



**UNIVERSITI PUTRA MALAYSIA**

***MULTISTEP BLOCK METHODS FOR SOLVING VOLTERRA  
INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND***

**NURUL ATIKAH BINTI MOHAMED**

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**By**

**NURUL ATIKAH BINTI MOHAMED**

**Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia,  
in Fulfillment of the Requirements for the Degree of Master of Science**

**June 2016**

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## DEDICATIONS

*Special dedicated to;*

*Mak & Abah*

*Beloved lecturers*

*Kak Til, Abang Din, Azyyati, Ainaa & Friends  
and to those who knowing me*



Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the Degree of Master of Science

## **MULTISTEP BLOCK METHODS FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF SECOND KIND**

By

**NURUL ATIKAH BINTI MOHAMED**

**June 2016**

**Chairman: Zanariah Abdul Majid, PhD**

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Numerical solutions of Volterra integro-differential equations (VIDEs) by using multistep block methods are proposed in this thesis. The two point one-step block method, two point two-step block method and two point three-step block method are derived by using the Lagrange interpolating polynomial. The generated multistep block methods will estimate the solution of VIDEs at two points simultaneously in a block by using constant step sizes. The source code for solving VIDEs are developed by using C programming.

In VIDEs the unknown functions appear under the differential and integral sign, so the combinations of multistep block methods with numerical quadrature rules are applied. The multistep block methods are used to solve the ordinary differential equation (ODE) part and quadrature rules are applied to calculate the integral part of VIDEs. The method developed has solved for linear and nonlinear second kind VIDEs.

The type of numerical quadrature rules used for solving the integral part of VIDEs is of Newton-Cotes type. Thus, the quadrature rules of suitable order are used to be paired with the multistep block methods. Two different approaches are proposed to solve for two cases where *kernel* equal or not equal one. The stability region of the combination methods are studied.

Numerical problems are presented to show the performance of the proposed method. The results indicated that the proposed method is suitable for solving both linear and nonlinear VIDEs.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Master Sains

## **KAEDAH-KAEDAH BLOK MULTILANGKAH BAGI PENYELESAIAN PERSAMAAN PEMBEZAAN-KAMIRAN VOLTERRA BENTUK KEDUA**

Oleh

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Penyelesaian berangka bagi persamaan pembezaan-kamiran Volterra (PPKV) dengan menggunakan kaedah-kaedah blok multilangkah dicadangkan di dalam tesis ini. Kaedah blok dua titik satu langkah, kaedah blok dua titik dua langkah dan kaedah blok dua titik tiga langkah ini diterbitkan dengan menggunakan interpolasi Lagrange polinomial. Kaedah-kaedah blok multilangkah yang terjana ini akan menganggar penyelesaian bagi PPKV pada dua titik serentak di dalam blok dengan menggunakan saiz langkah yang malar. Sumber kod bagi menyelesaikan PPKV dibina dengan menggunakan program C.

Di dalam PPKV fungsi yang tidak diketahui muncul dalam bentuk pembezaan dan dalam bentuk pengamiran, maka gabungan antara kaedah blok multilangkah dengan aturan kuadratur berangka digunakan. Kaedah blok multilangkah digunakan untuk menyelesaikan bahagian persamaan pembezaan biasa (PPB) dan aturan kuadratur digunakan untuk mengira bahagian kamiran bagi PPKV. Kaedah yang dibangunkan ini telah menyelesaikan jenis kedua linear dan tak linear PPKV.

Jenis aturan kuadratur berangka yang digunakan bagi menyelesaikan bahagian kamiran adalah dari jenis Newton-Cotes. Maka aturan kuadratur yang bersesuaian peringkat digunakan untuk dipasangkan dengan kaedah blok multilangkah. Dua pendekatan yang berbeza dicadangkan bagi menyelesaikan dua kes iaitu *kernel* sama atau tidak sama dengan satu. Rantau kestabilan bagi kaedah yang digabungkan juga dikaji.

Masalah berangka dibentangkan untuk menunjukkan prestasi bagi kaedah yang dicadangkan. Hasil kajian menunjukkan kaedah yang dicadangkan sesuai untuk menyelesaikan PPKV linear dan tak linear.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfillment of the requirement for the degree of Master of Science. The members of the Supervisory Committee were as follows:

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### **Declaration by Members of Supervisory Committee**

This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
- supervision responsibilities as stated in the Universiti Putra Malaysia (Graduate Studies) Rules 2003 (Revision 2012-2013) are adhered to.

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## TABLE OF CONTENTS

	Page
<b>ABSTRACT</b>	i
<b>ABSTRAK</b>	ii
<b>ACKNOWLEDGEMENTS</b>	iii
<b>APPROVAL</b>	iv
<b>DECLARATION</b>	vi
<b>LIST OF FIGURES</b>	x
<b>LIST OF ABBREVIATIONS/NOTATIONS</b>	xi
 <b>CHAPTER</b>	
<b>1 INTRODUCTION</b>	1
1.1 Introduction	1
1.2 Volterra Integro-Differential Equations	2
1.3 Motivation	3
1.4 Objective of the Thesis	3
1.5 Scope of the Study	4
1.6 Outline of Thesis	4
 <b>2 MATHEMATICAL CONCEPTS AND LITERATURE REVIEW</b>	5
2.1 Introduction	5
2.2 Lagrange Interpolation Polynomial	5
2.3 Linear Multistep Method	6
2.4 The Numerical of Volterra Integro-Differential Equations	7
2.5 Review of Previous Works	9
 <b>3 TWO POINT ONE-STEP BLOCK METHOD FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS USING CON- STANT STEP SIZE</b>	13
3.1 Introduction	13
3.2 Derivation of the Two Point One-Step Block Method	13
3.3 Order and Error Analysis of Two Point One-Step Block Method	16
3.4 Implementation of Two Point One-Step Block Method for Solving VIDEs	19
3.4.1 Case I, $K(x, s) = 1$	19
3.4.2 Case II, $K(x, s) \neq 1$	20
3.5 Stability Region of Two Point One-Step Block Method Combined with Simpson's 1/3 Rule	20
3.6 Algorithm 2P1BVIDE	23
3.7 Problem Tested	24

3.8	Numerical Results	25
3.9	Discussion	31
<b>4</b>	<b>TWO POINT TWO-STEP BLOCK METHOD FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS USING CON- STANT STEP SIZE</b>	<b>37</b>
4.1	Introduction	37
4.2	Derivation of the Two Point Two-Step Block Method	37
4.3	Order and Error Analysis of Two Point Two-Step Block Method	40
4.4	Implementation of Two Point Two-Step Block Method for Solving VIDEs	43
4.4.1	Case I, $K(x, s) = 1$	44
4.4.2	Case II, $K(x, s) \neq 1$	44
4.5	Stability Region of Two Point Two-Step Block Method Combined with Simpson's 1/3 Rule	45
4.6	Algorithm 2P2BVIDE	47
4.7	Numerical Results	48
4.8	Discussion	54
<b>5</b>	<b>TWO POINT THREE-STEP BLOCK METHOD FOR SOLVING VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS USING CON- STANT STEP SIZE</b>	<b>59</b>
5.1	Introduction	59
5.2	Derivation of the Two Point Three-Step Block Method	59
5.3	Order and Error Analysis of Two Point Three-Step Block Method	63
5.4	Implementation of Two Point Three-Step Block Method for Solving VIDEs	68
5.4.1	Case I, $K(x, s) = 1$	68
5.4.2	Case II, $K(x, s) \neq 1$	69
5.5	Stability Region of Two Point Three-Step Block Method Combined with Boole's Rule	71
5.6	Algorithm 2P3BVIDE	74
5.7	Numerical Results	75
5.8	Discussion	87
<b>6</b>	<b>CONCLUSION</b>	<b>93</b>
6.1	Summary	93
6.2	Future Work	93
	<b>REFERENCES</b>	<b>95</b>
	<b>BIODATA OF STUDENT</b>	<b>97</b>
	<b>LIST OF PUBLICATIONS</b>	<b>99</b>

## LIST OF FIGURES

Figure	Page
3.1 Two point one-step block method.	13
3.2 Stability region in the $h\xi, h^2\eta$ plane of the two point one-step block method combined with Simpson's 1/3 rule.	23
3.3 Graph of total functions call versus maximum errors between the RK3, ABM3 and 2P1BVIDE for solving Problem 1	33
3.4 Graph of total functions call versus maximum errors between the RK3, ABM3 and 2P1BVIDE for solving Problem 2	33
3.5 Graph of total functions call versus maximum errors between the ABM3 and 2P1BVIDE for solving Problem 3	34
3.6 Graph of total functions call versus maximum errors between the ABM3 and 2P1BVIDE for solving Problem 4	34
3.7 Graph of total functions call versus maximum errors between the ABM3 and 2P1BVIDE for solving Problem 5	35
3.8 Graph of total functions call versus maximum errors between the ABM3 and 2P1BVIDE for solving Problem 6	35
4.1 Two point two-step block method.	37
4.2 Stability region in the $h\xi, h^2\eta$ plane of the two point two-step block method combined with Simpson's 1/3 rule.	47
4.3 Graph of total functions call versus maximum errors between the RK4, ABM4 and 2P2BVIDE for solving Problem 1	56
4.4 Graph of total functions call versus maximum errors between the RK4, ABM4 and 2P2BVIDE for solving Problem 2	56
4.5 Graph of total functions call versus maximum errors between the ABM4 and 2P2BVIDE for solving Problem 3	57
4.6 Graph of total functions call versus maximum errors between the ABM4 and 2P2BVIDE for solving Problem 4	57
4.7 Graph of total functions call versus maximum errors between the ABM4 and 2P2BVIDE for solving Problem 5	58

4.8	Graph of total functions call versus maximum errors between the ABM4 and 2P2BVIDE for solving Problem 6	58
5.1	Two point three-step block method.	59
5.2	Stability region in the $h\xi, h^2\eta$ plane of the two point three-step block method combined with Boole's rule.	74
5.3	Graph of total functions call versus maximum errors between the ABM5 and 2P3BVIDE for solving Problem 1	89
5.4	Graph of total functions call versus maximum errors between ABM5 and 2P3BVIDE for solving Problem 2	89
5.5	Graph of total functions call versus maximum errors between the ABM5 and 2P3BVIDE for solving Problem 3	90
5.6	Graph of total functions call versus maximum errors between the ABM5 and 2P3BVIDE for solving Problem 4	90
5.7	Graph of total functions call versus maximum errors between the ABM5 and 2P3BVIDE for solving Problem 5	91
5.8	Graph of total functions call versus maximum errors between the ABM5 and 2P3BVIDE for solving Problem 6	91

## LIST OF ABBREVIATIONS

<i>IVPs</i>	initial value problems
<i>BVPs</i>	boundary value problems
<i>BVMs</i>	boundary value methods
<i>VIDEs</i>	Volterra integro-differential equations
<i>2P1BVIDE</i>	two point one-step block method for solving VIDEs
<i>2P2BVIDE</i>	two point two-step block method for solving VIDEs
<i>2P3BVIDE</i>	two point three-step block method for solving VIDEs
<i>ODEs</i>	ordinary differential equations
<i>RK3</i>	Runge-Kutta method of order 3 for solving VIDEs
<i>RK4</i>	Runge-Kutta method of order 4 for solving VIDEs
<i>GAM – 3</i>	Combination of boundary value methods and third order generalized Adams method by Chen and Zhang (2011)
<i>ETR<sub>2</sub> – 4</i>	Combination of boundary value methods and fourth order Extended Trapezoidal rule of Second Kind by Chen and Zhang (2011)
<i>GBDF – 5</i>	Combination of boundary value methods and fifth order generalized backward differentiation formula by Chen and Zhang (2011)
<i>ABM3</i>	Third order Adam-Bashforth-Moulton predictor-corrector method for solving VIDEs
<i>ABM4</i>	Fourth order Adam-Bashforth-Moulton predictor-corrector method for solving VIDEs
<i>ABM5</i>	Fifth order Adam-Bashforth-Moulton predictor-corrector method for solving VIDEs

## CHAPTER 1

### INTRODUCTION

#### 1.1 Introduction

The field of integral equations has attracted many researchers and various of studies have contributed towards the development of the field. Starting with the work of Abel in the 1820's many modern mathematicians such as Cauchy, Fredholm and Volterra are involved in this topic. An integral equation is an equation in which the unknown, generally a function of one or more variables, occurs under an integral sign (Linz, 1985). A standard integral equation is given in the following form

$$y(x) = f(x) + \lambda \int_{a(x)}^{b(x)} K(x,s)y(s) ds \quad (1.1)$$

where  $a(x)$  and  $b(x)$  are the region of integration,  $\lambda$  is the constant parameter and  $K(x,s)$  is called the *kernel* of the integral equation. One significant different between Fredholm equation and Volterra equation is that Fredholm equation has constant as integration limits, whereas Volterra equation has variable integral limits. This type of integral equations can be divided into two groups and referred as the first kind and the second kind.

Integro-differential equation emerged in many scientific and engineering applications mostly in an electrical circuit analysis. This type of equations appeared when initial value problems (IVPs) or boundary value problems (BVPs) are converted to integral equations. In integro-differential equation both differential and integral operator appeared together in the same equation. A standard integro-differential equation is given as

$$y^{(n)}(x) = f(x) + \lambda \int_{a(x)}^{b(x)} K(x,s)y(s) ds \quad (1.2)$$

The derivatives of the unknown functions may emerged in any order depending on the problems studied. The types of integro-differential equation which are regularly reviewed are of Fredholm and Volterra because of its various used in practical or real life applications.

It is necessary to choose numerical techniques for solving the integral equation since some of this equations cannot be solved analytically. A numerical methods are sufficient to obtain accurate approximation to the solution and the algorithm developed can be used to compute the approximation. It is well known that there are two widely used numerical methods for solving the numerical problems which are single step method and multistep method. Further it can be divided into the explicit and implicit method. Adam-Moulton method is an example of implicit linear multistep method while Adam-Bashforth method is an example of explicit linear multistep method.

## 1.2 Volterra Integro-Differential Equations

A research work by Volterra when he studied a population growth model resulted in the existence of this specific topic. In Volterra Integro-Differential Equations (VIDEs) both differential and integral operator appeared together in the same equation. VIDEs is given in the following form

$$y^n(x) = F(x, y(x), \int_0^x K(x, s, y(s)) ds), \quad (1.3)$$

where  $y^n$  is the  $n$ th derivative of  $y(x)$ . The value of  $n$  indicates the order of the equation and this study are focused on solving the first order VIDEs. VIDEs can be classify into two types which are the first kind and the second kind.

### 1. VIDEs of the first kind

$$\int_0^x K_1(x, t)y(s) ds + \int_0^x K_2(x, t)y'(s) ds = f(x), \quad K_2(x, s) \neq 0 \quad (1.4)$$

### 2. VIDEs of the second kind

(i)  $K(x, s) = 1$

$$y'(x) = f(x) + \int_0^x y(s) ds \quad (1.5)$$

(ii)  $K(x, s) \neq 1$

$$y'(x) = f(x) + \int_0^x K(x, s)y(s) ds \quad (1.6)$$

It is important to understand the concept of linearity in VIDEs. The VIDEs is classify as nonlinear if the unknown function of  $y(s)$  contain nonlinear function such as  $y^2(s)$ ,  $\sin(y(s))$  and  $e^{y(s)}$ . In this thesis both linear and nonlinear problems are solved.

After its establishment by Vito Volterra, VIDEs appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating (Wazwaz, 2011).

VIDEs of first order is written in the following form

$$y'(x) = F(x, y(x), z(x)), \quad y(0) = y_0, \quad 0 \leq x \leq a, \quad (1.7)$$

$$z(x) = \int_0^x K(x, s, y(s)) ds. \quad (1.8)$$

It is assume that  $F$  and  $K$  are uniformly continuous in all variables and that the following Lipschitz condition are satisfied:

$$|F(x, y_1, z) - F(x, y_2, z)| \leq L_1 |y_1 - y_2|, \quad (1.9)$$

$$|F(x, y, z_1) - F(x, y, z_2)| \leq L_2 |z_1 - z_2|, \quad (1.10)$$

$$|K(x, s, y_1) - K(x, s, y_2)| \leq L_3 |y_1 - y_2|. \quad (1.11)$$

Under these conditions eq.(1.7) and eq.(1.8) has a unique solution in  $0 \leq x \leq a$  (Linz, 1969).

During this several decades, researches had introduced many efficient numerical method for solving VIDEs. These includes linear multistep method (Linz, 1969), hybrid method (Makroglou, 1982), finite difference method (Raftari, 2010) and Runge-Kutta type method (Filiz, 2013). The integral term  $z(x)$  in eq.(1.8) are solved using numerical quadrature rule. Effective quadrature rules are needed to solve the integral part and it is usually grouped either under direct quadrature method or reducible quadrature rule. In this thesis we will consider the direct quadrature method. Examples of direct quadrature method are Newton-Cotes quadrature rule, Gregory rule and Gaussian quadrature formula. Throughout this thesis  $y(x_n)$  will denote the exact value where  $x_n = x_0 + nh$ .  $y_n$  and  $z_n$  will denote an approximate value of  $y$  and  $z$  at  $x_n$ .

### 1.3 Motivation

The study of VIDEs is important because of its physical applications in many areas. The search for more general, easier and accurate numerical method for solving VIDEs is a continuous and ongoing process. Since in literature there are no study that has been done on solving VIDEs using the multistep block method, it is a pleasure for us to solve the problem using the proposed method. By adapting multistep block method in solving VIDEs, hopefully more accurate results are gained by using less number of total steps, less number of function evaluations and with less execution time.

### 1.4 Objective of the Thesis

The main objective of this thesis is to solve VIDEs of second kind using multistep block method. It can be achieved by

1. Deriving the two point one-step block method and two point two-step block method by following the formulation in Majid and Suleiman (2011) to solve for differential part of VIDEs.
2. Verifying the error constant and zero stable of the multistep block methods.
3. Plotting the region of stability of the multistep block methods combined with

quadrature rules when solving VIDEs.

4. Validating the performance of the developed methods combined with quadrature rules when solving VIDEs.

## 1.5 Scope of the Study

This research will be focused on solving first order VIDEs of the second kind and both linear and nonlinear problems are tackled. Two different approaches are used to solve when  $K(x, s) = 1$  and  $K(x, s) \neq 1$ . The multistep block method of order three, order four and order five combined with quadrature rule are implemented for solving VIDEs. The combination of predictor and corrector formulas in the form of block are emphasized together with the used of constant step size.

## 1.6 Outline of Thesis

This thesis covers six chapters with the following contents:

Chapter 1 is the brief introduction of this thesis. VIDEs is introduced in this chapter together with the objective and the scope of the study. Review of previous works related to VIDEs is given in Chapter 2. Besides, basis definitions and properties of Lagrange interpolation polynomial and linear multistep method are presented. Relevant mathematical concepts on VIDEs are included.

In Chapter 3 the implementation of two point one-step block method for solving VIDEs are presented. First, the derivation of two point one-step block method is shown and the error constant and zero stable of the method are investigated. Then the implementation of the method and the algorithm is given. Various linear and non-linear numerical problems are tested and the numerical results are presented to show the efficiency of the proposed method. Lastly the numerical results are presented and discussed.

In the next chapter the two point two-step block method for solving VIDEs are given. The method is derived and the error constant and zero stable are determined to show that the method derived is stable. The implementation of the method in solving VIDEs and the algorithm used are also given. The numerical problems and the results are included and compared with the existing method.

In Chapter 5, VIDEs are solved by using the two point three-step block method. Start with the derivation of the method, determined the error constant and zero stable then continue with the implementation of the method. Next, the built algorithm 2P3BVIDE is shown. In order to verify the performance of the proposed method, some numerical problems are presented.

Lastly, Chapter 6 contains the summary of the study and the future work that can be suggested and extended from this research.

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## PUBLICATIONS

**Publications that arise from the study are:**

**Mohamed, N. A.** and Majid, Z. A. One-step Block Method for Solving Volterra Integro-Differential Equations. In *AIP Conference Proceedings*, 1682, 020018 (2015); doi: 10.1063/1.4932427.

**Mohamed, N. A.** and Majid, Z. A. Multistep Block Method for Solving Volterra Integro-Differential Equations. In *Malaysian Journal of Mathematical Sciences*, Accepted.



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