UNIVERSITI PUTRA MALAYSIA

EXPONENTIAL SUMS FOR SOME nth DEGREE POLYNOMIAL

SURIANA BINTI LASARAIYA

IPM 2016 19

EXPONENTIAL SUMS FOR SOME n^{th} DEGREE POLYNOMIAL

By

SURIANA BINTI LASARAIYA

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Master of Science

October 2016

DEDICATIONS

To all of my love;

Bapa & Mak
Sarinah
Lecturers
Family & Friends

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the Master of Science

EXPONENTIAL SUMS FOR SOME n^{th} DEGREE POLYNOMIAL

By

SURIANA BINTI LASARAIYA

October 2016

Chairman: Siti Hasana Binti Sapar, PhD
Institute : Mathematical Research

Let $f(x,y)$ be a polynomial in $Z_p[x,y]$ and p be a prime. For $\alpha > 1$, the exponential sums associated with $f(x,y)$ modulo a prime p^α is defined as $S(f;p^\alpha) = e_p\alpha(f(x,y))$, where the sum is taken over a complete set of residues modulo p^α. It has been shown that the exponential sums is depends on the cardinality of the set of solutions to the congruence equation associated with the polynomial $f(x,y)$. The objective of this research is to find an estimation of the exponential sums for some n^{th} degree polynomial at any point $(x-x_0, y-y_0)$. There are two conditions being considered, that is for $ord_p b^2 \neq ord_p ac$ and $ord_p b^2 = ord_p ac$.

The p-adic methods and Newton polyhedron technique is used to estimate the p-adic sizes of common zeros of partial derivative polynomials associated with n^{th} degree polynomial, where $n \geq 3$. Then, construct the combination of indicator diagram associated with some n^{th} degree polynomial. The indicator diagram is then examined and analyzed.

The information of p-adic sizes of common zeros that obtained is applied to estimate the cardinality of the set $V(f_x, f_y; p^\alpha)$. The results of the cardinality is then used to estimate the estimation of exponential sums associated to n^{th} degree polynomial, where $n \geq 3$.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Sarjana Sains

HASIL TAMBAH EKSPONEN BAGI BEBERAPA POLINOMIAL BERDARJAH n

Oleh

SURIANA BINTI LASARAIYA

Oktober 2016

Pengerusi : Siti Hasana Binti Sapar, PhD
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Katakan $f(x,y)$ suatu polinomial dalam $Z_p[x,y]$ dan p adalah suatu nombor perdana. Untuk $\alpha > 1$, hasil tambah eksponen yang disekutukan dengan $f(x,y)$ modulo suatu perdana p^α ditakrifkan sebagai $S(f;p^\alpha) = e_p\alpha(f(x,y))$, dengan hasil yang dinilaikan di dalam set reja lengkap modulo p^α. Ditunjukkan bahawa penganggaran hasil tambah eksponen adalah bersandarkan kepada kekardinalan set penyelesaian persamaan kongruen yang disekutukan dengan polinomial $f(x,y)$. Objektif kajian ini adalah untuk mendapatkan penganggaran hasil tambah eksponen bagi beberapa polinomial yang berdarjah ke-n pada sebarang titik $(x-x_0, y-y_0)$. Terdapat dua keadaan yang dipertimbangkan iaitu untuk $ord_p b^2 \neq ord_p ac$ dan $ord_p b^2 = ord_p ac$.

Kaedah p-adic dan teknik polihedron Newton digunakan untuk menganggarkan saiz p-adic pensifar sepunya polinomial terbitan separa yang disekutukan dengan beberapa polinomial yang berdarjah ke-n, dengan $n \geq 3$. Kemudian, bina kombinasi gambar rajah penunjuk yang disekutukan dengan beberapa polinomial yang berdarjah ke-n. Gambar rajah penunjuk kemudiannya diperiksa dan dianalisis.

Maklumat anggaran saiz p-adic pensifar sepunya yang diperoleh akan digunakan untuk mendapatkan anggaran kekardinalan bagi set $V(f_x, f_y; p^\alpha)$. Keputusan kekardinalan ini kemudiannya digunakan untuk mendapatkan penganggaran hasil tambah eksponen yang disekutukan dengan beberapa polinomial yang berdarjah ke-n, dengan $n \geq 3$.

ACKNOWLEDGEMENTS

Alhamdulillah.
The only word that I can say to express how grateful I am while doing this research. I am so excited when chosen to be doing this research entitled “Exponential Sums For Some n^{th} Degree Polynomial” and success.

I would like to express my special thanks of gratitude to my supervisor, Assoc. Prof. Dr. Siti Hasana Binti Sapar who gave me the golden opportunity to do this wonderful project. She really helped me a lot while doing this research and I came to know about so many new things. She also gave me the opportunity to join a few international conference on Mathematics held in Kuala Lumpur and Johor Bahru. Without her guidance and support, this thesis would not have been possible. I would like to extend my gratitude to my co-supervisor, Dr. Mohamat Aidil Mohamat Johari, for his advice and guidance throughout this research.

Secondly, I wish to express my sincere thanks to my parents and my sister for their love, encouragements and supports on both financially and mentally to complete my study within the semester.

Besides, I would like to thank Mybrainscience scholarship and Graduate Research Fund (GRF) for the financial support while doing this reasearch. My special thanks to Pn. Kalthom Bte Abu Bakar and Pn. Norlida Bte Zamberi as they always support me and give me motivation to complete my studies. Thank you very much.

Thanks are also addressed to my friends, Harisa, Syafiqah, Atikah, Mahirah, Kak Cik, Zila, Pokke and others for giving me so much idea and support during this journey. May Allah blessed all of you.

Last but not least, I would like to thank all lecturers and staffs of Universiti Putra Malaysia for their assistance throughout my study.

Thank you very much!

I certify that a Thesis Examination Committee has met on 13th October 2016 to conduct the final examination of Suriana Binti Lasaraiya on her thesis entitled "Exponential Sums For Some n^{th} Degree Polynomial" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

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Declaration by graduate student

I hereby confirm that:

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LIST OF ABBREVIATIONS

p	Prime number
α	Exponent of prime numbers
Z	Positive integer
Q	Rational number
C	Complex coefficient
R	Real numbers
Z_p	Ring of p-adic integer
Q_p	Field of rational p-adic numbers
Ω_p	Universal of p-adic field
Ω_p^n	Neighbourhood of points in the universal of p-adic field
$S(f;q)$	Exponential Sums
$N(f_x, f_y; P^{\alpha})$	Cardinality of set $V(f_x, f_y; P^{\alpha})$
$f(x)$	Polynomial with One-Variable
$f(x,y)$	Polynomial with Two-Variable
min	Minimum
max	Maximum
$ord_p a$	Highest power of p which divides a
inf	Infimum
mod	Modulo
exp	Exponent

CHAPTER 1

INTRODUCTION

1.1 Background

The exponential sums is defined as

$$S(P) = \sum_{x} e^{2\pi i f(x)} \tag{1.1.1}$$

where x runs all over integer from certain interval P, and $f(x)$ is a polynomial taking on real values under integer x. It is well known that bounds for such sums imply the corresponding estimates for the number of solutions of certain congruences.

Estimation of exponential sums is related in analytic number theory especially in solving the Waring's problems (Korobov (1992)) and also in cryptographic research especially related to communication theory (Paterson (1999)).

Koblitz (1977) discussed the Newton polygon method for polynomials and power series in $\Omega_p[x]$ where Ω_p denotes the completion of the algebraic closure of the field of p-adic numbers Q_p. This studies is basically to develop some basic ideas of p-adic analysis and presenting some applications that is related to this field, which in turn has stimulating an interest in this field.

Deligne (1974) showed that for a prime p,

$$|S(f;p)| \leq (m-1)^n p^{\frac{n}{2}} \tag{1.1.2}$$

where m denotes the total degree of polynomial f, n is the number of variables such that $n > 1$. However, equation 1.1.2 is not really clear at presenting on how large the class of polynomials that can satisfy the equation. Estimation of $|S(f;p)|$ has become more precise by using Deligne's results.

Then, Loxton and Smith (1982) upgrading the results of Deligne and shows that for a prime p and non-linear polynomial f in $Z[x]$ of degree $m+1$, the estimation of exponential sums is defined as

$$|S_F(p^{\alpha})| \leq m^n p^{\frac{n\alpha}{2}} (D(\nabla F)^5, p^{\alpha})^{\frac{n}{2}} \tag{1.1.3}$$

such that $D(\nabla F) \neq 0$ and $\alpha > 1$.

Later, Mohd Atan (1984) demonstrates the application of Newton polyhedron method in estimating the exponential sums associated with certain polynomials in $Z_p(x,y)$, where Z_p denotes the ring of p-adic integers. The polynomial that has been analyzed is

$$f(x,y) = ax^3 + bxy^2 + cx + dy + e.$$

It is found that if $\delta = max\{ord_p 3a, \frac{3}{2} ord_p b\}$, then

$$|S(f;p^{\alpha})| \leq 4p^{min\{2\alpha, 3\alpha+2\delta+1\}}.$$

Then, Mohd Atan (1986a) considered in extending the Newton polygon idea in the p-adic case of polynomials in two-variables and it is called as Newton Polyhedron method. He proved that for a prime p and a polynomial f in Ω_p, if (ξ,η) is a zero of f, then $(ord_p\xi, ord_p\eta, 1)$ is a normal to an edge in N_f and falls between the upward-pointing normals to the faces of N_f adjacent to this edges. He also proved that if $\hat{n} = (\lambda, \mu, 1)$ is a normal to E (non-vertical edge of N_f common to adjacent faces F_1 and F_2) and lies between the upward-pointing normals to F_1 and F_2, then there exist ξ and η such that $ord_p\xi = \lambda$, $ord_p\eta = \mu$ and $f(\xi,\eta) = 0$.

Mohd Atan (1986b) examined the combination of the indicator diagram that generated from both polynomials and proved that the p-adic sizes of both polynomials gives the coordinates of certain intersection points of segment of the indicator diagram. Besides, the author also proved a converse of an assertion in his previous studies in Mohd Atan (1986a) and written the following conjecture :

Conjecture 1.1.1 *Mohd Atan (1986b) Let f and g be a polynomials in $\overline{Q}_p[x,y]$ and let (λ,μ) be a point of intersection of their indicator diagram and suppose the edges through (λ,μ) do not coincide. Then, there are ξ and η in $\overline{Q}_p$ satisfying $f(\xi,\eta) = g(\xi,\eta) = 0$ and $ord_p\xi = \lambda$ and $ord_p\eta = \mu$.*

Mohd Atan (1988) focused in the use of Newton Polyhedron method to arrive at the estimates of cardinality. By considering the polynomials of $f(x,y) = 3ax^2 + by^2 + c$, it is showed that the cardinality are as follows :

$$card\ V_i(f;g;p^{\alpha}) \leq p^{\alpha+\delta}.$$

Mohd Atan (1990) discussed a method to estimate the exponential sums of a given polynomials. It is found that the exponential sums is depending on the cardinality of the set of solutions to the congruences equations modulo $p^{\frac{\alpha}{2}}$ for $\alpha > 0$. The estimation is obtained by using the newton polyhedron methods when α is an even. The exponential sums is as follows

$$|S(f;q)| \leq p^{n\theta+n(\gamma-\theta)} N(f;p^{\theta}).$$

However,

$$|S(f;q)| \leq \begin{cases} p^{\frac{n\alpha}{2}} N(f;p^{\frac{\alpha}{2}}) & \text{if } \alpha = even, \\ p^{n\alpha}\Sigma_{u(mod\, p^{\alpha})}|G(u)| & \text{if } \alpha = odd. \end{cases}$$

where $|G(u)|$ is the Gaussian Sums of quadratic form.

Mohd Atan and Abdullah (1993) used Newton polyhedron method to find the solution of $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + kx + my + n$ in $Z_p[x,y]$. The polynomial is differentiated with respect to x and y. It is found that the intersection point on the indicator diagram is at the point $(\frac{1}{2}ord_p(3a + b\alpha), \infty)$ for $\alpha > 0$.

By referring to the study of Deligne (1974) and Loxton and Smith (1982), Mohd Atan (1995) deduced that if α is even, let $\alpha = 2\theta$ then

$$|S(f;p^{\alpha})| \le p^{2(\alpha-\theta)} min\{p^{2\theta}, 4p^{\theta+\delta}\}.$$

While if α is an odd, let $\alpha = 2\theta + 1$, then

$$|S(f;p^{\alpha})| \le p^{\alpha+\frac{1}{2}} min\{p^{2\theta}, 4p^{\theta+\delta}\}$$

for a polynomial of $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3ex + my + n$.

Heng and Mohd Atan (1999) found that the cardinality of the set of solutions associated with a polynomial of cubic form $f(x,y) = ax^3 + bxy^2 + cx + dy + e$ is

$$N(f_x, f_y; p^{\alpha}) = \begin{cases} p^{2\alpha} & \text{if } \alpha \le \delta \\ 2p^{\alpha+\delta} & \text{if } \alpha > \delta \end{cases}$$

with $p > 3$, $\alpha > 1$ and $\delta = \min\{ord_p 3a, \frac{3}{2} ord_p b\}$.

Sapar and Mohd Atan (2002) continue the study on estimation of the cardinality of the set of solutions to congruences equations for some polynomials. The chosen cases is on the overlaping occurs on the vertices and the line segments in the indicator diagram associated with the second and third degree polynomials.

Sapar and Mohd Atan (2006) considering a polynomial of the form

$$f(x,y) = ax^5 + bx^4y + cx^3y^3 + dx^2y^3 + exy^4 + mx + ty + k$$

such that $ord_p b^2 > ord_p ac$ and $ord_p(10cm - 2de)^2 > ord_p(10dm - 4e^2)(2ce - d^2)$, then p-adic sizes of common zeros of partial derivatives of this polynomial is

$$ord_p \xi \ge \frac{1}{4}(\alpha - \delta) \ , \ ord_p \eta \ge \frac{1}{4}(\alpha - \delta)$$

with $\xi = max\{ord_p a, ord_p b, ord_p c, ord_p d, ord_p e, ord_p m\}$ and $ord_p f_x(0,0), ord_p$ $f_y(0,0) \ge \alpha > \xi$.

Later, Sapar and Mohd Atan (2007) investigated a polynomial of degree six by using the same technique. It is found that the p-adic sizes of common zeros are

$$ord_p \xi = ord_p x_0 \ge \frac{1}{5}(\alpha - \delta) \ , \ ord_p \eta = ord_p y_0 \ge \frac{1}{5}(\alpha - \delta)$$

for a polynomial

$$f(x,y) = ax^6 + bx^5y + cx^4y^2 + dx^3y^3 + ex^2y^4 + mxy^5 + ny^6 + sx + ty + k.$$

Then, Sapar and Mohd Atan (2009) found that if p is a prime, $p > 5$, $f(x,y) = ax^5 + bx^4y + cx^3y^2 + sx + ty + k$ a polynomial in $Z_p[x,y]$, $\alpha > \delta$ with $\delta = max\{ord_pa, ord_pb, ord_pc\}$ and $ord_pb^2 > ord_pac$, then subject to certain conditions, the p-adic sizes of a common zeros (ξ, η) of partial derivatives of this polynomial is

$$ord_p\xi \geq \frac{1}{4}(\alpha - \delta)$$

and

$$ord_p\eta \geq \frac{1}{4}(\alpha - \delta)$$

or

$$ord_p\eta \geq \frac{1}{4}(\alpha - \delta - \varepsilon)$$

for some $\varepsilon \geq 0$.

Yap (2010) focus is given on cases where the p-adic orders of common zeros occur on the overlapping segments of the indicator diagram. It is found that for the case involving one and two overlapping segment of the indicator diagram associated with the polynomial, the estimate of the associated multiple exponential sums is

$$|S(f;p^\alpha)| \leq \min\{p^{2\alpha}, 4p^{\frac{3}{2}\alpha+\delta}\}$$

and

$$|S(f;p^\alpha)| \leq \min\{p^{2\alpha}, 4p^{\frac{3}{2}\alpha+\delta+\varepsilon}\}$$

respectively for some $\varepsilon > 0$.

Yap et al. (2011) showed that p-adic sizes of such common zeros can be found explicitly on the overlapping segment of the indicator diagram associated with the polynomial. In this case, they considered a polynomial in a cubic form.

Recently, Sapar et al. (2013) investigated polynomial of degree nine. By using Newton polyhedron method, they found that there exist ξ and η such that $f_x(\xi, \eta) = 0$ and $f_y(\xi, \eta) = 0$. Sapar et al. (2014b) consider a polynomial of $f(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 + \frac{3}{2}ax^2 + bxy + \frac{1}{2}cy^2 + sx + ty + k$ such that $ord_pb^2 \neq ord_pac$. By using Newton polyhedron technique, if $ord_pf_x(0,0), ord_pf_y(0,0) \geq 2\delta$, then there exist (ξ, η) in Ω_p^2 such that $f_x(\xi, \eta) = 0$, $f_y(\xi, \eta) = 0$ and

$$ord_p\xi \geq \alpha - \delta \quad \text{or} \quad ord_p\xi \geq \alpha - \delta - \frac{1}{2}\varepsilon \quad \text{and} \quad ord_p\eta \geq \alpha - \delta \quad \text{or}$$
$$ord_p\eta \geq \alpha - \delta - \frac{1}{2}\varepsilon \quad \text{or} \quad ord_p\eta \geq \alpha - 2\delta \quad \text{or} \quad ord_p\eta \geq \alpha - 2\delta - \frac{1}{2}\varepsilon$$

for some $\varepsilon \geq 0$, $\alpha > 0$, $\delta = \max\{ord_pa, ord_pb, ord_pc, ord_pd\}$ and $ord_pbc > ord_pad$.

Then, Sapar et al. (2014c) consider all cases from the results above and found that the cardinality are as follow :

$$N(f_x, f_y; p^{\alpha}) = \begin{cases} p^{2\alpha} & \text{if } \alpha \leq \delta \\ 4p^{4\delta+\varepsilon} & \text{if } \alpha > \delta \end{cases}$$

for some $\varepsilon \geq 0$ as asserted.

1.2 Research Objectives

This study is to find the estimations of the exponential sums associated with polynomial in two variables, $f(x,y)$. The polynomial that we consider are of the form $f(x,y) = ax^n + bx^{n-1}y + cx^{n-2}y^2 + sx + ty + k$ where $n \geq 3$. Firstly, we have to find the p-adic sizes of common zeros of partial derivatives polynomials by examining the two following conditions :

- $ord_p b^2 \neq ord_p ac$.
- $ord_p b^2 = ord_p ac$.

The estimation of the cardinality of the sets of solutions to congruence equations associated to n^{th} degree polynomial will depends on the p-adic sizes that we obtained. Then, we can proceed to estimate the exponential sums of the polynomials by using the results of cardinality.

1.3 Research Methodology

The objective of this research is to find the estimation of exponential sums associated with some n^{th} degree polynomial. Newton polyhedron technique is the method used in finding the p-adic sizes of the polynomials. Then, indicator diagram is constructed and analyzed. After that, we estimate the cardinality of the polynomial that will be in turn used to estimate the exponential sums.

1.4 Outline of Thesis

This thesis covers six chapters as follows :

Chapter 1 gives a brief introduction of this study. The previous work done by many researchers also been mentioned in this chapter as well as the problem statement and the objectives of the research.

Chapter 2 focused on the method that is used in this research. A brief explanation on Newton polyhedron and indicator diagram will be discussed in this chapter. In order to understand the Newton polyhedron technique easily, one example of the polynomial is given.

Then, Chapter 3 will give the finding of the results of p-adic sizes of polynomials that we consider by using Newton polyhedron technique. We consider two conditions in this research, that are $ord_p b^2 \neq ord_p ac$ and $ord_p b^2 = ord_p ac$. Both conditions will give different estimation of the p-adic sizes.

Chapter 4 will give the estimating of the cardinality associated with the polynomials. The result of p-adic sizes of common zeros from Chapter 3 are being used in order to estimate the cardinality of the associated polynomials.

By using the result of the cardinality in Chapter 4, we will estimate the exponential sums of the associated polynomials in Chapter 5. There are two conditions that needed to be considered, that are when α is an even, and α is an odd.

Last but not least, Chapter 6 will discuss an application of exponential sums, briefly on BCH Codes in cryptography. Then, we come up with summary of the research and finally suggestion for future research.

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LIST OF PUBLICATIONS

The following are the list of publications that arise from this study.

Lasaraiya, S., Sapar, S. H. and Mohd Johari, M. A. Cardinality of Sets Associated to Certain Degree Seven Polynomials. In *Malaysian Journal Of Science*, 34 (2) : 210-224 (2015).

Lasaraiya, S., Sapar, S. H. and Mohd Johari, M. A. On The Cardinality Of The Set Of Solutions To Congruence Equation Associated With Polynomial Of Degree Eleven. In *AIP Conference Proceedings* 1750 050015-1-050015-10 (2016); doi: 10.1063/1.4954603.

Lasaraiya, S., Sapar, S. H. and Mohd Johari, M. A. On The Cardinality Of Twelfth Degree Polynomial. In *AIP Conference Proceedings* 1739 020008-1-020008-9 (2016); doi: 10.1063/1.4952488.

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