berperingkat s yang bersekutu dengannya.

Seterusnya, kami membincangkan hubungan antara $\Phi_{(2,k)}(n)$, bilangan paparan

bagi n sebagai hasil tambah k nombor berkuasa empat, dan $\Psi_{(2,k)}(n)$, bilangan paparan bagi n sebagai hasil tambah k nombor polinomial berbentuk $8\gamma^2 + 2\gamma$, dengan γ adalah nombor segi tiga. Bagi $1 \leq k \leq 7$, hubungannya diberi oleh

$$\Phi_{(2,k)}(8n + k) = 2^k \Psi_{(2,k)}(n).$$

Perbincangan masalah ini dilanjutkan dengan mempertimbangkan hubungan antara bilangan paparan bagi n sebagai hasil tambah nombor berkuasa $2m$ dan bilangan paparan bagi n sebagai hasil tambah nombor polinomial bersekutu berdarjah m bagi nombor segi tiga.

Ketiganya, kami mempertimbangkan hubungan antara $s_k(n)$, bilangan paparan bagi n sebagai hasil tambah k nombor segi empat dan $e_k(n)$, bilangan paparan bagi n sebagai hasil tambah k nombor segi lima berpusat. Bagi $1 \leq k \leq 7$, hubungannya diberi oleh

$$\alpha_k e_k(n) = s_k\left(\frac{8n - 3k}{5}\right), \text{dengan } \alpha_k = 2^k + 2^{k-1}\binom{k}{4}.$$

Kami lanjutkan analisis ini untuk bilangan paparan yang dipengaruhi oleh partisi λ bagi k dengan m bahagian. Jika bilangan paparan itu masing-masing diberi oleh $s_\lambda(n)$ dan $e_\lambda(n)$, maka

$$\beta_\lambda e_\lambda(n) = s_\lambda(\frac{8n - 3k}{5})$$

dengan

$$\beta_\lambda = 2^m + 2^{m-1}\left(\binom{i_1}{4} + \binom{i_1}{2}\binom{i_2}{1} + \binom{i_1}{1}\binom{i_3}{1}\right)$$

dan i_j menandakan bilangan bahagian bagi λ yang sama dengan j.

Tesis ini diakhiri dengan suatu kesimpulan dan cadangan masalah untuk kajian akan datang.

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I certify that a Thesis Examination Committee has met on **19 April 2013** to conduct the final examination of Mohamat Aidil bin Mohamat Johari on his thesis entitled "General Relation Between Sums of Figurate Numbers" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy. Members of the Thesis Examination Committee were as follows:

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

MOHAMAT AIDIL BIN MOHAMAT JOHARI

Date: 19 April 2013

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LIST OF ABBREVIATIONS

$s_k(n)$	number of representations of an integer n as a sum of k squares
$t_k(n)$	number of representations of an integer n as a sum of k triangular numbers
$s_\lambda(n)$	number of representations of an integer n as a sum of squares induced by λ
$t_\lambda(n)$	number of representations of an integer n as a sum of triangular numbers induced by λ
λ	partition
$c_k(m)$	number of representations of an integer m as a sum of k cubes
$p_k(m)$	number of representations of an integer m as a sum of k triangular pyramidal numbers
$c_k^{odd}(m)$	number of representations of an integer m as a sum of k odd cubes
$\alpha_{k,s}(n)$	number of representations of an integer n_s as $\sum_{i=1}^{k} x_i^s$, where x is a non-negative integer.
$\beta_{k,s}(n)$	number of representations of an integer n_s as a sum of k s-topic numbers where x is a non-negative integer.
$e_k(n)$	number of representations of an integer n as a sum of k centred pentagonal numbers
$e_\lambda(n)$	number of representations of an integer n as a sum of centred pentagonal numbers induced by λ

$\Phi_{(m,k)}(n)$	number of representations of an integer n as a $\sum_{i=1}^{k} x_i^{2m}$
$\Psi_{(m,k)}(n)$	number of representations of an integer n as a sum of k polynomials $P_m(\gamma)$ of degree m, where γ is a triangular number

CHAPTER 1

INTRODUCTION

1.1 A Brief History

Diophantine equations of the form

$$x_1^2 + x_2^2 + \ldots + x_n^2 = m$$

is one of topics of deep interest among researchers in mathematics since a long time ago. Fermat started his research by investigating the prime numbers that can be represented as a sum of two squares. In 1640, Fermat gave Two Squares Theorem which stated that each prime $p \equiv 1(mod 4)$ is a sum of two squares. This theorem was proved by Euler in 1754 (see Dickson (2005)). In 1738, Euler established the new result that a positive integer n can be represented as the sum of two squares if and only if n has no prime factor in its prime-power decomposition congruent to $3(mod 4)$ to an odd power(see Dudley (1978)).

For summation of three squares, Diophantus first studied a problem equivalent to finding three squares whose sum is $3a + 1$, and stated that for this problem, a must not be of the form $8n + 2$, which is however an insufficient condition. In 1621, Bachet subsequently excluded $8n + 2$ and $32n + 9$. In 1636, Fermat found that Bachet's condition failed to exclude $a = 37, 149$, etc., and gave the correct sufficient condition that a must not be of the form $((24k+7)4^n - 1)/3$. Thus $3a+1$ is not of the form $(24k+7)4^n$, or equivalently $(8m+7)4^n$.(see Wolfram Mathworld (Retrieved 18/10/2011a)

In 1636, Fermat stated that no integer of the form $8k + 7$ is the sum of three rational squares. Later in 1638, Descartes proved this statement for integer squares. In 1658, Fermat made an assertion that $2p$, where p is any prime of the form $8n-1$ is the sum of three squares without proof. In 1775, Lagrange made some progress on Fermat's assertion, but could not completely prove it. In 1785, Legendre remarked that Fermat's assertion is true for all odd numbers but he gave an incomplete proof that either every number or its double is a sum of three squares.(see Wolfram Mathworld (Retrieved 18/10/2011a)

In 1774, Beguelin had concluded that every integer congruent to 1, 2, 3, 5 or $6(mod 8)$ is a sum of three squares, but he did not give an adequate proof (see Dickson (2005)). Then, in 1798, Legendre proved that every positive integer greater than zero can be represented as the sum of three squares if and only if the integer is not of the form $4^a(8n+7)$, where a, b are non-negative integer (see Mollin (2008)).

For sum of four squares, many a researchers tried to prove that every positive integer can be written as a sum of four squares. In 1621, Bachet verified that it was true for integer up to 325, but he was unable to prove it. Fermat said that he can prove it. However no details of the proof was given (see Dickson (2005)). Descartes said that it was a very difficult problem. Euler started working on this problem in 1730. In 1743 and 1751, Euler came out with two results that are important to prove the assertion. Finally, in 1770, based on ideas of Euler, Lagrange proved in Lagrange's four-square theorem that every positive integer can be written as the sum of at most four squares (see Dudley (1978)).

One of the popular topics often discussed by mathematicians is Waring's Problem. In 1770, Waring's problem was proposed by Edward Waring. Originally, Waring only speculated that every non-negative integer is a sum of 4 squares, 9 cubes and 19 biquadrates without giving any proof. A few years later, he suggested that similar property also hold for higher powers without being more explicit (see Mollin (1989)). In 1909, Hilbert proved the general conjecture using an identity in 25-fold multiple integrals. This proof is known as the Hilbert-Waring Theorem (see Rademacher and Toeplitz (1957)).

Let $g(k)$ denote the minimum number of s of kth powers of positive integers needed to represent all integers where k is any given positive integer. Waring conjectured that $g(2) \geq 4$, $g(3) \geq 9$ and $g(4) \geq 19$, and these values were the best possible.

In 1770, Lagrange gave a proof that $g(2) = 4$ in Lagrange's four-square theorem (see Hardy (1999)). From 1909 to 1912, $g(3) = 9$ was established by Wieferich and Kempner. By using Lagrange's four square theorem and Lioville polynomial identity, Liouville proved that $g(4) \leq 53$. Hardy and Littlewood showed that $g(4) \leq 21$ (see Wolfram Mathworld (Retrieved 18/10/2011b). In 1986, Balasubramanian et al. showed that $g(4) = 19$ (see Balasubramanian et al. (1986a) and Balasubramanian et al. (1986b)). For $g(5)$, Maillet came with a proof that $g(5) \leq 192$ in 1896, then Wieferich proved that $g(5) \leq 59$ in 1909 (see Wolfram Mathworld (Retrieved 18/10/2011b)). Chen (1964) proved that $g(5) = 37$.Earlier, Pillai (1940) established that $g(6) = 73$.

Hardy and Littlewood introduced $G(k)$, which is defined to be the least positive integer s such that every sufficiently large integer can be represented as a sum of at most s kth powers of positive integers. Since squares are congruent to 0, 1, or $4 (mod 8)$, no integer congruent to $7 (mod 8)$ can be represented as a sum of three squares, implying that $G(2) \geq 4$. Since $G(k) \leq g(k)$ for all k, this shows that $G(2) = 4$.

In 1909, Landau showed that $G(3) \leq 8$ and in 1939 Dickson showed that only 23 and 239 requiring nine cubes. Wieferich proved that only 15 integers requiring eight cubes. This established $G(3) \leq 7$ (see Wells (1986)).

In 1933, Hardy and Littlewood established that $G(4) \leq 19$. This result was improved in 1936 to 16 or 17 (see Wolfram Mathworld (Retrieved 18/10/2011b)). Later, Davenport (1939b) showed that $G(4)$ was exactly 16. Vaughan (1986) made a big improvement on a method of Hardy and Littlewood and obtained an improved result for $n \geq 5$. Later, Brudern (1990) made an improvement to these results and gave $G(5) \leq 18$ and Wooley (1992) gave $G(n)$ for $n = 6$ to 20. In 1993, Vaughan and Wooley established $G(8) \leq 42$(see Vaughan and Wooley (1993a) and Vaughan and Wooley (1993b)).

In this thesis, we will discuss relations between representations of an integer n as the sum of k figurate numbers. Our motivation to study further this problem is to find a relation between the number of representations of n as the sum of k squares and number of representations of n as the sum of k triangular numbers.

1.2 Problem Statement

To determine the number of representations of an integer n as sums of powers of other integers has been an interesting and active area of research in the field of number theory. Our research interest is concentrated in searching relations between two numbers of representations of an integer n as sums of figurate numbers. In earlier works carried out by past researchers, relations between the number of representations of an integer n as sum of squares and triangular numbers were established. In our research, we extend this problem to find and establish a relations between the number of representations of integer n as sum of cubes and triangular pyramidal numbers. This will be further extended to find a relation between the number of representations of an integer n in the form $\sum_{i=1}^{k} x_i^s$ and its associated polytopic numbers of order s. We will also searching for the relation between the number of representations of an integer n as sums of powers of squares and its associated polynomials of triangular numbers. Finally, we will determine a relation between the number of representations of an integer n as sums of squares and sums of centred pentagonal numbers.

1.3 Research Objectives

Firstly we work to determine the relation between the number of representations of an integer n as a sum of k cubes and number of representations of n as a sum of k triangular pyramidal numbers. After that we extend this problem and obtain general relation between the number of representations of an integer n as sums of s-th powers with $s > 3$ and the associated polytopic numbers of order s.

Secondly, we also work to determine relation between the number of representations of an integer n as a sum of k-th powers and the number of representations of n as a sum of k terms of the polynomials $P_2(\gamma) = 8\gamma^2 + 2\gamma$, where γ is the triangular number. We also extend this result by considering the general relation between the number of representations of an integer n as a sum of k $2m$-th powers and the number of representations of n as a sum of k terms determined by associated polynomial of degree m evaluated at a triangular number.

Thirdly, we work to determine a relation between the number of representations of an integer n as a sum of k squares and the number of representations of n as sums of k centred pentagonal numbers for $1 \leq k \leq 7$. We extend this problem by considering the relation between the number of representations of an integer n as sums of squares and number of representations of n as sums of centred pentagonal numbers induced by a partition λ of $1 \leq k \leq 7$

1.4 Thesis Outline

This thesis consists of six chapters. In Chapter 1, we give a brief history on sum of squares and Waring's problem. The definitions of triangular pyramidal numbers and centred pentagonal numbers are also given in this chapter. In Chapter 2, we give some literature review on the works done by previous researchers where the results motivate us to extend the problem discussed in this thesis.

In chapter 3, we find and establish a relation between the number of representations of an integer n as sum of cubes and number of representations for sum of triangular pyramidal numbers. We also extend this problem to the relation between number of representations for $\sum_{i=1}^{k} x_i^s$ and its associated polytopic numbers of order s.

In Chapter 4, we give the relation between the number of representations of an integer n as sum of m-th powers of squares and the number of representations by certain associated polynomials of degree m evaluated at triangular numbers.

In chapter 5, we give a relation between the number of representations of an integer n as sum of squares and sum of centred pentagonal numbers. Here we use generating function and combinatorial methods to prove our result. A short discussion and some open problem are presented for further research in Chapter 6.

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4. M. A. M. Johari, K. A. M. Atan and S. H. Sapar. 2012. A Relation Between $\sum_{k}^{i=1} x_i^{2m}$ and Polynomials of Triangular Number. . (submitted)