UNIVERSITI PUTRA MALAYSIA

SOLVING MATRIX DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS USING DIFFERENTIAL TRANSFORMATION METHOD AND CONVOLUTIONS

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SOLVING MATRIX DIFFERENTIAL AND INTEGRO-DIFFERENTIAL EQUATIONS USING DIFFERENTIAL TRANSFORMATION METHOD AND CONVOLUTIONS

By

ÖMER ALTUN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

September 2016

DEDICATIONS

To;
My mother, late father and
To my wife

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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September 2016

Chair: Professor Adem Kılıçman, PhD
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The differential transform method (DTM) was introduced to solve linear and nonlinear initial value problems which appear in electrical circuit analysis. In this method we construct approximate solutions which is close to the exact solutions that differentiable and having high accuracy with minor error. However, DTM is differ with the traditional high order Taylor series where we need long computation time and derivatives. Thus the DTM is applied to the high order differential equations as alternative way to get Taylor series solution. In final stage this method yields truncated series solution in the practical applications and most of time coincides with the Taylor expansion.

In this work we study the differential equations systems by using the differential transformation method (DTM). Further, we apply the convolutions to matrices and study their fundamental properties using differential transformation method. We also provide many different applications of matrix convolutional equations such as coupled matrix convolution equations by DTM. In the applications, we proved that the solutions converge to the exact solutions. Finally we propose to generate matrix integro-differential equations by using convolutions and differential equations.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

MENYELESAIKAN MATRIKS PEMBEZAAN DAN PERSAMAAN PEMBEZAAN-KAMIRAN DENGAN MENGGUNAKAN KAEDAH PENJELMAAN PEMBEZAAN DAN KONVOLUSI

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Kaedah penjelmaan pembezaan (DTM) telah diperkenalkan untuk menyelesaikan linear dan tak linear masalah nilai awal yang muncul dalam analisis litar elektrik. Dalam kaedah ini kita membina penyelesaian anggaran yang hampir dengan yang tepat penyelesaian yang boleh beza dan mempunyai ketepatan yang tinggi dengan kesilapan kecil. Walau bagaimanapun, DTM adalah berbeza dengan yang kaedah tradisional perintah siri Taylor di mana kita perlu masa yang lama untuk pengiraan dan terbitan tinggi. Oleh itu DTM yang digunakan untuk persamaan pembezaan peringkat tinggi sebagai cara alternatif untuk mendapatkan penyelesaian siri Taylor. Akhirnya kaedah ini menghasilkan penyelesaian siri dipenggal dalam aplikasi praktikal dan kebanyakan masa bertepatan dengan cara Taylor series.

Dalam kerja ini kami mengkaji sistem persamaan pembezaan dengan menggunakan kaedah penjelmaan pembezaan (DTM). Selanjutnya, kami menggunapakai konvolusi ke atas matriks dan mengkaji sifat asas mereka menggunakan kaedah penjelmaan pembezaan. Kami juga menyediakan pelbagai aplikasi yang berbeza persamaan konvolusi matriks seperti persamaan-persamaan matriks konvolusi kembar oleh DTM. Dalam penggunaan ini, kami telah membuktikan bahawa penyelesaian menumpu kepada penyelesaian yang tepat. Akhirnya kami mencadangkan untuk menjana matrik persamaan pembezaan-kamiran dengan menggunakan konvolusi dan persamaan pembezaan.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

The differential equations are used to model the real world applications problems in science and engineering that involves several parameters as well as the change of variables with respect to others. Most of these problems will require the solution of initial and boundary conditions, that is, the solution to the differential equations are forced to satisfy. However to model the most of the real world problems is very complicated and difficult to find the exact solution. Thus there are two type of methods to solve the differential equations. One is to find the exact solution by analytic method and another is by numerical method to approximate the solutions. In order to get the analytic solution we can apply some integral transforms.

Commonly applied in engineering, physics and even astronomy, various integral transforms have been included in the literature, see Kilicman and Gadain (2010), Kilicman et al. (2011) and Kılıçman and Eltayeb (2009). Many works such, as those of Fourier, Hankel, Laplace, and Mellin, have been complied on both the theory and applications, see Kilicman (2001); Kilicman and Ariffin (2002). Due to their particular importance, integral transforms were extensively used to solve the various type of differential equations.

Recently the fractional integral transform were introduced to solve the some differential equations in engineering and they are the generalization of the classical transforms. For example, the optics problems can be solved by using fractional Fourier transform, see Kılıçman (2003). Now we recall the following definition, see Davies (2012) and Rahman (2007).

Definition 1.1 The transform

$$f(s) = \int_a^b g(x)\, K(s,x)\, dx$$

is said to be an integral transform and $K(s,x)$ is known as the kernel of the transform.

In the integral transform theory the kernel plays a significant role. By changing the kernels one can obtain different types of transforms:

- If $K(s,x) = e^{-sx}$ then we have Laplace transform,

$$L(s) = \int_0^\infty f(x)\, e^{-sx}\, dx$$

- if the kernel is given by $K(s,x) = xJ_\nu(sx)$ then obtain the Hankel transform

$$H_\nu(\alpha) = \int_0^\infty f(x)\, x J_\nu(sx)\, dx$$

  and similarly,

- if $K(x,s) = \dfrac{1}{x-s}$ then obtain the Hilbert transform

$$H(s) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{f(x)}{x-s}\, dx$$

  if the integrals exists.

- the Mellin transform is defined by considering the kernel x^{s-1} and given by

$$M[f(x);s] = f^*(s) = \int_0^\infty x^{s-1} f(x) dx.$$

  The largest open strip (a,b) where the integral converges is known as fundamental strip. Note that each of the transform can be used for different type of the differential equations, see Kılıçman (2008).

As we can see in the above examples that some of the kernels are smooth enough to carry out the integration. However, there are also some singular integral operators such as the Hilbert transform H. In order to carry the integration process there exists two ways in the literature. Either we consider principal Hadamard finite part or we use convolution to smoothing the singularity.

Each of the integral transform can be used to solve some particular differential equations either in ordinary differential equations or partial differential equations. For example, since Laplace transform reduced the differential equations to the algebraic equations which is very suitable to solve linear ODEs with constant coefficients while the Mellin transform is able to reduce the differential equations to the difference equations thus it is more suitable for the differential equations having polynomial coefficients.

Furthermore, the various transforms have many interconnections that exist between them. For instance, by changing one variable in the Mellin transform, it becomes a bilateral version of the Laplace. It should be noted though that the convergence and other properties of the Laplace and the Mellin transforms are also different to such an extent due to the difference in the ranges of integration between the standard case and the bilateral one. Other connections between all the usual transforms also demonstrate similar distinctions; hence more studies are needed to reveal all the possible connections, see Eltayeb and Kılıçman (2008).

However, the present study will fall in the second category which is numerical solution and it is also known as differential transform method. We also consider several different type of problems such as existence and uniqueness of the solutions in the analytic methods as well as the error analysis in the approximation.

Zhou was the first person to suggest and apply the concept of differential transform to linear and nonlinear problems in electric circuit analysis, see Zhou (1986). This method attempts to formulate an analytic solution as a finite order polynomial. The idea of transformation technique is based on the Taylor series expansion also known as the DTM , and has proven to be handy in reaching to analytical solutions for the differential equations.

DTM involves in the applications to boundary conditions and the governing differential equations are all transformed into a set of algebraic equations by using the differential transform of the original functions. The sought solution of the problem is derived from these algebraic equations. This is different part than the higher-order Taylor series method because symbolic computation of the derivatives of the functions need to be applied therein for large orders which also computationally takes a long time for higher orders. As an alternative, analytic Taylor Series solutions of ordinary or partial differential equations can be obtained by DTM since it is an iterative procedure, see Che Hussin and Kiliçman (2011); Hussin and Kilicman (2011).

In the literature, in quest of obtaining exact solutions of linear and nonlinear partial differential equations, the two and three-dimensional differential transformation method was implemented by Ayaz in Ayaz (2003). Results were compared to the decomposition method, as it was found that DTM has less computational effort. Recently, in Arikoglu and Ozkol (2005, 2006, 2007), fractional differential equations were solved by Arikoglu and Ozkol, who used DTM by applying the fractional differential equations to many different types of problems such as the Ricatti, Bagley-Torvik and composite fractional oscillation equations, see Neta and Igwe (1985). Later, a numerical solution was presented by Erturk and Momani in Ertürk and Momani (2007) and compared ADM and DTM. The results proved that DTM is extremely efficient and accurate. A more general form of non-linear higher order boundary value problems were solved using the DTM where its accuracy was compared with the Adomian decomposition method Che Hussin and Kiliçman (2011). The DTM was applied to solve fractional order nonlinear boundary value problems in Hussin and Kilicman (2011).

DTM with convolutions term is used in some integro differential equations in this study. Furthermore, the convolution is suggested to be used as a new method solving the partial differential equations(PDEs) that might have singularities. When the operator has some singularities, this new method can be applied for smoothing to remove the singularities.

1.2 Differential Transformation Method(DTM)

Let f be smooth enough in an open interval $(x_0-\varepsilon, x_0+\varepsilon)$ for $\varepsilon > 0$ then we have the following definition.

Definition 1.2 *The differential transform of the function $y(x)$ for the k^{th} derivative is defined by:*

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{1.1}$$

where $y(x)$ is an original function and $Y(k)$ is the transformed function. The inverse differential transform of $Y(k)$ is given by

$$y(x) = \sum_{k=0}^{\infty}(x-x_0)^k Y(k). \tag{1.2}$$

Remark 1.1 The substitution of (1.1) into (1.2) yields:

$$y(x) = \sum_{k=0}^{\infty}(x-x_0)^k \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{1.3}$$

which is also known as Taylor's series for $y(x)$ at $x = x_0$.

The basic definitions of differential transformation can also be extended to the matrix form as follows.

Definition 1.3 *If $u(x) \in R^{n\times n}$ can be expressed by Taylor$'$s series about fixed point x_i, then $u(x)$ can be represented as*

$$u(x) = \sum_{k=0}^{\infty}\frac{u^{(k)}(x_i)}{k!}(x-x_i)^k. \tag{1.4}$$

If $u_n(x)$ is the n-partial sums of a Taylor$'$s series (1.2), then

$$u_n(x) = \sum_{k=0}^{n}\frac{u^{(k)}(x_i)}{k!}(x-x_i)^k + R_n(x) \tag{1.5}$$

where $u_n(x)$ is the n-th Taylor polynomial for $u(x)$ around x_i and $R_n(x)$ is remainder term.

Remark 1.2 If $U(k)$ is defined as

$$U(k) = \frac{1}{k!}\left[\frac{d^k}{dt^k}u(x)\right]_{x=x_i}, \qquad \text{where} \qquad k = 0, 1, \ldots, \infty \tag{1.6}$$

then Eq (1.4) is reduced to

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_i)^k \tag{1.7}$$

and the n-partial sums of a Taylor′s series (1.7) is also reduced to

$$u_n(x) = \sum_{k=0}^{n} U(k)(x - x_i)^k + R_n(x). \tag{1.8}$$

The $U(k)$ defined in Eq (1.6), is called the differential transform of function $u(x)$.

Remark 1.3 The above definitions indicate that if $x_i = 0$, then solution (1.7) reduces to

$$u_n(x) = \sum_{k=0}^{n} U(k)x^k + R_{n+1}(x). \tag{1.9}$$

Let the functions $U(k)$, $V(k)$ and $W(k)$ in $R^{n\times n}$, be the differential transform of $u(x)$, $v(x)$ and $w(x)$ respectively, then the following theorems hold. Next, we recall several related theorems and their proofs which can be seen in Che Hussin and Kiliçman (2011), Hussin and Kilicman (2011) and Kiliçman and Altun (2014).

Theorem 1.1 If $u(x) = \alpha v(x)$ then, $U(k) = \alpha V(k)$. Further, if $w(x) = c_1u(x) \pm c_2v(x)$ where $c_1, c_2 \in \mathbb{R}$, then

$$W(k) = c_1U(k) \pm c_2V(k).$$

Proof: The proof is followed by linearity of 1.3.

Theorem 1.2 If $w(x) = \frac{d^m}{dx^m}u(x)$, then $W(k) = \frac{(k+m)!}{k!}U(k+m)$.

Proof: From definition 1.3, it is easy to see that

$$\frac{d^k}{dx^k}w(x) = \frac{d^k}{dx^k}\left[\frac{d^m}{dx^m}u(x)\right] = \frac{d^{k+m}}{dx^{k+m}}u(x).$$

Thus

$$\left[\frac{d^k}{dx^k}w(x)\right]_{x=x_i} = \left[\frac{d^{k+m}}{dx^{k+m}}u(x)\right]_{x=x_i} = (k+m)!U(k+m),$$

then from (1.6), one gets $W(k) = \frac{(k+m)!}{k!}U(k+m)$.

Theorem 1.3 If $w(x) = u(x)v(x)$, then $W(k) = \sum_{l=0}^{k} U(l)V(k-l)$.

Proof: Using the Leibnitz rule, we obtain

$$\frac{d^k}{dx^k}w(x) = \frac{d^k}{dx^k}\left[u(x) * v(x)\right] = \sum_{l=0}^{k}\binom{k}{l}\frac{d^l}{dx^l}u(x)\frac{d^{k-l}}{dx^{k-l}}v(x),$$

therefore

$$\left[\frac{d^k}{dx^k}w(x)\right]_{x=x_i} = \sum_{l=0}^{k}\binom{k}{l}\, l!(k-l)!U(l)V(k-l),$$

then follows from (1.6), that

$$W(k) = U(k)V(k) = \sum_{l=0}^{k} U(l)V(k-l).$$

Theorem 1.4 If $w(x) = \frac{d^m}{dx^m}u(x)\frac{d^n}{dx^n}v(x)$, then

$$W(k) = \sum_{l=0}^{k}\frac{(l+m)!(k-l+n)!}{l!(k-l)!}U(l+m)V(k-l+n).$$

In particular,

If $f(x) = \frac{dr(x)}{dx}$ then, $F(k) = (k+1)R(k+1)$.

If $f(x)=\dfrac{d^2r(x)}{dx^2}$ then, $F(k)=(k+1)(k+2)R(k+2)$.

If $f(x)=\dfrac{d^nr(x)}{dx^n}$ then, $F(k)=(k+1)(k+2)\ldots(k+n)R(k+n)$.

Proof: By using the definitions, it follows that

$$\frac{d^k}{dx^k}w(x)=\frac{d^k}{dx^k}\left[\frac{d^m}{dx^m}u(x)\frac{d^n}{dx^n}v(x)\right]=\sum_{l=0}^{k}\binom{k}{l}\frac{d^{m+l}}{dx^{m+l}}u(x)\frac{d^{n+k-l}}{dx^{n+k-l}}v(x),$$

therefore

$$\left[\frac{d^k}{dx^k}w(x)\right]_{x=x_i}=\sum_{l=0}^{k}\binom{k}{l}(l+m)!(k-l+n)!U(l+m)V(k-l+n),$$

then it follows that

$$W(k)=\sum_{l=0}^{k}\frac{(l+m)!(k-l+n)!}{l!(k-l)!}U(l+m)V(k-l+n).$$

Remark 1.4 *If* $w(x)=\dfrac{d^m}{dx^m}v(x)\dfrac{d^n}{dx^n}u(x)$, *then*

$$W(k)=\sum_{l=0}^{k}\frac{(l+m)!(k-l+n)!}{l!(k-l)!}V(l+m)U(k-l+n).$$

Theorem 1.5 Let $U(k)$, $V(k)$ and $W(k)$ be transform of $u(x)$, $v(x)$ and $w(x)$ respectively, then

(i) If $w(x)=u(x)\dfrac{d^n}{dx^n}v(x)$, then $W(k)=\displaystyle\sum_{l=0}^{k}\frac{(k-l+n)!}{(k-l)!}U(l)V(k-l+n)$.

(ii) If $w(x)=\dfrac{d^m}{dx^m}u(x)v(x)$, then it follows that $W(k)=\displaystyle\sum_{l=0}^{k}\frac{(l+m)!}{l!}U(l+m)V(k-l)$.

Proof: It is obvious from Theorem 1.4.

Now operations of differential transforms follows as in Long and Dinh (1995):

Theorem 1.6 If $f(x)=x^n$ then $F(k)=\delta(k-n)$ where,

$$\delta(k-n)=\begin{matrix}1 & \text{if } k=n\\ 0 & \text{if } k\neq n.\end{matrix}$$

Theorem 1.7 If $f(x) = e^{(-\lambda x)}$ then, $F(k) = \frac{(-\lambda)^k}{k!}$.

Proof: The proof is straightforward by using definition of differential transform.

Theorem 1.8 If $f(x) = (1+x)^n$ then, $F(k) = \frac{n(n-1)\dots(n-k+1)}{k!}$.

Proof: We can easly proof by using the binomial expansion.

The following theorem is the differential transform of the convolution that we will apply in the development of the study.

Theorem 1.9 If $h(t) = \int_0^t f(t-x)g(x)dx$ then for the differential transform of $h(t)$ in $x = 0$, is given by

$$H(k) = \sum_{l=0}^{k-l} \frac{l!(k-l-1)!}{k!} F(l)G(k-l-1)\bigg|_{x=0}, k = 1, 2, \dots \tag{1.10}$$

where F and G are the differential transform of functions $f(x)$ and $g(x)$ in $x = 0$ respectively.

Proof: The proof of the theorem is given in Tari (2012).

1.3 Convergence Analysis

In this study, we consider approximating the numerical solution of differential equation by using the DTM and investigate some of the properties. Thus we need to study the convergence analysis and that is also a crucial point for estimating of the error in approximation.

In this section, we show convergence.

Theorem 1.10 Let the a function $u : R^{n\times n} \to R^{n\times n}$, and n- times continuously differentiable on an interval I and $s_0, s \in I$. Then formulas (1.6) and (1.7) hold, with

$$R_n(x) = \frac{1}{n!}\int_{s_0}^{x} u^{(n+1)}(s)(x-s)^n ds, \tag{1.11}$$

$$\|R_n(x)\|_\infty \leq \frac{|x-s_0|^{n+1}}{(n+1)!}\left\|u^{(n+1)}(s)\right\|_\infty \tag{1.12}$$

where $\|.\|_\infty$ is uniform norm.

Proof: By using the definition 1.3, it follows as

$$R_n(x) = u(x) - u(s_0) - \sum_{k=1}^{n} u^{(k)}(s_0)\frac{(x-s_0)^k}{k!}.$$

Now define a function $h : R^{n\times n} \to R^{n\times n}$ such that

$$h(s) = u(a) - u(s) - \frac{u'(s)}{1!}(a-s) - \ldots - \frac{u^{(n)}(s)}{n!}(a-s)^n, \quad s \in R \tag{1.13}$$

Then $h(s_0) = R_n(a)$ and $h(a) = 0$, further h is relatively continuous and finite on T, and thus differentiable. Differentiating (1.13), we see that all cancels out except for one term

$$h'(s) = -\frac{u^{(n+1)}(s)}{n!}(a-s)^n, \quad s \in T - Q \tag{1.14}$$

then it can be found

$$-h(s) = \int_s^a \frac{u^{(n+1)}(s)}{n!}(a-s)^n ds, \quad s \in T$$

and

$$\int_{s_0}^a \frac{u^{(n+1)}(s)}{n!}(a-s)^n ds = -h(a) + h(s_0) = R_n(a), \quad s \in T.$$

As $x = a$, (1.11) is proved. Next, let $M = ||u^{(n+1)}(s)||_\infty$, If $M = +\infty$, the (1.11) is valid. If $M < +\infty$, define

$$g(s) = M\frac{(s-a)^{n+1}}{(n+1)!}$$

for $s \geq a$, and

$$g(s) = -M\frac{(a-s)^{n+1}}{(n+1)!}$$

for $s \leq a$. In both cases,

$$g'(s) = M\frac{|s-a|^n}{n!} \geq ||h'(s)||_\infty, \quad s \in T - Q$$

then we get,

$$||h(s_0) - h(a)||_\infty \leq ||g(s_0) - g(a)||_\infty,$$

or

$$||R_n(a)||_\infty \leq M\frac{|a-s_0|^{n+1}}{(n+1)!}.$$

Thus the Eq. (1.11) follows, because a is arbitrary value, see the details Abazari and Kılıcman (2012).

1.4 Differential Equations and DTM

In this section, we consider applying the differential transformation method with convolutions term. Further, by using the convolution we proposed a new method to solve integro-differential equations with singularity as well as we examined existence, uniqueness, and smoothness. To commence, some of the definitions pertaining to the differential transformation method are recalled:

Consider $y(x)$ is k times differentiable in $(x_0 - \varepsilon, x_0 + \varepsilon)$ for $\varepsilon > 0$ then we have the following definition.

Definition 1.4 *The transform of $y(x)$ is defined as follows:*

$$Y(k) = \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{1.15}$$

where $y(x)$ is a function and $Y(k)$ is the transform. The inverse of $Y(k)$ is defined

$$y(x) = \sum_{k=0}^{\infty} (x - x_0)^k Y(k). \tag{1.16}$$

The substitution of (1.15) into (1.16) yields

$$y(x) = \sum_{k=0}^{\infty} (x - x_0)^k \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0}. \tag{1.17}$$

Suppose that the power series $\sum_{k=0}^{\infty} b_k (x-c)^k$ has radius of convergence $\varepsilon > 0$. Then the series converges to a function, f on the interval $(c - \varepsilon, c + \varepsilon)$.

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k. \tag{1.18}$$

Thus, by differentiating $f(x)$, it can be written

$$f'(x) = \sum_{k=0}^{\infty} b_k k (x-c)^{k-1} \tag{1.19}$$

$$f''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(x-c)^{k-2} \tag{1.20}$$

$$f'''(x) = \sum_{k=0}^{\infty} b_k k(k-1)(k-2)(x-c))^{k-3} \tag{1.21}$$

$$f^{(k)}(c) = k! b_k. \tag{1.22}$$

Solving (1.22) for b_k, obtain

$$b_k = \frac{1}{k!}\left(f^{(k)}(c)\right). \tag{1.23}$$

By substituting equation (1.23) into (1.18), yields

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k. \tag{1.24}$$

Theorem 1.11 Suppose a function f has $(n+1)$ derivatives on $(c-\varepsilon, c+\varepsilon)$, for $r > 0$. Then, for $x \in (c-\varepsilon, c+\varepsilon)$, $f(x) \approx P_n(x)$ and the error between $P_n(x)$ and $f(x)$ is given

$$R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1} \tag{1.25}$$

for some number z between x and c, see Anton et al. (2009).

Theorem 1.12 Suppose that the function f has derivatives of all orders in the interval $(c-\varepsilon, c+\varepsilon)$, for some $\varepsilon > 0$ and $\lim_{x\to\infty} R_n(x) = 0$, for all $x \in (c-\varepsilon, c+\varepsilon)$. Then, Taylor series for $f(x)$ expansion about $x = c$ converges to $f(x)$, that is

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(f^{(k)}(c)\right)(x-c)^k \tag{1.26}$$

for all $x \in (c-\varepsilon, c+\varepsilon)$.

We provide some simple examples of a system of linear differential equations.

$$\begin{aligned} x_1'(t) &= x_1(t) - 2x_2(t) \\ x_2'(t) &= 2x_1(t) - x_2(t). \end{aligned}$$

In the literature these kind of system is known as a coupled system since knowledge of x_2 is required in order to find x_1 and likewise knowledge of x_1 is required to find x_2.

Note that differential equations can easily be written in the form of a system. For example, if we have a second order linear differential equations $y'' - y' + 2y = f(t)$ then by making substitution

$$\begin{aligned} x_1(t) &= y(t), \\ x_2(t) &= y'(t), \end{aligned}$$

then by differentiating both side we obtain

$$\begin{aligned} x_1'(t) &= y'(t) \\ x_2'(t) &= y''(t) = y' - 2y + f(t). \end{aligned}$$

Now if we replace the substitution then we have the following system

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= x_2(t) - 2x_1(t) + f(t). \end{aligned}$$

Following examples show that any differential and integro-differential equations can be written in the matrix form or system of equations.

Example 1.1 *Consider a circuit RCL which consists of the input $u(t)$, and the voltage $v_c(t)$ as output. Then it can easily be written as 2nd-order system of two DEs:*

$$\begin{cases} L\dot{i} + Ri + v_c = u \\ C\dot{v}_c = i \end{cases} \tag{1.27}$$

$$\begin{cases} L\dot{i} + Ri + v_c = u \\ C\dot{v}_c = i \end{cases} \tag{1.28}$$

Define $x_1(t) = i(t)$ and $x_2(t) = v_c(t)$, and the state equations

$$\begin{cases} \dot{x}_1 = -\frac{R}{L}x_1 - \frac{1}{L}x_2 + \frac{1}{L}u \\ \dot{x}_2 = \frac{1}{C}x_1 \end{cases} \tag{1.29}$$

and the output

$$y = v_c = x_2.$$

In the vector form we can write

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = A\bar{x} + B\bar{u} = \begin{bmatrix} -R/L & -1/L \\ 1/C & 0 \end{bmatrix} \bar{x} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} u \tag{1.30}$$

and

$$y = x_2 = [0, 1]\bar{x}.$$

Example 1.2 *Consider a system with input $u(t)$ and output $y(t)$ which is represented by a 3rd order DE:*

$$\frac{d^3y}{dt^3} + a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0 y = u.$$

Then we define three state variables $x_1 = y$, $x_2 = \dot{y}$ and $x_3 = \ddot{y}$ and the state equation:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = x_3 \\ \dot{x_3} = -a_0x_1 - a_1x_2 - a_2x_3 + u \end{cases} \tag{1.31}$$

and output equation

$$y = x_1.$$

In matrix form these equations is given by

$$\dot{\bar{x}} = \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = A\bar{x} + B\bar{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \tag{1.32}$$

$$y = x_1 = [1, 0, 0]\bar{x}.$$

In the following section we review the integro differential equations and also provide some examples.

1.5 Integro Differential Equations

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics where they are not only useful but often indispensable even for numerical computations.

It seems that it is appropriate to illustrate by means of a simple example the intimate connection between the mathematical theory, which forms the subject-matter of this study, and the ′practical′ problems of applied sciences. It is well known that, if the speed of a rotating shaft is gradually increased, the shaft, at a certain definite speed (which may at times be far below maximal speed allowed), will undergo rather large unstable oscillations. Of course, this phenomenon occurs when the speed of the shaft is such that, for a suitable deformation of the shaft, the corresponding centrifugal force just balances the elastic restoring forces of the shaft, see Ozdemir Ozgumus and Kaya (2010).

In order to determine the possible $'$critical$'$ speeds of the shaft, we may utilize a simple, yet general, result from the theory of elastic beams: For an arbitrary elastic beam under arbitrary end conditions, there always exists a uniquely defined *influence function* G which yields the deflection of the beam in a given direction γ at an arbitrary point P of the beam caused by a unit loading in the direction γ at some other point Q. For, if the cross-sections of the beam are placed in one-to-one correspondence with the points of the segment $0 \leq x \leq 1$, then G is a symmetric function $G(x,y)$ of the abscissae x and y of P and Q respectively, see Yalcin et al. (2009). Consequently by the superposition principle of elasticity, if $p(x)$ is an arbitrary continuous load distribution along the beams then the corresponding deflection is

$$z(x) = \int_0^1 G(x,y)\,\mu(y)\,dy, \quad (0 \leq x \leq 1) \tag{1.33}$$

This equation (1.33) is known as integral equation; more precisely, it is called a linear homogeneous Fredholm equation of second kind with the kernel $G(x,y)\mu(y)$. If the kernel is complex or complicated we transform the kernel in different format. By using the fact that $\mu(x) > 0$ we can transform equation (1.33) into a similar one with a symmetric kernel that might be handle easy. For example, if we set

$$\Phi(x) = \sqrt{\mu(x)}z(x) \tag{1.34}$$

and $\omega^2 = \lambda$,
then we obtain the equation

$$\Phi(x) - \lambda \int_0^1 K(x,y)\,\phi(y)\,dy = 0 \quad (0 \leq x \leq 1) \tag{1.35}$$

whose kernel

$$K(x,y) = \sqrt{[\mu(x)\,\mu(x)]}G(x,y) \tag{1.36}$$

is obviously symmetric.

Remark 1.5 The advantage of this transformation is that a symmetric kernel generally possesses an infinity of eigenvalues (also known as characteristic or proper values),i.e. values of λ for which the equation has non-zero solutions. On the other hand, a non-symmetric kernel may or may not have eigenvalues.

1.6 Objectives and Scope of the Study

This research consists of two parts. In the first part, we study the matrix differential equations before and after convolutions as well as the effect on the solutions. In the second part, we solve integro differential equations by generating integro-differential equations. Thus the main objectives of this research are summarized as follows:

(i) to provide the convolutions and existence of solutions to Volterra Equations, Volterra Integro-Differential Equations and some examples.

(ii) to study existence of the solutions of non-linear higher order boundary value problems using differential transformation method and convolution method.

(iii) to determine relations between differential equations and integro-differential equations after convolutions.

(iv) to prove existence of solution for the type of integro-differential equations and boundary values in the form of $y^{(n)}(x) = \int_0^t e^{-\lambda t} * (y(x-t))^m \, dt$ and $y^{(n)}(x) = \int_0^t p(t) * (y(x-t))^m \, dt$.

(v) to propose to solve singular integro - differential equations by smoothing on using convolution.

To verify the proposed theorems, the proofs are provided by using the standard proving techniques. Some examples are provided to illustrate the proposed theorem.

1.7 Outline of Thesis

The thesis is divided into six chapters as follows.

Chapter 1 of this thesis is about introduction, necessary definitions, theorems, problem statement and objectives.

Chapter 2 explains the literature review and the historical development of the study.

In Chapter 3, we provided the convolutions and existence of solutions such as Volterra Equations, Volterra Integro-Differential Equations and some related examples.

Chapter 4 is on the existence of the solutions of non-linear higher order boundary value problems using differential transformation method and convolution technique.

Chapter 5 is on the applications of the differential transformations to the non-linear higher order matrix equations and also matrix equation systems.

Chapter 6 is on the future study and open problems.

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