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GEOMETRY OF TWO-QUBIT SYSTEM AND HOPF FIBRATION

WONG WEN WEI

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GEOMETRY OF TWO-QUBIT SYSTEM AND HOPF FIBRATION

By

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

GEOMETRY OF TWO-QUBIT SYSTEM AND HOPF FIBRATION

By

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March 2013

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Complex Hopf fibration and quaternionic Hopf fibration provide distinct ways of describing the state space for two-qubit quantum states. In this research, we have studied the geometry of quantum states for a two-level quantum system, along with the correspondence between complex Hopf bundle and quaternionic Hopf bundle. In the first part of our study, we investigate the behaviour of local coordinates for both Hopf bundles under different degree of entanglement such as entangled states and non-entangled states. Fubini-Study metric for complex projective space is also obtained. Its form suggests that for the intermediate entangled states, complex projective space $\mathbb{C}P^3$ can be described as a set of flat three-tori parametrized by a three-sphere. The local inhomogeneous coordinate of $\mathbb{C}P^3$ (base space of complex Hopf fibration) is found to carry the description of both subsystems A and B, whereas in the case of maximally entangled state, basis elements of local coordinates $\mathbb{C}P^3$ is not linearly independent of each other.

Next, we construct a base space map between $\mathbb{C}P^3$ and S^4, which is denoted as η map. After the mapping, we obtained phases in the base space manifold, different sections and coordinate charts are related by transition functions. We found that there is an inherent symmetry of coordinate transformation corresponds to different sections of $\mathbb{C}P^3$, which is expressed in terms of transition functions having the $U(1)$ group structure. Also under η map, phases and transition function in S^4 is doubled over that of $\mathbb{C}P^3$, indicating subtle symmetric changes after the mapping. The base space coordinates of quaternionic Hopf bundle are consist of two parts, whereby the first part is invariant to the coordinate transformation in $\mathbb{C}P^3$ but sensitive to the coordinate transformation in S^4.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

GEOMETRI SISTEM DUA QUBIT DAN SERATAN HOPF

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Seratan Hopf kompleks and seratan Hopf berkuaternion memberikan dua cara berbeza untuk memerihalkan ruang keadaan bagi keadaan kuantum dua qubit. Dalam penyelidikan ini, kita telah mengkaji geometri keadaan kuantum bagi sistem kuantum dua aras bersama-sama dengan kesepadanan antara berkas Hopf kompleks dan berkas Hopf berkuaternion. Dalam bahagian pertama kajian kami, kami mengkaji sifat koordinat setempat untuk kedua-dua berkas Hopf bagi darjah belitan berbeza seperti keadaan terbelit dan keadaan tak terbelit. Metrik Fubini-Study bagi ruang unjuran kompleks juga diperolehi. Bentuknya menyarankan bahawa untuk keadaan terbelit perantaraan, ruang unjuran kompleks $\mathbb{C}P^3$ dapat diperihal sebagai set 3-torus datar yang diparameterkan oleh sfera tiga dimensi. Koordinat tak homogen setempat $\mathbb{C}P^3$ (ruang dasar bagi seratan Hopf kompleks) didapati membawa perihalan kedua-dua subsistem A dan B, manakala dalam kes keadaan terbelit maksimum, unsur asas koordinat setempat $\mathbb{C}P^3$ adalah tidak bebas linear antara satu sama lain.

Seterusnya, kita bangunkan satu pemetaan ruang dasar antara $\mathbb{C}P^3$ dan S^4, yang diberi tatatanda pemetaan η. Selepas pemetaan, kami mendapati fasa-fasa di manifold ruang dasar, keratan dan carta koordinat berbeza terhubung dengan fungsi peralihan. Kami mendapati bahawa adanya simetri transformasi koordinat sedia ada berpadanan dengan keratan rentas $\mathbb{C}P^3$ yang berbeza, terhurai dalam sebutan fungsi peralihan yang berstruktur kumpulan $U(1)$. Juga di bawah pemetaan η, fasa dan fungsi peralihan dalam S^4 adalah dua kali ganda bagi kes $\mathbb{C}P^3$, yang menunjukkan perubahan simetri tersirat selepas pemetaan. Koordinat ruang dasar berkas Hopf berkuaternion dapat dibahagi dua yang mana bahagian pertama adalah tak varian di bawah transformasi koordinat $\mathbb{C}P^3$ tetapi sensitif kepada transformasi koordinat S^4.

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I certify that a Thesis Examination Committee has met on 14 May 2013 to conduct the final examination of Wong Wen Wei on his thesis entitled "**Geometry of Two-Qubit System and Hopf Fibration**" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

WONG WEN WEI

Date: 14 March 2013

"I believe there is no philosophical high-road in science, with epistemological sign-posts. No, we are in a jungle and find our way by trial and error, building our road behind us as we proceed."

Max Born

TABLE OF CONTENTS

Page

ABSTRACT ii
ABSTRAK iv
ACKNOWLEDGEMENTS vi
APPROVAL vii
DECLARATION ix
LIST OF FIGURES xiv
LIST OF TABLES xv
LIST OF ABBREVIATIONS xvi
LIST OF NOTATIONS xvii

CHAPTER
1 INTRODUCTION 1
1.1 Geometric Quantum Mechanics 1
1.2 Quantum Bits 3
1.3 Background Theory of Projective Spaces 4
1.4 Problem Statement 6
1.5 Objectives 7
1.6 Importance of Study 7
1.7 Structure of Thesis 8

2 LITERATURE REVIEW 9
2.1 Hopf Fibration of Single Qubit System 9
2.2 Complex Hopf Fibration and Symplectic Geometry 11
2.3 Local Orbit of Bipartite Entangled States 12
2.4 Quaternions and Hopf Fibration 14
2.5 Canonical Connection in Quaternionic Hopf Fibration 15

3 HOPF FIBRATION OF SINGLE QUBIT SYSTEM 17
3.1 Short Introduction of Lie Groups 17
3.2 Manifold 19
3.3 Fibre Bundle 21
3.3.1 Transition Function of Fibre Bundles 22
3.4 Principal Bundle 23
3.4.1 Sections of Principal Bundle 24
3.5 Hopf Fibration: A General Introduction 24
3.6 Geometry of Single Qubit 26
3.7 Conclusion 29

4 COMPLEX AND QUATERNIONIC HOPF FIBRATIONS 30
4.1 Introduction 30
4.2 Quaternionic Numbers 31

4.3 Introduction to the Two-Qubit System 33
4.3.1 Schmidt Coefficient 34
4.3.2 Entanglement Measure and Concurrence 35
4.3.3 Submanifolds of Fixed Entanglement 36
4.4 Complex Projective Space and Quantum States 37
4.5 Quaternionic Hopf Fibration 40
4.5.1 Separable States in Quaternionic Hopf Fibration 45
4.5.2 Separable States in Complex Hopf Fibration 48
4.5.3 Maximally Entangled States in Quaternionic Hopf Fibration 50
4.5.3.1 Bell States 51
4.5.4 Maximally Entangled States in Complex Hopf Fibration 53
4.6 Conclusion 54

5 BASE SPACE MAPS AND SYMMETRIES IN HOPF FIBRATIONS 55
5.1 Introduction 55
5.2 The η Maps 55
5.2.1 Separable States 64
5.2.2 Maximally Entangled States 66
5.3 Inverse Maps 68
5.3.1 Separable States 71
5.3.2 Maximally Entangled States 72
5.4 Conclusion 73

6 CONCLUSION AND FUTURE WORK 75
6.1 Future Work 78

APPENDICES
A Visualizing the Hopf Fibration in single qubit 83
A.1 Stereographic Projection 83
A.2 Gnomonic Projection 87
A.3 Orthogonal Projection 89
A.4 Mathematica Codes 91

B Derivations of Fubini-Study Metrics in Chapter 4 98
B.1 General manifold of $\mathbb{C}P^3$ 98
B.2 Submanifold of Separable States 101
B.3 Maximally Entangled States 103

C Details of calculation of a generic Hopf fibration 104
C.1 First Coordinate Chart of Hopf Bundle 104
C.2 Second Coordinate Chart of Hopf Bundle 107

D Pauli Matrices for Two-Qubit States 111

BIODATA OF STUDENT 113
LIST OF PUBLICATIONS 115

LIST OF FIGURES

Figure		Page

A.1 Stereographic projection of S^1 onto a line $\mathbb{R}^1$. 85

A.2 Visualization of the nested tori 87

A.3 Villarceau circles 88

A.4 Gnomonic projection of S^1 onto a line $\mathbb{R}^1$. 88

A.5 Gnomonic projection and the great circles 90

A.6 Orthogonal projection of S^1 onto a line $\mathbb{R}^1$. 90

A.7 Mathematica Codes-Stereographic Projection 92

A.8 Gnomonic projection of the S^3 in three-dimensional space-top view 93

A.9 Stereographic projection of the S^3 in three-dimensional space-front view 93

A.10 Mathematica Codes-Gnomonic Projection 94

A.11 Gnomonic projection of the S^3 in three-dimensional space-top view 95

A.12 Gnomonic projection of the S^3 in three-dimensional space-front view 95

A.13 Mathematica Codes-Orthogonal Projection 96

A.14 Orthogonal projection of the S^3 in three-dimensional space-top view 97

A.15 Orthogonal projection of the S^3 in three-dimensional space-front view 97

LIST OF TABLES

Table		Page
4.1	Bell states and the coordinates under Hopf maps	53
6.1	Transition functions between local sections in $\mathbb{C}P^3$	76
6.2	Transition functions between $C_{\alpha 1}$ in $\mathbb{C}P^3$	76

LIST OF ABBREVIATIONS

M.E.S.	Maximally Entangled State
Qubit	Quantum Bits

LIST OF NOTATIONS

$\mathbb{R}$	Real Numbers
$\mathbb{C}$	Complex Numbers
$\mathbb{H}$	Quaternionic Numbers
ρ	Density Matrix
Tr	Partial Trace
ρ_α	Partially Traced Density Matrices
ξ	Hopf Maps
$\|\psi\rangle$	Ket Vector
$\langle\psi\|$	Bra Vector
$\sigma_x, \sigma_y, \sigma_z$	Pauli Matrices
$\otimes$	Tensor Product
$\times$	Cartesian Product
ξ	Quaternionic Hopf Map
ϵ	Complex Hopf Map
$GL(N)$	General Linear Group of Dimension N
$U(N)$	Unitary Group of Dimension N
$SU(N)$	Special Unitary Group
$SO(N)$	Special Orthogonal Group
$\mathbb{C}P^n$	Complex Projective Space
$\mathbb{H}P^n$	Quaternionic Projective Space
q	Quaternion
s_α	Cross Sections of $\mathbb{C}P^3$
B	Base Space
E	Total Space

P	Principal Bundle
F	Fibre Space
U	Open Neighbourhood
h_0	Map from S^7 to $\mathbb{C}^2 + \infty$
h_1	Map from $\mathbb{C}^2 + \infty$ to S^4
$t_{\alpha\beta}, t', g_{\alpha\beta}$	Transition Functions
$Q_{\alpha\beta}, Z_\alpha$	Complex Coefficients
$s_{\alpha\beta}$	Sections of Quaternionic Hopf Maps

To my parents.

CHAPTER 1
INTRODUCTION

This chapter gives an overview of our work as well as introductions to some of the central ideas pertaining to this research. Attention is given to quantum entanglement and qubits along with their physical and geometrical properties.

1.1 Geometric Quantum Mechanics

Quantum entanglement, the spooky action at the distance as it is called by Einstein, is a phenomenon whereby two physical entities are correlated in such a way that one cannot describe the one quantum system without the knowledge of another, and they are sharing a single superposition state until a measurement is made [25]. Mathematically, quantum entanglement is associated with the existence of vectors in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ that are not of the tensor product form $\vec{\psi}_1 \otimes \vec{\psi}_2$, where $\vec{\psi}_1$ and $\vec{\psi}_2$ are two vectors correspond to their respective quantum subsystem. Another important property of entanglement is that it can persist over a long distance, which indicates that the entanglement is a non-local correlation. This unintuitive behaviour has seen potential application in real world, such as quantum teleportation. Furthermore, quantum entanglement has played a major role in the development of various fields of study in physics, in particular, quantum information science and quantum computing.

As a comparatively new coherent discipline of physics, quantum information began to emerge in the 1980s and blossomed in 1990s, during which many breakthroughs such as quantum teleportation and formulation of the entanglement of formation measure are made [35, 36]. Besides entanglement, other properties of quantum theory that have made quantum information to be quite different from classical information are the non-deterministic nature of quantum process (at least according to Copenhagen interpretation) and uncertainty principle. The latter has the implication that measuring one observable A will inevitably influence the outcomes of succeeding measurement of an observable B [28], and that to acquire information about a quantum system will disturb the quantum system itself. Classical physics (in particular, Newtonian physics) saw no such limitation.

Closely associated with these properties of quantum theory is the investigation on the geometry of quantum mechanics, which is pioneered by von Neumann in his discovery that a quantum system can be considered as a point in a Hilbert space [33]. In recently years, the works of Kibble and his collaborators have shown that quantum mechanics exhibits a natural Hamiltonian phase-space dynamics, which means that quantum theory possesses an intrinsic mathematical structure that has an equivalent counterpart in Hamiltonian phase-space dynamics [5]. The state space of a quantum system is represented by complex projective space $\mathbb{C}P^n$, and its relation to quantum states is widely studied [1, 2, 7, 8, 14, 20]. For a two-qubit system, the underlying projective space is $\mathbb{C}P^3$, which was studied by [2] and [20] (see chapter 2).

The quantum state space can also be described in the language of fibre bundle and related geometrical concepts. For a two-qubit quantum system, quaternionic Hopf fibration is a useful mathematical tool for reducing Hilbert space of the composite system. Basically, it is a fibre bundle whereby the total space is a unit sphere S^7 and a map from that total space to a base space S^4. Quaternionic Hopf fibration was studied by Mosseri and Dandoloff [21] for two qubits, and it was extended to the study of three-qubit quantum system in the works of Bernevig [4] and Pinilla [26].

1.2 Quantum Bits

In quantum information and quantum computing, one of the underlying fundamental concepts other than entanglement is the qubit (or quantum bit) representation of physical states as the information units. In analogous to the classical bits having the state either 0 or 1, a qubit for a two-level quantum system also has two possible states, written in Dirac notation as $|0\rangle$ and $|1\rangle$. However, qubit is different from a classical bit as it can be in a superposition state of $|0\rangle$ and $|1\rangle$. This superposition can be understood as a linear combination of states:

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{1.1}$$

where the states $|0\rangle$ and $|1\rangle$ are the orthonormal basis vectors which may represent the spin up and spin down of a spin 1/2 particle (such as electron) in a vector space, with $\alpha, \beta \in \mathbf{C}$. Note that in this context, a qubit is treated as an abstract mathematical object instead of actual physical states and this would allow a construction of a general theory of quantum computation and quantum information, independent of any particular physical realization.

Despite the fact that there could be an infinite number of linear combinations for the equation (1.1), quantum mechanics tells us that there is restriction about the amount of information we could extract from the equation [23]. For instance, measurement of a qubit yield either $|0\rangle$ or $|1\rangle$, with respective probabilities $|\alpha|^2$ and $|\beta|^2$. Since probabilities are required to sum to one, complex coefficients can be written as

$$|\alpha|^2 + |\beta|^2 = 1, \tag{1.2}$$

which is known as the normalization condition. To introduce entanglement into a system, we require the state space to describe a composite system where at least two qubits are involved. The state space of a separable (without entanglement) composite physical system is the tensor product of the state space of the component physical systems. This is however not true for entangled qubits, taking the example of this state:

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{1.3}$$

This quantum state cannot be described as a product state of two component states. A composite system that fulfill this property is called an entangled state.

1.3 Background Theory of Projective Spaces

The linear representation of state that is described in the equation (1.1) and the normalization condition in the equation (1.2) have profound geometrical meanings. The earlier equation implies that it corresponds to the dependence of the physical properties of the system on the wave function up to an overall complex factor, while the latter equation (1.2) implies that in general, a normalized quantum state is a unit vector in a complex vector space. Based on these conditions, we can define the pure states in the Hilbert space $\mathcal{H}$ as

$$|\Psi\rangle = \sum_{i,j} a_{ij} |i\rangle \otimes |j\rangle, \tag{1.4}$$

where $|i\rangle$ and $|j\rangle$ represent the bases of quantum state of each system in the respective Hilbert space, i.e. $|i\rangle \in \mathcal{H}_A$ and $|j\rangle \in \mathcal{H}_B$, a_{ij} are complex coefficients.

Each component state falls on the complex line and it admits an equivalence relation $a_{ij} \sim z a_{ij}$, $z \in \mathbb{C}/\{0\}$. This construction defines a complex projective space $\mathbb{C}P^n$, which is the set of all lines that pass through the origin in $\mathbb{C}^{n+1}$ space. If we further impose the normalization condition, the space of all lines will become a $(2n+1)$-sphere. Geometrically speaking, we can think of the quantum state as a complex line modulo a global phase factor,$e^{i\theta} \in S^1$, therefore every points in $\mathbb{C}^{n+1}$ that varied by a factor of $e^{\mathbf{i}\theta}$ is identified. To illustrate the case more clearly, we will give an example on $\mathbb{C}P^3$, which is the space of two subsystems. Mathematically speaking, the complex projective space can be described as quotient of the S^7 space by the action of S^1, $S^1 \hookrightarrow S^7 \rightarrow \mathbb{C}P^3$. This is the definition of a complex Hopf fibration (more rigorous definition about Hopf fibration will be discussed for the cases of single qubit and two qubits in chapter 3 and chapter 4 respectively). It is by itself a principal fibre bundle of which the S^1 as a group acts on the total space S^7.

For the two qubit system, there is also a type of Hopf fibration that is not solely based on complex number field but also quaternionic number field $\mathbb{H}$, this Hopf fibration is known as quaternionic Hopf fibration (it is called just Hopf fibration if no comparison to the complex counterpart is made), it can be described with a surjective map from the total space S^7 to the base space S^4, which is also called quaternionic Hopf map and is denoted as ξ. For two-qubit system, this means that we can write the homogeneous coordinates of a point as $(q_0, q_1, \ldots, q_n)$, where q_α are quaternions such that not all of them are zero. Similar to complex projective space, we may define the quaternionic projective space by identifying all the points that are differed by one quaternion,

$$(q_0, q_1, \ldots, q_n) \sim q(q_0, q_1, \ldots, q_n), q \in \mathbb{H}, \tag{1.5}$$

this construction is denoted as $\mathbb{H}P^n$, which is a projective space of dimension n over $\mathbb{H}$ field. For example, $\mathbb{H}P^1$ is the quaternionic projective line for which $(q_0) \sim Q(q_0)$. Moreover, since quaternion is the extension of complex number, there are some analogies between quaternionic projective line and complex projective line $\mathbb{C}P^1$ in their respective roles in Hopf fibration. For example, $\mathbb{H}P^1$ is isomorphic to S^4, base space of the quaternionic Hopf fibration, while $\mathbb{C}P^1$ is isomorphic to the base space of complex Hopf fibration, which happens to be S^1.

1.4 Problem Statement

As far as current literature is concerned, there is no effort in bridging the quaternionic Hopf fibration and complex Hopf fibration with respect to two-qubit system. It is therefore considered the best of our interest to see the correspondence between these two fibre bundles that are different in formalism yet bear the description of the same quantum system. Our aim here is to extend the previous research of [2, 4, 19, 21], by exploring the geometry of quaternionic Hopf fibration in detail. At a first glance, both complex and quaternionic Hopf fibration share the same total space, S^7, hence we could construct a map from the local section of $\mathbb{C}P^3$ in S^7 to S^4 to identify their correspondence. The geometry of two qubits under complex Hopf fibration will also be studied and compared with the quaternionic one, using the above mentioned base space maps. Our next aim is to develop how symmetries in term of phases and transition functions are carried along with the fibres when such mapping is constructed.

1.5 Objectives

- To compare and contrast the local coordinates of complex Hopf fibration and quaternionic Hopf fibration for two-qubit state space.
- To identify the correspondence between complex Hopf fibration and quaternionic Hopf fibration for two qubits with respect to their degree of entanglement.
- To calculate the transition functions over local sections of $\mathbb{C}P^3$ and the coordinate charts of $\mathbb{H}P^1$.

1.6 Importance of Study

The study of the geometric properties of two-qubit quantum system will provide a new insight on the physical nature of the quantum system. The research goal is to investigate the geometric relation between the quaternionic Hopf fibration and complex Hopf fibration, by using suitable maps that connects the base space and total space of each fibration. This will give a clearer understanding about the correspondence between complex Hopf fibration and quaternionic Hopf fibration for different states of entanglement.

1.7 Structure of Thesis

This thesis is organized in six chapters. The second chapter consists of review sessions of several papers pertaining to our research. The central paper is Mosseri's paper on quaternionic Hopf fibration [21], other papers which touch on various topics surrounding Hopf fibration are also included in this literature reviews, such as Urbantke's paper on the single qubit and Hamiltonians [30], Symplectic Geometry [29], Bengtsson's paper on two-qubit complex projective space [2], and Peter Levay's work on the connection in Hopf bundle [19].

The third chapter is about theory and methodology, where we aim to give an overview of the mathematical ideas and geometrical concepts that are important in our study. These include a brief introduction to the concept of fibre bundle and principal bundle, Hopf fibration for a single qubit and some other mathematical tools.

Our main calculations and results are discussed in chapter 4 and chapter 5. The main content of chapter 4 is the investigation of the state space of two qubits for both quaternionic and complex projective space, where details regarding both spaces are explored including the Fubini-Study metric for complex projective space. In chapter 5, we try to construct a map from the six-dimensional complex projective space to quaternionic projective space, and study the transition functions and fibres structures.

In chapter 6, conclusion of our research is given and some remarks regarding future work are discussed.

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