UNIVERSITI PUTRA MALAYSIA

DERIVATIONS OF SOME CLASSES OF LEIBNIZ ALGEBRAS

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By

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May 2013

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Let L be an algebra over a field K. It is called a Leibniz algebra if its the bilinear binary operation $[\cdot,\cdot]$ satisfies the following Leibniz identity:

$$\Big[x,[y,z]\Big] = \Big[[x,y],z\Big] - \Big[[x,z],y\Big], \ \forall \ \ x,y,z \in L.$$

The thesis deals with derivations of Liebniz algebras introduced by Loday in 1993. Leibniz algebra is a generalization of Lie algebra. The class of Leibniz algebras in dimensional two and three has been classified by Loday and Omirov, respectively. Starting from dimensions four and above there are classifications of subclasses (in dimension four nilpotent case and in dimensions (5-8) there are classifications of filiform Leibniz algebras). According to the classification results above, in the thesis the derivations of each of the classes are given. It is known that the class of filiform Leibniz algebras in dimension n is split into three disjoint sub classes denoted by FLb_n, SLb_n and TLb_n.

The main results of the research are follows:

- An algorithm to find the derivations of any class of algebras is given, which is implemented to low- dimensional cases of Leibniz algebras.

- Descriptions of derivations of low-dimensional Leibniz algebras (in dimensions two and three this description is given for all Leibniz algebras, in dimension four the derivations of nilpotent part are given and in dimensions 5 -8 the derivations of filiform Leibniz algebras are found). In each of the cases the Leibniz algebras which are characteristically nilpotent are identified.

- The derivations of naturally graded Leibniz algebras is described and the derivation algebra dimensions are provided.

- For the classes FLb_n, SLb_n and TLb_n estimations of dimensions of the derivation algebras, depending on n, are given .

- A computer programm for finding the derivations of algebras is provided.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENERBITAN ALJABAR LEIBNIZ BERDIMENSI RENDAH

Oleh

AL-NASHRI AL-HOSSAIN AHMED I.

Mei 2013

Pengerusi:Profesor Madya I. S. Rakhimov, PhD

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Tesis ini berkaitan dengan penerbitan aljabar Leibniz yang diperkenalkan oleh Loday pada 1993. Aljabar Leibniz ialah satu yang diitlakkan dari aljabar Lie. Aljabar Leibniz bagi dimensi dua dan tiga masing-masing telah dikelaskan oleh Loday dan Omirov . Bermula dimensi empat dan ke atas adalah pengelasan subkelas (dalam dimensi empat kes nilpotent dan dalam dimensi 5 hingga 8 terdapat pengelasan ke atas aljabar Leibniz filiform). Berikutan daripada hasil pengelasan di atas, dalam tesis ini penerbitan untuk setiap kelas tersebut diberi. Diketahui bahawa kelas aljabar Leibniz filiform dalam n dimensi adalah terbahagi kepada tiga subkelas tak bercantum yang disimbolkan sebagai FLb_n, SLb_n dan TLb_n. Hasil utama penyelidikan ini adalah seperti berikut:

- Satu algorithm untuk mencari penerbitan bagi kelas aljabar diberi, yang mana diaplikasikan kepada kes aljabar Leibniz berdimensi rendah.
- Menghuraikan penerbitan bagi aljabar Leibniz berdimensi rendah (dalam dimensi dua dan tiga penghuraian ini diberikan untuk semua aljabar Leibniz, dalam dimensi empat penerbitan bagi bahagian nilpotent diberi dan dalam

dimensi 5 hingga 8 penerbitan bagi aljabar Leibniz filiform ditemui). Dalam setiap kes, aljabar Leibniz yang bersifat nilpotent adalah dikenalpasti.

- Penerbitan bagi aljabar Leibniz tergred secara semulajadi dihuraikan dan dimensi bagi penerbitan aljabar disediakan.

- Untuk kelas FLb_n, SLb_n dan TLb_n satu anggaran dimensi bagi penerbitan aljabar ini yang bergantung kepada n diberi.

- Satu pengaturcaraan komputer untuk mencari penerbitan bagi aljabar ini disediakan.

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I certify that a Thesis Examination Committee has met on 7 may 2013 to conduct the final examination of AL-Nashri, AL-Hossan Ahmed I. on his thesis entitled "DERIVATIONS OF SOME CLASSES OF LEIBNIZ ALGEBRAS" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor Of Philosophy.

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DECLARATION

I declare that the thesis is my original work except the quotations and citations which have been duly acknowledged. I also declare that it has not been previously and not concurrently, submitted for any other degree at Universiti Putra Malaysia or any other institution.

AL-NASHRI AL-HOSSAIN AHMED I.

Date: 7 May 2013

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CHAPTER 1

INTRODUCTION AND LITERATURE REVIEW

1.1 Introduction

The interest in the study of derivations of some classes of algebras goes back to a paper by Jacobson (1955). There Jacobson proved that any Lie algebra over a field of characteristic zero which has non degenerate derivations is nilpotent. In the same paper, he asked for the converse. In the paper by Dixmier and Lister (1957), the authors have given a negative answer to the converse of Jacobson's hypothesis by constructing an example of a nilpotent Lie algebra all of whose derivations are nilpotent (hence degenerate). Lie algebras whose derivations are nilpotent endomorphisms have been called characteristically nilpotent. The result of Dixmier and Lister is assumed to be the origin of the theory of characteristically nilpotent Lie algebras. A few years later in 1959, Leger and Togo published a paper showing the importance of the characteristically nilpotent Lie algebras. The results of Leger and Togo have been extended for the class of non associative algebras.

The theory of characteristically nilpotent Lie algebras constitutes an independent research object since 1955. Until then, most studies about Lie algebras were oriented to the classical aspects of the theory, such as semi simple and reductive Lie algebras (see Tits (1966)). The core of this thesis is the extensions of some results on derivations of Lie algebras to the case of Leibniz algebras. First we give an algorithm for finding derivations. We apply this algorithm in low-dimensional cases of Leibniz algebras. Then we study filiform case for arbitrary dimension. The importance of this class has been mentioned by Vergne. InVergne (1970) introduced the concept of naturally graded filiform Lie algebras as those admitting a gradation associated with the lower central series. In her paper, she also classified them, up to isomorphism.

Apart from that, several authors have studied algebras which admit a connected gradation of maximal length, this is, whose length is exactly the dimension of the algebra. So, Loday and Pirashvili (1993), Ayupov and Omirov (2001), Millionschikov (2002), Rakhimov and Bekbaev (2010) and Albeverio et al. (2006b), continued this direction by giving an induction classification method and finally, Rakhimov et al. (2009) gave the full list of these algebras (over an arbitrary field of zero characteristic).

It is well-known that the natural gradation of nilpotent Lie and Leibniz algebras is very helpful in investigation of their structural properties. This technique is more effective when the length of the natural gradation is sufficiently large. The case when it is maximal, the algebra is called *filiform*. For applications of this technique, see, for instance, Vergne (1966) Goze and Khakimdjanov (1994) (for Lie algebras) and Ayupov and Omirov (2001) (for Leibniz algebras) case. Recall that an algebra L over a field K is called Leibniz algebra if it satisfies the following Leibniz identity:

$$\Big[x,[y,z]\Big] = \Big[[x,y],z\Big] - \Big[[x,z],y\Big]$$

where $[\cdot,\cdot]$ denotes the multiplication in L. It is not difficult to see that the class of Leibniz algebras is "non-antisymmetric" generalization of the class of Lie algebras. A derivation is a function on an algebra which generalizes certain features of the derivation operator. Specifically, given an algebra A over a ring or field K, an K-derivation is an K-Linear map D from A to itself that satisfies Leibniz's law: $D(xy) = (Dx)y + x(Dy)$. More generally, a K-linear map D from A into an A-module M, satisfying the Leibniz law is also called a derivation. The collection of all K-derivation of A to itself is denoted by $Der(A)$. The collection of all K-derivation of A into an A-module M is denoted by $Der(A, M)$. Plainly, the derivations of A to M are exactly the elements of $Z^1(A, M)$ 1-cocycles, while the so-called inner derivations are 1-coboundaries denoted by $B^1(A, M)$. Thus the derivations of A to M form an A-module with operations defined point wise, and the set of inner derivations is a submodule of derivations. The quotient of these modules is the first cohomology group $H^1(A, M)$. Derivations occur in many different contexts in diverse areas of mathematics. If the algebra A is noncommutative then the commutator with respect to an element of the algebra A defines a linear endomorphism of A to itself, which is a derivation over K. Furthermore, the K-module $Der(A)$ forms a Lie algebra with respect to Lie bracket defined by the commutator: $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. The Leibniz algebra is a generalization of Lie algebra, so it makes sense to study the problems on derivations of Lie algebras for the class of Leibniz algebras.

1.1.1 Basic concepts

In this section, we introduce some basic concepts and notations that are used throughout this thesis. Most of the Lie algebra concepts can be found in any standard book on Lie algebras such as Jacobson, (1962), Bourbaki, (1971), Gilmore, (1974) and on nilpotent Lie algebras, Goze and Khakimjanov (1996). For Leibniz algebra, the notations and concepts are referred by papers such as Loday and Pirashvili (1993), Ayupov and Omirov (2001), Khakimdzhanov (2000), Albeverio et al. (2008), Albeverio et al. (2006b), Rakhimov and Bekbaev (2010), Ladra et al. (2011b) and Ladra et al. (2011a).

Definition 1.1 Let K be a field, V be a vector space over K with a binary operation $[\cdot,\cdot] : V \times V \longrightarrow V$. If the binary operation is bilinear, then V is said to be an algebra over K. In other words an algebra is a vector space with a bilinear binary operation $[\cdot,\cdot]$ satisfying the conditions:

$$[\alpha_1 x + \alpha_2 y, z] = \alpha_1[x, z] + \alpha_2[y, z],$$

$$[z, \alpha_1 x + \alpha_2 y] = \alpha_1[z, x] + \alpha_2[z, y]\,,$$

where $x, y, z \in V$ and $\alpha_1, \alpha_2 \in K$.

Definition 1.2 A Lie algebra L over a field K is an algebra satisfying the following conditions:

$$[x, x] = 0, \ \forall x \in L \tag{1.1}$$

$$\Big[[x, y], z\Big] + \Big[[y, z], x\Big] + \Big[[z, x], y\Big] = 0, \forall x, y, z \in L \tag{1.2}$$

Remark 1.1

1. Identity (1.2) is called the Jacobi identity.
2. Relation (1.1) implies the anticommutativity of the multiplication of L:

$$[x, y] = -[y, x] \ \ \forall \ x, y \in L$$

In fact, we have

$$0 = [x + y, x + y] = [x, y] + [y, x]$$

Conversely, if the characteristic of the field K is different from 2, anticommutativity of the bracket implies $[x, x] = 0$.

Definition 1.3 An algebra L is a Leibniz algebra if the Leibniz identity

$$\Big[x, [y, z]\Big] = \Big[[x, y], z\Big] - \Big[[x, z], y\Big]$$

holds.

Remark 1.2 From now on all algebras are assumed to be over the fields of complex numbers $\mathbb{C}$. Note that if the identity $[x, x] = 0, \forall x \in L$, holds then the Leibniz identity becomes the Jacobi identity. Thus, the Leibniz algebras are the "noncommutative" analogue of the Lie algebras.

Definition 1.4 Let L, L_1 be two algebras over a field K. A linear mapping $\psi : L \longrightarrow L_1$ is a homomorphism if

$$\psi([x,y]_L) = [\psi(x), \psi(y)]_{L_1}, \; \forall x, y \in L\,.$$

Remark 1.3

1. The mapping ψ is an isomorphism if it is a bijective homomorphism (one to one and onto).
2. The mapping ψ is an automorphism if ψ is an isomorphism and $L = L_1$.

For a Leibniz algebra L we define

$$L = L^1, \; L^{k+1} = [L^k, L], \; k \geqslant 1\,.$$

Clearly,

$$L^1 \supseteq L^2 \supseteq ...$$

Definition 1.5 A Leibniz algebra L is called nilpotent if there is a positive integer $s \in \mathbb{N}$ such that $L = L^1 \supset L^2 \supset L^3 \supset \cdots \supset L^s = \{0\}$. The smallest integer s such that $L^s = \{0\}$ is called the nil-index of L.

Definition 1.6 A Leibniz algebra L is called filiform if $dimL^i = n - i$, where $n = dimL$, for $2 \leqslant i \leqslant n$.

It is obvious that a filiform Leibniz algebra is nilpotent. The definition of filiform Leibniz algebra first appeared in a paper by Ayupov and Omirov in 2001. Given a nilpotent Liebniz algebra L with nil-index $k+1$, we denote

$$L_i = L^i / L^{i+1}, \;\; 1 \leqslant i \leqslant k, \; \text{and} \;\; grL = \oplus_{i=1}^{k} L_i.$$

Using $[L^i, L^j] \subseteq L^{i+j}$, it is easy to establish that $[L_i, L_j] \subseteq L_{i+j}$. The graded algebra grL is called the naturally graded Leibniz algebra.

Definition 1.7 A linear transformation d of a Leibniz algebra L is called a derivation if for any $x, y \in L$

$$d([x,y]) = [d(x), y] + [x, d(y)]$$

holds.

The space of all derivations of the algebra L equipped with the multiplication defined as the commutator, forms a Lie algebra which is denoted by $Der(L)$.
It is clear that the operator of right multiplication R_a by an element a of the algebra L that is ($R_a(x) = [x, a]$) is also derivation. Derivations of this type are called inner derivations.

1.1.2 Motivation

The main motivation of the research, the concept of derivation and its applications appear in analysis, differential equations, differential geometry, differential algebra, mechanics, physics and other many areas of science. In algebra the derivations are important invariant in studying the cohomological problems. In the ring theory it is an independent research area. The study of derivations of Lie algebras caused the solution of some important structural and cohomological problems. Extensions of the results on derivations to Leibniz algebras are desirable. Particularly, there is an important class of Lie algebras called charteristically nilpotent which forms one of the irreducible components of variety of Lie algebras. Finding these algebras for Leibniz algebras case gives the description one of the irreducible components. One of the important invariant of algebras is the dimension of the derivation algebra. The dimension of the derivation algebra and the dimension of the automorphism group are complements to each other. These two invariants are useful in geometric classification of Leibniz algebras.

1.1.3 Objectives of the research

The main objectives of this research are:

1. To describe the derivation algebras of low dimensional complex Leibniz algebras.
2. To identify if low dimensional complex Leibniz algebra is characteristically nilpotent or not.
3. To find basic derivations of filiform Leibniz algebras.
4. To study derivations of filiform Leibniz algebras.

1.1.4 Outline of thesis

The dissertation contains eight chapters. Now, we briefly mention the layout of the thesis.

- Chapter 1 is a review of some important results about Lie and Leibniz algebras. We introduce the basic definitions of filiform Lie and Leibniz algebras which will be used throughout the thesis.
- Chapter 2 contains an algorithm for finding derivations of algebras and applies it to find derivations of low-dimensional Leibniz algebras.
- Chapter 3 focuses on studying the derivations of 5-dimensional filiform Leibniz algebras and we indicate the characteristically nilpotent Leibniz algebras in this class.

- Chapter 4 gives a description of derivations and determines if different types of algebras on 6-dimensional of filiform Leibniz algebra are characteristically nilpotent or not.

- Chapter 5 describes derivations and gives which classes if different types of algebras of 7-dimensional of filiform Leibniz algebra is characteristically nilpotent.

- Chapter 6 studies derivations and explains if different types of algebras on 8-dimensional of filiform Leibniz algebras are characteristically nilpotent or not.

- Chapter 7 gives the derivations of naturally graded filiform Leibniz algebras. Then we provide the upper and lower bounds of dimensions of derivation algebras of n-dimensional filiform Leibniz algebras.

- Chapter 8 makes some conclusions on results of the thesis and suggest a few problems for future work in these areas.

1.2 Literature Review

Schenkman (1951) had published his derivation tower theorem for centerless Lie algebras, which described in a nice manner the derivation algebras. This theory was not applicable to the nilpotent algebras, as the adjoint representation is not faithful. This fact led to the assumption that the structure of derivations for nilpotent Lie algebras is much more difficult than for classical algebras.

The first paper about the structure of characteristically nilpotent Lie algebras, short CNLA, is due to Leger and Togo in 1959. They proved the equivalence of the sequence condition of Dixmier and Lister and the nilpotence of the Lie algebra of derivations. Although this paper does not give any additional example of such an algebra, it reduces the search to the class of nilpotent Lie algebras. On the other side, the deduced properties of a CNLA excluded the 2-step nilpotent or metabelian Lie algebras. The last author, Togo, published in 1961 an excellent work which contained much of the information known about derivation algebras of Lie algebras. Among others, he introduced special classes of algebras which were shown to be non CNLA (see Togo (1967)). The importance of CNLAs within the variety of nilpotent Lie algebra laws was soon recognized by the author, and he also formulated an interesting question which is nowadays not satisfactorily solved: *the problem of Togo*. He asked for the existence of CNLA of derivations, that is, algebras for which both the derivations and the derivations of these are nilpotent. Very little is known about the general structure of such Lie algebras, though its existence has been verified by Ancochea and Campoamor (2001). The deformations theory for algebraic structures of Gerstenhaber (1964), originally developed to study the rigidity of algebraic structures, has become since then a powerful tool to determine the nilpotence of derivations. Vergne (1966) applied the cohomology theory of Lie algebras to the study of the variety of nilpotent Lie algebras, obtaining in particular interesting results about its irreducible components. In particular, she showed the existence of only two naturally graded filiform Lie algebras, L_n and Q_n, the second existing only in even dimension. In particular, the first algebras has been a central research object for the last thirty years. Studying its deformations, lots of families of CNLA have been constructed (see Khakimdjanov (1991)). In 1970 Dyer gave a nine dimensional example of CNLA (see Dyer (1970)), which was interesting in its own as it had an unipotent automorphism group. This property is not satisfied by the original example of Dixmier and Lister, and showed than even CNLA can have quite different behaviors. By that time, it was perfectly known that such algebras could exist only from dimension 7 on, as a consequence of the classification in 1958 of the six dimensional algebras by Morozov (1958). In 1972 Favre discovered the lowest dimensional CNLA known until then (see Favre

(1972)), which additionally was of the same nature as Dyer's example. At the same time, Leger and Luks investigated the metabelian Lie algebras and proved several results about their rank, and establishing that rank one algebras were given if the existence of a characteristic ideal containing the derived subalgebra is assured. These results can be interpreted as a constructive proof that the original example of 1957 is the known CNLA with lowest characteristic sequence. Luks applied in 1976 computational methods to prove the existence of CNLA in any dimension greater or equal to seven. Four years later, Yamaguchi constructed families of CNLA in arbitrary dimension, constructions that have been completed and generalized in later years by Khakimdjanov (1991). The topological study of the variety of Lie algebra laws led Carles (1984) to study the topological properties of CNLA. Among other results he states that the set of CNLA is constructible for $n \geq 7$. For the particular dimension 7, he also proves that CNLA do not form an open set. Recently in Ancochea and Campoamor (2001) this result has been generalized to any dimension. Another interesting approach to the CNLA has been deformation theory applied to the Borel subalgebras of complex simple Lie algebras, like the one done by Khakimdjanov in 1988 to prove that almost all deformations of the nilradical of Borel subalgebras of complex simple Lie algebras are characteristically nilpotent. This has shown that these algebras are, in fact, in abundance within the variety of nilpotent laws. Goze and Khakimdjanov (1994) proved, that for any dimension $n \geq 9$ an irreducible component of the filiform variety F_n contains an open set consisting of CNLA. Filiform Lie algebras, specifically the model filiform Lie algebra L_n, has been also the fundamental source for constructing families of CNLAs. In particular, its cohomology has been calculated, which has allowed to describe its deformations in a precise manner and characterize those deformations which are characteristically nilpotent (see Khakimdzhanov (1990)). Ancochea and Campoamor (2001) considered the nilpotent Lie algebras, which structurally "look like Q_n". As known, this algebra cannot appear in odd dimension. This is a consequence of the so called centralizer property (see Campoamor (2000)), which codifies information about the structure of the commutator subalgebra and the ideals of the central descending sequence. Now the centralizer property can be generalized to any naturally graded nilpotent Lie algebra, and defines a class of algebras which can be interpreted as those which are the "easiest nilpotent Lie algebras to deform for obtaining CNLAs". The key to this is extension theory combined with deformation theory. This approach also leads to certain questions about the rigidity of a nilpotent Lie algebras. In 1970 Vergne postulated the nonexistence of nilpotent Lie algebras that are rigid in the variety $\mathfrak{L}_n$ for $n \neq 1$. In his study about the structure of rigid Lie algebras Carles (1984), established that if a nilpotent Lie algebra is rigid, then it necessarily must be a CNLA. This result seems to confirm the validity of the conjecture, although there is no known procedure to prove it.

Another type of results are about affine structures over Lie algebras. This kind of structures are of great importance not only for purposes of cohomology theory Burde (1999), but also for representation theory of nilpotent Lie algebras. The interesting point is that CNLA can admit an affine structure, such as it was proven for the example of Dixmier and Lister by Scheuneman (1974). Although practically nothing is known about CNLA with affine structures, the cohomological method developed by Burde (1999) could be an important source for studying these algebras. (In this thesis we deal with derivations of Leibniz algebras). The Leibniz algebra is a generalization of Lie algebra, so it makes sense to study the Jacobson's question for the class of Leibniz algebras as well. In Loday (1993) introduced a non-antisymmetric generalization of Lie algebra called Leibniz algebra. Leibniz algebras play an important role in Hochschild homology theory, as well as in Nambu mechanics. Leibniz algebra identity becomes the Jacobi identity subject to antisymmetricity. Therefore, a Lie algebra is Leibniz. Leibniz algebras (co)homologies properties are similar to those of the classical Chevalley-Eilenberg (co)homology theory for Lie algebras. Since a Lie algebra is Leibniz, it is interesting to study Leibniz cohomology of Lie algebras. It provides new invariants for Lie algebras. Hagiwara and Mizutani (2002), computed Leibniz cohomology of certain Lie algebras to obtain the Godbillon-Vey invariants for foliations. A Paper by Lodder (1998) studies some interrelations with manifolds, which could lead to possible applications of Leibniz cohomology in geometry. Leibniz algebras appear to be related in a natural way to several topics such as differential geometry, homological algebra, classical algebraic topology, algebraic K-theory, loop spaces, noncommutative geometry, quantum physics etc., as a generalization of the corresponding applications of Lie algebras to these topics. The (co)homology theory, representations and related problems of Leibniz algebras were studied by Cuvier (1991), Loday and Pirashvili (1993), Liu and Hu (2004) and others. A good survey about these all and related problems is Loday et al. (2001). The problems related to the group theoretical realizations of Leibniz algebras are studied by Kinyon and Weinstein (2001) and others. Deformation theory of Leibniz algebras and related physical applications of it, is initiated by Fialowski et al. (2009). Problems concerning Cartan subalgebras, solvability, weight spaces were studied by Albeverio et al. (2006a), and Omirov (2006). The notion of simple Leibniz algebra was suggested by Dzhumadildaev and Abdykassymova (2001), who obtained some results concerning special cases of simple Leibniz algebras. The aim of the present work is two-fold. Firstly, it aims at giving precise examples of derivation algebras of some classes of Leibniz algebras. Secondly, it provides many examples of characteristically nilpotent Leibniz algebras. The first paper, concerning the derivations of Leibniz algebras and the characteristically nilpotent Leibniz algebras, was a paper by Omirov (2006), where the author describes the derivations of the so-called naturally graded filiform Leibniz algebras. In this thesis we shall use the following terminology:

- Representative: The representatives of the isomorphism classes.
- Derivation: The basic derivations.
- Dim: Dimensions of the algebra of derivations.
- CN: Characteristically nilpotent
- Yes: The algebra is characteristically nilpotent
- No: The algebra is not characteristically nilpotent

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