UNIVERSITI PUTRA MALAYSIA

MULTI PURSUER DIFFERENTIAL GAME OF OPTIMAL APPROACH WITH INTEGRAL CONSTRAINTS ON PLAYER CONTROLS

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MULTI PURSUER DIFFERENTIAL GAME OF OPTIMAL APPROACH WITH INTEGRAL CONSTRAINTS ON PLAYER CONTROLS

By

NORSHAKILA BT ABD RASID

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirement for the Degree of Master of Sciences

December 2013

DEDICATION

To

Muhamad Arham Hashim

Sarah Safiyyah

My Father and My Mother

My lovely Family

For their support, encouragement and love

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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Chair: Associate Professor Gafurjan Ibragimov, PhD
Faculty: Science

A game contains three important elements, which is players, set of strategies, and payoff that quantitatively identified either win or lose for each players in term of the amount called value game. Pursuit-evasion game is a common type of game that describes how to guide one or a group of pursuers to catch one or a group of moving evader in a given environment.

This thesis studied a pursuit-evasion differential game of optimal approach of finite or countable pursuers with one evader in the Hilbert space. The movements of players are described by the ordinary differential equation of first and second order. The control functions of players are subject to integral constraints, such constraints arise in modelling the constraints on energy. Important point to note is resource energy for the control of each pursuer need not to be greater than that of evader. Duration of the game q is fixed. The payoff functional is the greatest lower bound of distances between the pursuers and evader when the game is finish. The pursuers try to minimize the functional and, the evader tries to maximize it. The formula to calculate the value of the game is given and construction of optimal strategies of the players. To solve the main theorem of the problem in this thesis relies on the solutions of auxiliary differential game in half space, and some properties of balls in half space. In the first part of proof of the theorem, the method of counterfeit or fictitious pursuers is used.

The thesis shows the sufficient condition for the pursuers to catch the evader explicitly, and also prove the admissibility of the strategy. It is shows that the strategies of players ensure the value of the game.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Sarjana Sains

PERMAINAN PEMBEZAAN BEBERAPA PEMANGSA BAGI PENDEKATAN OPTIMA DENGAN KAWALAN PEMAIN ADALAH TERTAKLUK KEPADA KEKANGAN KAMIRAN

Oleh

NORHAKILA BT ABD RASID
Disember 2013

Pengerusi: Profesor Madya Gafurjan Ibragimov, PhD
Fakulti: Sains

Permainan mengandungi tiga unsur utama, iaitu pemain, set strategi, dan satu ganjaran yang mengenal pasti secara kuantitatif sama ada menang atau kalah bagi setiap pemain di dalam bentuk nilai yang disebut sebagai nilai permainan. Permainan mangsa-pemangsa merupakan jenis pemainan umum yang menerangkan bagaimana memberi panduan kepada satu atau sekumpulan pemangsa untuk menangkap satu atau sekumpulan mangsa di dalam persekitaran yang diberi.

Dalam tesis ini, kami mengkaji permainan pembezaan mangsa-pemangsa bagi pendekatan optima untuk bilangan pemangsa terhingga atau yang boleh dibilang dengan satu mangsa di dalam ruang Hilbert. Pergerakan pemain ini dijelaskan melalui persamaan pembezaan biasa peringkat pertama dan kedua. Fungsi kawalan bagi pemain adalah tertakluk kepada kekangan kamiran, kekangan tersebut wujud dalam merangka kekangan yang berkaitan tenaga. Perkara penting yang perlu diberi penekanan disini adalah sumber kawalan tenaga untuk setiap pemangsa tidak semestinya lebih besar daripada sumber kawalan tenaga mangsa. Tempoh permainan masa θ telah ditetapkan. Fungsi ganjaran adalah batas bawah terbesar dari jarak pemangsa dan mangsa apabila pemainan tamat. Pemangsa cuba meminimumkan fungsi ganjaran dan mangsa cuba memaksimumkannya. Kami mencadangkan satu formula untuk mengira nilai permainan dan membina strategi yang optimum bagi pemain-pemain tersebut. Bagi menyelesaikan permasalahan teorem utama didalam tesis ini, bergantung kepada penyelesaian dari permainan pembezaan bantuan didalam ruangan separa, dan mengambil kira sedikit ciri atau sifat bola didalam ruangan separa. Bahagian pertama pembuktian teorem tersebut, kami menggunakan kaedah pemangsa khayalan atau tiruan.

Tesis ini menunjukkan keadaan yang memadai untuk pemangsa menangkap mangsa dengan jelas dan nyata. dan membuktikan strategi tersebut boleh diterima pakai. Menunjukkan bahawa strategi bagi pemain-pemain tersebut menjamin nilai permainan.

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I certify that a Thesis Examination Committee has met on 30 December 2013 to conduct the final examination of Norshakila Abd Rasid on her thesis entitled "Multi Pursuer Differential Game of Optimal Approach with Integral Constraints on Player Controls" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Sciences.

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DECLARATION

I hereby confirm that:

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TABLE OF CONTENT

LIST OF FIGURE

LIST OF ABBREVIATIONS

In this page, some of the abbreviations involved in this thesis are listed

l_2	Hilbert space
P	Pursuer
E	Evader
$\dot{x}, \ddot{x}$	First and second derivatives of x
u_i	Control of the ith pursuer
v	Control of the evader
x_{ik}	k coordinates of the ith pursuer
y_k	k coordinates of the evader
U_i	Strategy of the ith pursuer
V	Strategy of the evader
θ	Duration of the game
γ	Value of the game
z_i	The ith fictitious pursuer
$\mathbb{R}^n$	n-dimensional field of real number
$H(x_0, r)$	Ball of radius r centered at x_0
$S(x_0, r)$	Sphere of radius r centered at x_0
inf	Infimum
sup	Supremum

CHAPTER 1
INTRODUCTION

1.1 Game

There are many type of game from general to specific , with a wide range of descriptions from verbal to formal. A game can be defined formally from mathematical point of view. Each game involves a number of players, a set of strategies for each of the players, and a set of score that explain the result of win or lose for each of the player quantitatively. A game starts with a plan with various strategies for each player. The plan may be complicated in order to give power to players to take action in probable situation, and the number of moves depends on the type of the game. It is good to have a plan for a game which have limited move compare to the complicated one. In more complicated games such as chess, no definite set of moves exist, hence the number of strategies could be huge [3].

1.2 Differential Game

The term "Differential Games" is applied to a group of problems in applied mathematics. Initially, the study of differential game is for military purpose. Differential game arises in the study procedures in which one controlled object will be pursued by another. In the differential game fields, we consider some principle that describes the game which is known as set of differential equations. Hence its known as differential game. Two different players, which has different goal will choose their own action bounded by control constraints. The most common example of differential games in military problem is Pursuit-Evasion (PE) game. So here, pursuit-evasion differential game defined procedures how one or many pursuers try to catch one or many evader.

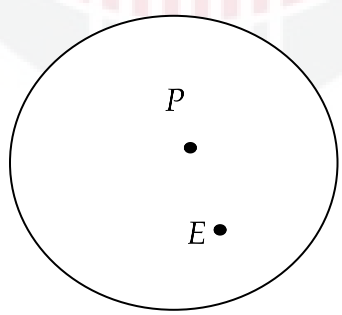
Isaacs [23] denoted E as an Evader and P as a Pursuer, and we will follow his notation.

Through the differential game that we discuss in this thesis, each player has their own goal, which is opposite to each other. The term of goal here are known as payoff functional. The payoff functional of a game is the greatest lower bound of distances between the evader and the pursuers when the game is terminated (at a given time $t = \theta$). The goal of the pursuers is to minimize the payoff, whereas the evader's goal is to maximize the payoff. These "games" are modelled mathematically by first defining state variables that represent the position (and perhaps velocity) of the participants, determining (differential) equations of motion for the rivals, and then describing sets in the state space called target sets. (For example, a target set for a pursuer may include points in the state space where the distance between the pursuer and the evader is small). Each participant in the game tries to drive the state variables of the game into a particular target set by controlling key variables called, naturally, controls. Definition of strategy in the field of differential game is a guideline to the players, to choose the best action in any possible circumstance. As the games proceed, players will select the best action (strategy) within their control constraint, regarding changes in the state variables. When the game arrives at specific time, the termination of the game will occur. This terminal value, which is an optimal situation of the players, could give as a function of the strategies. The value of the game is the payoff of the terminal state.

Throughout this thesis, we consider zero-sum pursuit-evasion differential game (In zero-sum differential game there are two players competing to each other to achieve their own goal) with multi-pursuers in the Hilbert space l_2. Integral constraints are imposed on controls of players and duration of the game is fixed. Movements of players are described by linear differential equations.

1.3 Lion and Man Problem

The classical Lion and Man problem is a game posed as to determine a strategy for a pursuer (lion) to capture the evader (man) in a given environment. Capturing means that the man and lion become at the same position after a finite time. The aim of the lion is to capture the man for any trajectory and any control of the man as evader. The man wins the game if it can avoid capture. Both the Lion and Man have identical motion capabilities. Capture strategies are important in surveillance where we would like to detect and capture equally agile intruders. Suppose P as a Pursuer (lion) and E as an Evader (man).



Let P = Pursuers(Lion); E = Evader (Man)

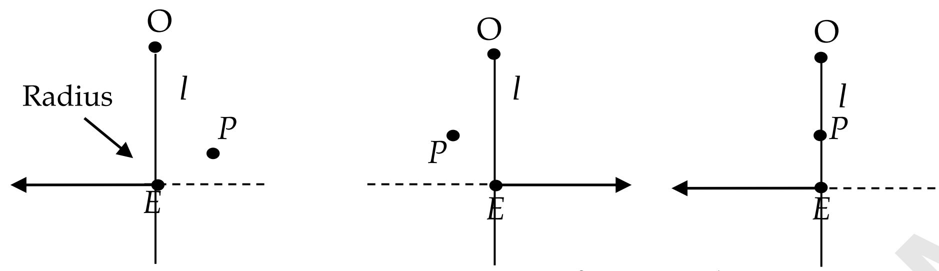


Figure 1.1: Movement of Lion and Man.

From Figure 1.1, assume that the situation where *P* and *E* are confined to a circular arena and move with maximum speeds bounded by 1. Moreover, *P* and *E* have perfect information about each other's position, but they have contrary objectives; *P* wants to decrease his distance from *E* to value zero in finite time, while *E* wants to avoid being captured by *P*. If the position of the pursuer coincides with that of the evader then pursuit is said to be completed.

Theorem 1.1. In the Lion and Man game, evasion is possible [34].
Proof. The proof of this theorem includes three parts.

1. Construction of the strategy of the evader.
Let the position of *P* and *E* at t are denoted by *P(t)* and *E(t)* respectively. Assume that evader is inside the circle. Evader change directions of his velocity *E* at times $t_1,...,t_n$ where t_i, $i=1,...,n$ is a time, at which distance of the point $E_i = E(t_i)$ from the circumference equals $r/(i+1)$, where *r* is distance of E_0 from the circumference at $t_0 = 0$. At each time t_i, we pass a straight line l_i passing through the point *O* and E_i (see Figure 1.1). There are three possible cases. The pursuer $P_i = P(t_i)$ is on the right of l_i, or on the left of l_i, or on the line l_i. If the pursuer is on the right of l_i, then *E* moves to the left perpendicularly to l_i. If P_i is on the left of l_i, then *E* moves to the right perpendicularly to l_i. Finally, if P_i is on l_i, then *E* moves either to the left or to the right of l_i. In the last case, for definiteness we take the left direction. Without restriction of generality assume that P_i always is on the right or on l_i, i.e., P_i is not on the left of l_i. Then *E* moves in the left direction.

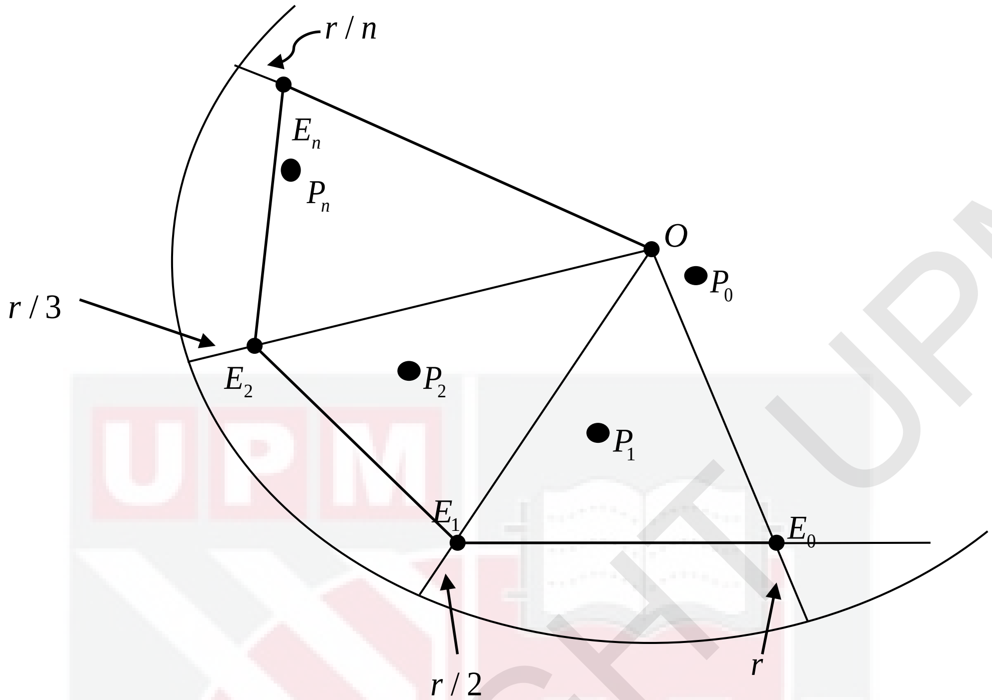


Figure 1.2: Evasion of the Evader.

2. Evasion is possible on each section.

Let $E_i = E(t_i)$ we show that on each section E_iE_{i+1} evasion is possible. Note that $P(t_i)$ is not on the left of l_i (see Figure 1.3).

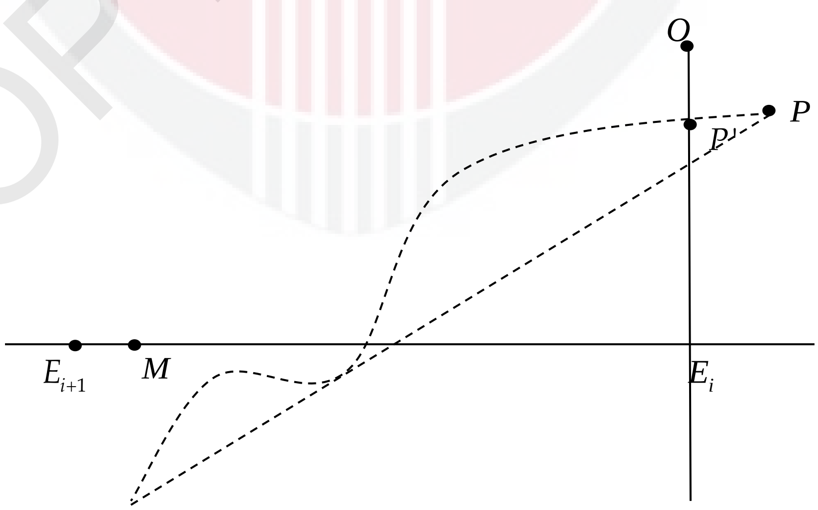


Figure 1.3 : *P* meets *E* at some point *M*

Assume the contrary: Let *P(t)* = *E(t)* = *M* at some point $M \subset E_i E_{i+1}$. Then

$$t = \frac{E_i M}{1} = \frac{PM}{1} \geq \frac{PM}{1} \geq \frac{P'M}{1} > E_i M = t,$$

where $\overset{?}{P}M$ is length of the curve *PM*, *P'* is intersection point of the straight line *PM* with OE_i. Then, it is a contradiction. Therefore on each section $E_i E_{i+1}$ the evader will not be captured by the pursuer.

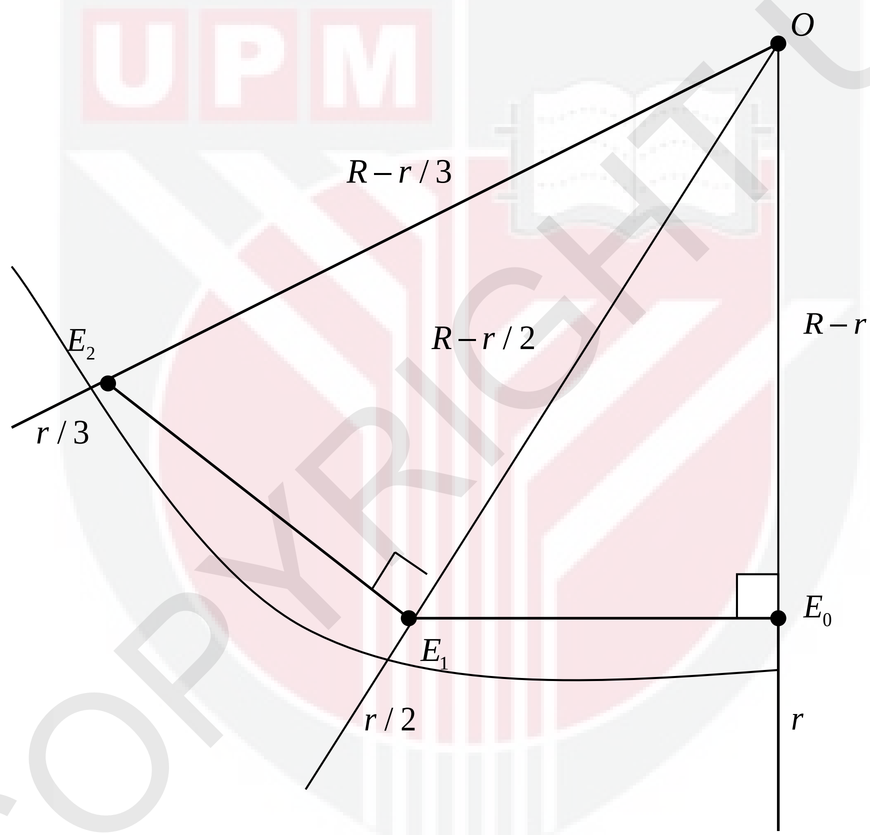


Figure 1.4 : Estimation the total time.

3. Estimation of total time.

The previous estimations shows that evasion is possible on each section of $E_i E_{i+1}$. To traverse the evader *E* spends time equal to

$$t_i = \frac{E_i E_{i+1}}{1} = E_i E_{i+1}.$$

Thus

$$t_1 = E_1E_2 = \left(\left(R - \frac{r}{2}\right)^2 - (R-r)^2\right)^{1/2}$$

$$t_2 = E_2E_3 = \left(\left(R - \frac{r}{3}\right)^2 - \left(R - \frac{r}{2}\right)^2\right)^{1/2}$$

$$\vdots$$

$$t_n = E_nE_{n+1} = \left(\left(R - \frac{r}{n+1}\right)^2 - \left(R - \frac{r}{n}\right)^2\right)^{1/2}.$$

We show that $t_n \geq \dfrac{r}{n+1}$.

Indeed,

$$R^2 - \frac{2Rr}{n+1} + \frac{r^2}{(n+1)} - R^2 + \frac{2Rr}{n} - \frac{r^2}{n^2} \geq \frac{r^2}{(n+1)^2}.$$

After simplification we have

$$\frac{2R}{n} - \frac{2R}{n+1} \geq \frac{r}{n^2},$$

Then

$$2R(n+1) - 2Rn \geq \frac{r(n+1)}{n},$$

so

$$2Rn \geq rn + r.$$

As $R \geq r$ and $n \geq 1$, then $Rn + Rn \geq rn + r$. Therefore

$$t_1 + t_2 + t_3 + \ldots \geq \frac{r}{2} + \frac{r}{3} + \frac{r}{4} + \ldots = r\sum_{n=2}^{\infty}\frac{1}{n} = \infty\ .$$

since the series $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

Thus, for the time $t_1 + t_2 + ... + t_n$ the evader will not be captured. Moreover $t_1 + ... + t_n \to \infty$ as $n \to \infty$. Consequently, evasion is possible in the Lion and Man game.

1.4 Objective thesis

We study a pursuit-evasion differential game of optimal approach of finite or countably many pursuers with one evader in the Hilbert space l_2. Game is described by equations

$$\ddot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad \dot{x}_i(0) = x_i^1,$$

$$\ddot{y} = v, \quad y(0) = y^0, \quad \dot{y}(0) = y^1,$$

On control functions of players integral constraints are imposed.

$$\int_0^{\theta} \|u_i(s)\|^2 \, ds \le \rho_i^2,$$

$$\int_0^{\theta} \|v(s)\|^2 \, ds \le \sigma^2,$$

Such constraints arise in modeling the constraint on energy. The control resource energy of any pursuer ρ_i is not necessarily greater than that of the evader σ. The payoff functional is the greatest lower bound of distances between the pursuer and evader when the game is terminated. Pursuers try to minimize the payoff, and the evader tries to maximize it. The objectives of the research are:

- To find the value of the game, γ.
- To construct the strategy of the pursuers guaranteeing the value γ.
- To construct strategy for the evader guaranteeing the value γ.

1.5 Outlines of thesis

This thesis will be presented in six main chapters which attempts mainly to construct optimal strategies of players in a linear differential game in the Hilbert's space l_2.

Chapter 1 consists of introduction to differential game problems, and we present “Lion and Man” game problem.

In Chapter 2, we give a background of this research.

Chapter 3 studies control, trajectory, and construction of P-strategy.

In Chapter 4, we used some basic result to prove our main theorem. We discussed about method by Ibragimov [21] and Ibragimov and Mehdi [22], which we used to prove our main result. We fix the index i and study an auxiliary differential game of two players P_i and E

Chapter 5 studies a pursuit-evasion differential game of optimal approach of finite or countable pursuers to one evader in the Hilbert space l_2. On control functions of the players integral constraints are imposed. Such constraints arise in modeling the constraint on energy. Duration of the game θ is fixed. The payoff functional is the greatest lower bound of distances between the pursuers and evader when the game is terminated. The pursuers try to minimize the functional and, the evader tries to maximize it. We find formula for the value of the game and construct explicitly optimal strategies of the players. For this aim, instead of second order differential equation we consider an equivalent first order differential equation with the same payoff functional. Subsequently, then we give a lemma with proof and solve the optimal pursuit problem under an important assumption. Finally, we prove the main theorem by taking help from two lemmas in three parts.

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