UNIVERSITI PUTRA MALAYSIA

CLASSIFICATION OF SECOND CLASS FILIFORM LEIBNIZ ALGEBRAS IN DIMENSION NINE

FATANAH DERAMAN

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CLASSIFICATION OF SECOND CLASS FILIFORM LEIBNIZ ALGEBRAS IN DIMENSION NINE

By

FATANAH DERAMAN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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DEDICATIONS

With Special Love,
Deraman Ismail and Rohani Ibrahim
Also goes my beloved,
Meriam Hj Jalal
For,
Mardziah Deraman and Shukri Abd. Karim
Sara Shukri
Ariff Fahmi Deraman and Azie Azwa Kamaruddin
Muhammad Naim Deraman
Siti Amnah Deraman
Nurul Aunie Deraman
Wafi Arawi Deraman
Muhammad Farees Deraman
Ilham Aqil Deraman
Last but not least,
Zahela Ismail and Zulkurnain Ismail For all their tough and value to bring me until here,
my dear teachers.

Satu itu Alif, Alif itu Ikhlas
Ikhlas itu perlu diletakkan pada tempat Pertama, Juga Perlu di Utamakan

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

CLASSIFICATION OF SECOND CLASS FILIFORM LEIBNIZ ALGEBRAS IN DIMENSION NINE

By

FATANAH DERAMAN

August 2013

Chair: Professor Isamiddin S. Rakhimov, PhD

Faculty: Science

This study is centered on part of structural property of Leibniz algebras which is classification of a subclass of filiform Leibniz algebras over complex field. The class of filiform Leibniz algebras has been divided into three disjoint classes. The first two subclasses denoted by FLb_n and SLb_n arise from naturally graded non-Lie filiform Leibniz algebras. The third class, TLb_n arises from naturally graded filiform Lie algebras. In particular, the study is focused on the classification problem of second class filiform Leibniz algebras, SLb_n. The method employed in the classification is to first construct multiplication table of SLb_9. Then, adapted transformation and elementary base change are used to create an isomorphism criterion. From the isomorphism criterion, the isomorphism classes of algebras from SLb_9 are deduced. Some of these classes are single orbits while others are parametric family orbits. An invariant algebraic approach is used to describe the parametric family of orbits. As a result, it is concluded that there are 38 families of non-isomorphic classes of algebras in SLb_9. This comprises 17 single orbits and 21 parametric families of orbits.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENGELASAN ALJABAR LEIBNIZ FILIFORM KELAS KEDUA DALAM MATRA SEMBILAN

Oleh

FATANAH DERAMAN

Ogos 2013

Pengerusi: Profesor Isamiddin S. Rakhimov, PhD

Fakulti: Sains

Tesis ini menumpukan kajian kepada salah satu bahagian dalam ciri-ciri struktur aljabar Leibniz yang merupakan pengelasan bagi subkelas aljabar Leibniz filiform terhadap medan kompleks. Aljabar Leibniz filiform ini telah dibahagikan kepada tiga kelas yang tidak bercantum. Dua subkelas yang pertama dilambangi FLb_n dan SLb_n yang terbit daripada pengelasan secara asli aljabar Leibniz filiform bukan Lie. Kelas ketiga, TLb_n terbit daripada pengelasan secara asli aljabar Lie filiform. Secara khususnya kajian ini berfokus kepada pengelasan kelas kedua aljabar Leibniz filiform, SLb_n. Kaedah yang digunakan dalam kajian pengelasan ini adalah pertamanya, membina sifir pendaraban dalam SLb_9. Kemudian transformasi yang telah diadaptasi dan pertukaran asas permulaan digunakan untuk menghasilkan kriterium isomorfisma. Daripada kriterium isomorfisma ini, isomorfisma kelas aljabar akan diperoleh dari SLb_9. Sebahagian daripada kelas aljabar ini adalah orbit tunggal, manakala yang lain adalah keluarga orbit berparameter. Hasilnya boleh dirumuskan sebagai 38 keluarga aljabar tak berisomorfisma di dalam SLb_9 yang terdiri daripada 17 orbit tunggal dan 21 keluarga orbit berparameter.

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I certify that a Thesis Examination Committee has met on **22 August 2013** to conduct the final examination of Fatanah Binti Deraman on her thesis entitled "Classification of Second Class Nine Dimensional Filiform Leibniz Algbras" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the of Master Of Science.

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

FATANAH DERAMAN

Date: 22 August 2013

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF ABBREVIATIONS

Lb_n	The class of n dimensional filiform Leibniz algebras
LN_n	The class of n dimensional nilpotent Leibniz algebras
FLb_n	First class of n dimensional complex filiform Leibniz algebras
SLb_n	Second class of n dimensional complex filiform Leibniz algebras
TLb_n	Third class of n dimensional complex filiform Leibniz algebras
$GL_n(\mathbb{K})$	Group of $n \times n$ nonsingular matrices over $\mathbb{K}$
$Alg_n(\mathbb{K})$	Algebraic variety of n dimensional algebra structures over a field $\mathbb{K}$
G_{ad}	Subgroup of all adapted transformations of $GL_n(\mathbb{K})$
NGF_i	i class of naturally graded filiform Leibniz algebras

CHAPTER 1

INTRODUCTION

This thesis presents the results on classification problems of second class of filiform Leibniz algebras in dimension nine. This class arises from naturally graded non-Lie filiform Leibniz algebras. In general, the classification problem of algebras is an age long problem which cannot be overemphasized. Therefore, we have to consider some subclasses under some conditions. Nilpotency and filiformity are some of such kind conditions.

This chapter contains preface to get a hold of literature review, research objectives and sketch of the thesis chapters. The methods used in this study gathered tools from different known concepts such as group action, linear transformation, elementary base change, just to mention but a few.

This chapter also contains an engrossing and fast-paced historical review combine with current study which brought an answer for the readers who are anxious about the present research.

1.1 Literature review

According to Helgason (1994), classifications and simplifications of transformation groups have been initiated by Sophus Lie. He also introduced a class of non-commutative and non-associative algebras which is known now as the class of Lie algebra. The extensive studies on the cohomological problems of algebras by Jean-

Louis Loday in 1990s, leads to a generalization of the Lie algebras called Leibniz algebras (Loday and Pirashvili, 1993). The Leibniz algebras have been found to be closely related to Lie algebras. As for the classification problem of finite dimensional Lie algebras, it can be divided into three parts. They are the classification of nilpotent Lie algebras, the description of solvable Lie algebras with given nilradical and the description of Lie algebras with given radical. The classification of nilpotent complex Lie algebras is only known in dimension up to seven. The description of solvable Lie algebras has been reduced to the description of orbits of certain unipotent linear groups. The description of solvable Lie algebras has been reduced to the description of semisimple subalgebras whose classification has been known in the work of Cartan-Killing. In dimension six there are several classification results over a field of characteristic zero and an arbitrary field has been given by Morozov (1958), Shedler (1964), respectively. Romdhani (1989) studied on dimension seven. All these results were obtained using a variety of isomorphism invariants. There are some other results on classification of subclasses of nilpotent Lie algebras such as Scheuneman (1967), Gauger (1973), Galitski and Timashev (1999) and Gomez et al. (1998) (filiform Lie algebras) and others.

In recent year, there has been an increasing interest on investigation of Leibniz algebras problems. Albeverio et al. (2005) describe the irreducible components of the nilpotent complex Leibniz algebras varieties of dimension less than five. They also construct degenerations between one parametric families of nilpotent Leibniz algebras and study the rigidity of these families. In the same years, Omirov

(2005) studied analogues of zero filiform and filiform Leibniz algebras in the finite dimensional case. The Leibniz superalgebras are generalizations of Leibniz algebras. This area can be investigated by applying restrictions on their characteristic sequence (Ayupov et al., 2009). By using Mathematica software, Casas et al. (2009) develop a computer program to check the Leibniz identity. Cabezas et al. (2011) present the classification of a subclass of naturally graded Leibniz algebras. Meanwhile, Casas et al. (2013) extend the classification of solvable Lie algebras with naturally graded filiform Lie algebra to the case of Leibniz algebras. Namely, the classification of solvable Leibniz algebras whose nilradical is naturally graded filiform Leibniz algebras.

The structure theory of Leibniz algebras is studied through derivations, classification, Cartan subalgebras, weight spaces, solvability and etc. The derivations of naturally graded Leibniz algebras are described by Omirov (2005). Albeverio et al. (2006b) give the classification of nilpotent Leibniz algebras up to dimension four. A further study on the classification of low dimensional filiform Leibniz algebras over a field of p-adic numbers is done by Ayupov and Kurbanbaev (2010). Rikhsiboev and Rakhimov (2011) completed the classification of three dimensional complex Leibniz algebras. Besides that, the details on properties of Cartan subalgebras, weight space and solvability of finite dimensional Leibniz algebras are discussed in Albeverio et al. (2006a) and Zargeh (2012).

The notion of filiformity for Leibniz algebras was introduced by Ayupov and

Omirov (2001). In the same paper the authors deal with the class of Leibniz algebras arising from naturally graded complex non-Lie filiform Leibniz algebras. The authors showed that this class is split into two subclasses. Later, Gomez and Omirov (2006) gave the isomorphism criteria for these two classes and they mentioned an existence of one more class of filiform Leibniz algebras which arises from naturally graded filiform Lie algebras. The third class of filiform Leibniz algebras has been treated by Omirov and Rakhimov (2009). They gave an adapted basis with respect to this class and were written in simple table form. As for the first two classes Rakhimov and Bekbaev (2010), suggested an approach based on algebraic invariant to classify these two classes in fixed dimensional case. This classification approach uses the notion of group action. Furthermore, it was stated that the results can also be used in geometric classification of filiform Leibniz algebras.

Sozan et al. (2010) gave the classification of nine dimensional first class of complex filifom Leibniz algebras. The classification up to dimension six of the third class of complex filiform Leibniz algebras has been given by Hassan (2010). The classification problem of lower dimensional cases of the first and second classes have been studied by Rakhimov and Said Husain (2011a,b). The aim of this thesis is to give a complete classification of the second class of filiform Leibniz algebras in dimension nine.

1.2 Objectives

The main objectives of this research are:

1. To give a multiplication table of nine dimensional second class of complex filiform Leibniz algebras (SLb_9).

2. To create isomorphism criterion for SLb_9.

3. To list the isomorphism classes and their respective orbits.

4. To give invariants in parametric orbits case.

1.3 Outline of chapters

This thesis contains six chapters. It is divided into three main sections which are introduction, results and conclusion. In Chapter 1, the historical background and objectives are discussed. Meanwhile, Chapter 2 focuses on the basic concepts and the methodology employed in this research. The main results are presented in Chapter 3, Chapter 4 and Chapter 5. Basically, Chapter 3 focuses on isomorphism criterion. In Chapter 4 and Chapter 5 the isomorphism classes and their respective invariants are given. Finally, Chapter 6 summarizes the results and gives recommendations for further research.

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