SELECTED PURSUIT AND EVASION DIFFERENTIAL GAME PROBLEMS IN HILBERT SPACE

ABBAS BADAKAYA JA'AFARU

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By

ABBAS BADAKAYA JA'AFARU

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

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DEDICATIONS

To my mother and father,

as well as to my wife and children.

My dedication is with most extreme love and sincere gratitude.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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May 2012

Chair: Associate Professor Gafurjan I. Ibragimov, PhD

Faculty: Science

This thesis deals with the solution of some pursuit and evasion differential game problems described by some models in Hilbert space. The models arise from the solution of pursuit and evasion game problems described by some partial diffrential equations. Three different type of models are considered, where for each model, we solve pursuit and evasion problem with some forms of constraints on controls of the players.

The first model is the infinite system of first order differential equations

$$\dot{z}_k(t) + \lambda_k z(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\quad k = 1, 2, \dots\ ,$$

where $z_k, u_k, v_k, z_{k0} \in \mathbb{R}^1, z_0 = (z_{10},\ z_{20}, \dots) \in l^2_{r+1}$, $u = (u_1, u_2, \dots)$ is the control parameter of the pursuer, $v = (v_1, v_2, \dots)$ is that of the evader and $\lambda_1, \lambda_2, \dots$ is a bounded sequence of negative numbers. For this model, we present solution of optimal pursuit problem, where the controls of the players are subjected to integral

constraints.

Secondly, we consider

$$\dot{z}_k(t) + \lambda_k(t) z_k(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\quad k = 1, 2, \dots ,$$

where $z_0 = (z_{10},\ z_{20}, \dots) \in l_2,\ \lambda_k(t),\ k = 1, 2, \dots,$ are bounded, non-negative continuous functions such that $\lambda_k(0) = 0,\ k = 1, 2, \dots,$ on the interval $[0,\ T]$ and all other parameters are defined as in the first model. In this case, we solve pursuit and evasion problems with integral, geometric, and mix constraints on control functions of the players.

The third model is given by

$$\ddot{z}_k(t) + \lambda_k z(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\ \dot{z}_k(0) = z_{k1},\ k = 1, 2, \dots ,$$

where $z_k, u_k, v_k \in \mathbb{R}^1,\ k = 1, 2, \dots,\ z_0 = (z_{10},\ z_{20}, \dots) \in l^2_{r+1},\ z_1 = (z_{11},\ z_{21}, \dots) \in l^2_r,\ u = (u_1, u_2, \dots)$ is the control parameter of the pursuer and $v = (v_1, v_2, \dots)$ is the control parameter of the evader. Conditions for the solvability of pursuit and evasion problems described by this model are obtained.

Furthermore, we also study control problems related to each of the three models. In the case of first and third models, necessary and sufficient conditions for which the state of the systems can be transfered to the origin are presented. Sufficient conditions are given for the control problem described by the second model for the cases of geometric and integral constraints on the control functions.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

MASALAH-MASALAH PERMAINAN PEMBEZAAN MENGENAI MANGSA DAN PEMANGSA YANG TERPILIH DARI RUANG HILBERT

Oleh

ABBAS BADAKAYA JA'AFARU

May 2012

Pengerusi: Profesor Madya Gafurjan I. Ibragimov, PhD

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Tesis ini berkenaan dengan penyelesaian permainan pembezaan mengenai mangsa dan pemangsa yang diperihalkan dari model ruang Hilbert. Model ini diperolehi dari penyelesaian masalah permainan mangsa dan pemangsa dari persamaan pembezaan separa dengan menggunakan kaedah penguraian. Tiga jenis model yang berbeza dipertimbangkan. Untuk setiap model, kami selesaikan masalah mangsa dan pemangsa dengan beberapa bentuk kekangan pada kawalan para pemain.

Model yang pertama adalah sistem tidak terhingga bagi persamaan pembezaan peringkat pertama

$$\dot{z}_k(t) + \lambda_k z(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\quad k = 1, 2, \dots\ ,$$

di mana $z_k, u_k, v_k, z_{k0} \in \mathbb{R}^1, z_0 = (z_{10},\ z_{20}, \dots) \in l^2_{r+1}$, $u = (u_1, u_2, \dots)$ adalah kawalan paramater kepada pemangsa, $v = (v_1, v_2, \dots)$ adalah kawalan paramater kepada mangsa, and $\lambda_1, \lambda_2, \dots$ adalah satu urutan batasan nombor negatif. Un-

tuk model ini, kami mempersembahkan penyelesaian kepada masalah pengejaran yang optimum, di mana kawalan kepada para pemain tertakluk kepada kekangan kamiran.

Keduanya, kami mempertimbangkan

$$\dot{z}_k(t) + \lambda_k(t) z_k(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\ \ k = 1, 2, \dots\ ,$$

di mana $z_0 = (z_{10},\ z_{20}, \dots) \in l_2,\ \lambda_k(t),\ k = 1, 2, \dots,$ adalah terbatas, fungsi selanjar bukan-negatif sedemikian hingga $\lambda_k(0) = 0,\ k = 1, 2, \dots,$ di atas selang $[0,\ T]$ dan semua parameter yang lain ditafsirkan sebagai model yang pertama. Dalam kes ini, kami selesaikan masalah mangsa dan pemangsa dengan kamiran, geometrik dan kekangan campuran pada fungsi kawalan pemain.

Model ketiga diberikan sebagai

$$\ddot{z}_k(t) + \lambda_k z(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\ \dot{z}_k(0) = z_{k1},\ k = 1, 2, \dots$$

yang mana $z_k, u_k, v_k \in \mathbb{R}^1,\ k = 1, 2, \dots,\ z_0 = (z_{10},\ z_{20}, \dots) \in l^2_{r+1},\ z_1 = (z_{11},\ z_{21}, \dots) \in l^2_r,\ u = (u_1, u_2, \dots)$ adalah kawalan parameter pemangsa,$v = (v_1, v_2, \dots)$ adalah kawalan parameter mangsa. keadaan keboleh-selesaian masalah mangsa dan pemangsa yang diterangkan oleh model ini diperoleh.

Dengan demikian, kami mengkaji masalah kawalan yang berkaitan dengan ketiga-tiga model tersebut. Dalam kes model-model yang pertama dan ketiga, syarat perlu dan cukup bagi keadaan sistem tersebut boleh di pindahkan kepada yang asal ditunjukkan. Syarat cukupa dijelaskan untuk masalah kawalan yang digambarkan oleh model yang kedua untuk kedua-dua kes kekangan iaitu geometrik dan kamiran terhadap fungsi kawalan.

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I certify that a Thesis Examination Committee has met on (**insert the date of viva voce**) to conduct the final examination of **Abbas Badakaya Ja'afaru** on his thesis entitled "**Selected Pursuit and Evasion Differential Game Problems in Hilbert Space**" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the **Doctor of philosophy**.

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

(Signature)

ABBAS BADAKAYA JA'AFARU

Date:

TABLE OF CONTENTS

LIST OF ABBREVIATIONS

In this part, we define some abbreviations involved in the thesis.

$a \in S$	a is an element of the set S
$a \notin S$	a is not an element of the set S
$\mathbb{R}$	Set of Reals Numbers
$\mathbb{R}^n$	n-dimentional real Euclidean space,
$\mathbb{C}$	Set of Complex Numbers
IS	Infinite System
FODE	First-Order Differential Equations
SODE	Second-Order Differential Equations
l_r^2	$\left\{\alpha = (\alpha_1, \alpha_2, \dots) : \sum_{k=1}^{\infty} \lvert\lambda_k\rvert^r \alpha_k^2 < \infty\right\}$
(Hilbert space)	
$L_2(0,\ T)$	The set of square integrable functions on $[0,\ T]$
$L_2(0,\ T,\ l_r^2)$	$\left\{w(t) = (w_1(t), w_2(t), \dots) : \sum_{k=1}^{\infty} \lvert\lambda_k\rvert^r \int_0^T w_k^2(t)dt < \infty\right\}$
	and $w_k(\cdot) \in L_2(0,\ T)$
$C[a, b]$	Space of continous functions on $[a, b]$
$C(0, T, l_r^2)$	Space of continuous functions in the norm of l_r^2
	with values in l_r^2.
ODEs	Ordinary Differential Equations
PDEs	Partial Differential Equations

$\max\{S\}$	The maximum of the set $S \subset \mathbb{R}$; the largest element in S
$\min\{S\}$	The minimum of the set $S \subset \mathbb{R}$; the smallest element in S
∂S	Boundary of the set S
$\rightarrow$	Converges (approaches) to; into
$[a,\ b]$	$\{x = (1-t)a + tb : 0 \le t \le 1\}$
$(a,\ b)$	$\{x = (1-t)a + tb : 0 < t < 1\}$
$\dot{z}, \ddot{z}$	First and second derivatives of z
z_t, z_{tt}	First and second partial derivatives of z
$z(\cdot)$	Function $z(t),\ 0 \le t \le T$
$\|\|f\|\|_p = \left(\int_R \|f\|^p dx\right)^{1/p}$	p-norm
$L^p(\mathbb{R})$	Completion of the space of continues functions with bounded support in p-norms.

CHAPTER 1

INTRODUCTION

This chapter contains an introduction which presents an overview about differential games problem. It also includes some basic definitions and standard results used in this thesis. Objectives of the thesis are also stated in this chapter.

1.1 Introduction

Differential game constitutes a group of important mathematical problems related to game theory and optimal control theory. It is a game that consists of two players, a pursuer and an evader, with conflicting goals. The goal of the pursuer is to capture the evader in some sense, while that of the evader is to prevent this capture. For example, capture could be minimizing the distance as much as possible between the two players. The game consists of a model describing the behavior of the players which is determined by the player's input through their respective control functions contained in the model. The model is usually a system of differential equations and each player attempts to control the state of the system so as to achieve his goal.

Differential game relates to optimal control theory in the sense that optimal control problems consists of a single control function in the model and a single criterion to be optimized. Differential game theory generalizes this to two controls and two criteria, one for each player. Therefore, optimal control problem are regarded as differential game involving only one player. Technically, control problem can be extended to a differential game problem by introducing control function of the second player to the game model. In both optimal control and differential game problems, the control functions are normally subjected to constraints to reflect a natural phenomenon. Usually, a constraints could either be geometric or integral. If player's control parameter belongs to a subset of $\mathbb{R}^n$, then it is said to be subjected to a geometric constraint. A constraint is referred to as integral if the resources of the player are bounded.

Differential game problem that requires finding conditions for which the pursuer can catch the evader is called pursuit problem. In the other hand, evasion problem requires finding conditions for which the evader can escape catch from the pursuer. Pursuit and evasion differential game is played in an environment(space) where the solution of the system of the differential equations(game model) exists.

Numerous applications of differential games signify it's importance. It has been applied to solve practical problems related to military operations, economics and engineering among others. For example, it has been employed for missile guidance system and military strategy. It has successfully been used to solve problems related market, financial and economy strategies. Most recent, addition of stochasticity to differential games helps in its application to the study of capitalism eg. Leong and Huang (2010). Other applications includes searching building for Intruders, traffic control and surgical operations to mention but a few.

In this thesis, we study some pursuit and evasion differential game problems described by three different models in Hilbert space. In each case, we first study control problem and then extend it to pursuit and evasion problems. Existence-uniqueness of solution to each of the three models in the Hilbert space is also discussed. Generally, we use analytical techniques as method of solution to the problems.

1.2 Preliminaries

In this section, we present basic definitions; examples and some standard results which will be used in the subsequent chapters.

1.2.1 Hilbert space

Definition 1.1 : *Let X be a complex linear space. An inner-product on X is a function $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$ which satisfies the following axioms:*

i. $\langle y, x \rangle = \overline{\langle x, y \rangle}$ *, the complex conjugate of* $\langle y, x \rangle$.

ii. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

iii. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$.

iv. $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

where $x, y \in X$ *and* α *is a complex number.*

An *inner-product space* is a linear space with an *inner-product* on it and is denoted as $(X, \langle \cdot, \cdot \rangle)$. (Chandrasekhara, 2002)

Example 1.1 *Euclidean space* $\mathbb{R}^n$ *with the dot product*

$$\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{k=1}^{n} x_k y_k,$$

is an inner product space.

(Pedersen, 2000)

Definition 1.2 : *A sequence* $\{x_n\}$ *in Hilbert space* $(X, \ \langle \cdot, \ \cdot \rangle)$ *is called Cauchy sequence if for every positive real number* $\varepsilon > 0$*, there is a positive integer* $N(\varepsilon) > 0$ *such that* $||x_m - x_n|| < \varepsilon$ *for all natural numbers* $m, n > N(\varepsilon)$.

(Thomson et al., 2001)

Definition 1.3 : *A complete inner-product space is called a Hilbert space. In other words, a Hilbert space is an inner-product space in which every Cauchy sequence in the space converges to a point in the space.*

(Ponnusamy, 2002)

Example 1.2 : $\mathbb{C}^n$ *is a Hilbert space with inner-product*

$$\langle x, y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k,$$

where $x = (x_1, x_2, \ldots, x_n)$ *and* $y = (y_1, y_2, \ldots, x_n)$. *are in* $\mathbb{C}^n$

Example 1.3 : *Let E be a measurable subset of $\mathbb{R}$. Then the space of all square integrable functions denoted by $L_2(E)$ with inner product define by*

$$\langle f, g\rangle = \int_E f\bar{g}d\mu,$$

is Hilbert space.

Definition 1.4 : *Let X be a linear space. A norm on X is a real-valued function $||\cdot||$ on X satisfying the following axioms:*

a. $||x|| > 0 \ \forall x \in X.$

b. $||x|| = 0 \ \Leftrightarrow x = 0$, *the zero element in* X.

c. $||x+y|| \leq ||x|| + ||y|| \ \forall x, y \in X.$ *(triangular or subadditivity)*

d. $||\alpha x|| = |\alpha| ||x|| \ \forall x \in X$ *and for all scalars* α. *(absolute homogeneity)*

A linear space X with a norm $||\cdot||$ on it is called a *normed space* (or a *normed linear space*). It is denoted by $(X, \ ||\cdot||)$). The *norm* is also referred to as the length of the vector x. (Chandrasekhara, 2002)

Definition 1.5 : *The sequence space* l^p $(1 \leq p \leq \infty)$ *for which norm for the sequence* $\{z_n\} \in l^p$ *defined by*

$$||z||_p = \begin{cases} \left(\sum_{n=1}^{\infty} |z_n|^p\right)^{1/p} < \infty, \ if \ 1 \le p < \infty, \\ \sup_{1 \le n < \infty} |z_n| < \infty, \ if \ p = \infty, \end{cases}$$

is normed space.

The norm defined for $1 \le p < \infty$ is called l^p norm (or simply $p - norm$) and for $p = \infty$ is called l^∞ norm or simply supnorm (Ponnusamy, 2002).

Theorem 1.1 : *Every inner-product space is a normed linear space with norm defined by* $||x|| = \sqrt{\langle x, x\rangle}$.

For the proof of this theorem, see (Chandrasekhara, 2002).

1.2.2 Important Inequalities

Theorem 1.2 :(Cauchy-Schwarz Inequality) *Let* $(X, \langle \cdot, \ \cdot \rangle)$ *be an inner-product space. Cauchy-Schwarz inequality states that*

$$|\langle x, y\rangle|^2 \le \langle x, x\rangle \cdot \langle y, y\rangle,$$

for all vectors $x, y \in X$.

(Chandrasekhara, 2002)

Example 1.4 : *In Euclidean space* $\mathbb{R}^n$ *with the standard inner product, the Cauchy-Schwarz inequality is*

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \le \left(\sum_{k=1}^{n} x_k^2\right)\left(\sum_{k=1}^{n} y_k^2\right).$$

Example 1.5 : *For the inner product space of square-integrable complex-valued functions, Cauchy-Schwarz inequality is*

$$\left|\int f(x)\overline{g(x)}dx\right|^2 \le \int |f(x)|^2 dx \int |g(x)|^2 dx.$$

where $\overline{g(x)}$ *is the conjugate of* $g(x)$

(Ponnusamy, 2002)

Theorem 1.3 : (Minkowski inequality) *Suppose that* $p \geq 1$ *and* $f, g \in L^p(\mathbb{R})$. *Then*

$$||f+g||_p \leq ||f||_p + ||g||_p.$$

For the proof, refer to (Michael, 2000).

Example 1.6 : (Minkowski's sum Inequality) *If* $p \geq 1$ *and* $x_k, y_k \in \mathbb{R}, k = 1, 2, \ldots,$ *then*

$$\left[\sum_{k=1}^{n} |x_k + y_k|^p\right]^{1/p} \leq \left[\sum_{k=1}^{n} |x_k|^p\right]^{1/p} + \left[\sum_{k=1}^{n} |y_k|^p\right]^{1/p}.$$

This result is true for infinite sum and for the details , see (Chandrasekhara, 2002).

Example 1.7 : (Minkowski's integral Inequality). *If* $p \geq 1$ *and* $f, g \in C[a,\ b],\ k = 1, 2, \ldots,$ *then*

$$\left[\int_a^b |f(x) + g(x)|^p dx\right]^{1/p} \leq \left[\int_a^b |f(x)|^p dx\right]^{1/p} + \left[\int_a^b |g(x)|^p dx\right]^{1/p}.$$

(Ponnusamy, 2002)

1.2.3 Measurable functions

Definition 1.6 : *Let* X *be a set. A collection* Σ *of subsets of* X *is called* $\sigma-$ *algebra on set* X, *if the following holds*

i. X *belongs to* Σ,

ii. if A *belongs to* Σ *then complement of* A *belongs to* Σ,

iii. if A_k *is a sequence of elements of* Σ, *then the union of the* $A_k s$ *belongs to* Σ.

Elements of the $\sigma - algebra$ are referred to as measurable sets.

Definition 1.7 : *Let X be a set and let Σ be a $\sigma - algebra$ defined on X. A set X together with Σ is called a measurable space and is denoted as $(X,\ \Sigma)$.*

Definition 1.8 : *Let $(X,\ \Sigma_1)$ and $(Y,\ \Sigma_2)$ be measurable spaces. A measurable function is a function $f : (X,\ \Sigma_1) \to (Y,\ \Sigma_2)$ such that $f^{-1}(E) \in \Sigma_1$ for all $E \in \Sigma_2$.i.e., preimage of any measurable set is measurable.*

(Cohn, 1980)

Definition 1.9 : *Let Σ be a $\sigma - algebra$ over a set X. A function $\mu : \Sigma \to R$ is called a measure if it satisfies the following:*
i. $\mu(E) \geq 0$ *for all* $E \in \Sigma$,
ii. $\mu(\cup_{k\in I} E_k) = \sum_{k\in I} \mu(E_k)$, *for all countable* $\{E_k\}_{k\in I}$ *pairwise disjoint members of* Σ,
iii. $\mu(\emptyset) = 0$.

Definition 1.10 : *In measure theory, a property holds almost everywhere, if the set of elements for which the property does not hold has a measure zero.*

For example, If $f : [a, b] \to R$ is a monotonic function, then it is differentiable almost everywhere. (Aliprantis and Burkinshaw, 1981)

1.3 Objectives of the Thesis

The following are the objectives of the thesis:

- To obtain solution of optimal pursuit differential game problem described by the infinite system

$$\dot{z}_k(t) + \lambda_k z(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\quad k = 1, 2, \ldots\ ,$$

where $z_k, u_k, v_k, z_{k0} \in \mathbb{R}^1, z_0 = (z_{10},\ z_{20}, \dots) \in l^2_{r+1},\ u = (u_1, u_2, \dots)$ is the control parameter of the pursuer, $v = (v_1, v_2, \dots)$ is that of the evader; $\lambda_1, \lambda_2, \dots$ is a bounded sequence of negative numbers. The case for which integral constraints imposed on the controls functions of the players.

- To obtain solution of pursuit and evasion differential game problems described by the infinite system

$$\dot{z}_k(t) + \lambda_k(t) z_k(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\quad k = 1, 2, \dots\ ,$$

where $z_0 = (z_{10},\ z_{20}, \dots) \in l_2,\ \lambda_k(t),\ k = 1, 2, \dots,$ are bounded, non-negative continuous functions on the interval $[0,\ T]$ such that $\lambda_k(0) = 0,\ k = 1, 2, \dots,$ and other parameters are defined as in above. We consider the cases where the control functions of the players are subjected to both integral, geometric and mixed constraints.

- To obtain solution of pursuit and evasion differential game problems described by the infinite system

$$\ddot{z}_k(t) + \lambda_k z(t) = -u_k(t) + v_k(t),\ z_k(0) = z_{k0},\ \dot{z}_k(0) = z_{k1},\ k = 1, 2, \dots\ ,$$

where $z_k, u_k, v_k z_{k0} \in \mathbb{R}^1,\ k = 1, 2, \dots,\ z_0 = (z_{10},\ z_{20}, \dots) \in l^2_{r+1},\ z_1 = (z_{11},\ z_{21}, \dots) \in l^2_r,\ u = (u_1, u_2, \dots)$ is the control parameter of the pursuer, $v = (v_1, v_2, \dots)$ is that of the evader. The control functions of the players are subjected to integral constraints.

1.4 Organization of the Thesis

The remaining part of the thesis is organized as follows:

Chapter 2 contains review of related literature. We review a history for the emergence of the differential game in brief and reported some important works by many researchers related to our works. The report includes both methods and results to some relevant differential game and optimal control problems.

In chapter 3, the proof of existence-uniqueness for the solution to the three considered differential equation models are presented.

We present optimal solution to pursuit problem described by infinite system of first-order differential equations with negative coefficients in chapter 4. Moreover, the chapter includes solution to the control problem described by this system.

Chapter 5 focuses on the solutions to pursuit and evasion problems described by infinite system of first-order differential equations with variable coefficients, where geometric, integral and mixed constraints are imposed to the control functions of the players. Furthermore, the chapter houses solution to control problems for cases involving geometric and integral constraints on the control function.

Solution to both pursuit and evasion as well as control problem described by infinite system of second order equations with negative coefficients are presented in chapter 6. Here, integral constraints on the control function of the players are considered.

We summarize the thesis in Chapter 7, and explore some possible future research directions related to this work. This chafter if followed by bibiliography, biodata of the author to this thesis and finally the list of publications extracted from the thesis.

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BIODATA OF STUDENT

The author Abbas Badakaya Ja'afaru was born in Kazaure town of Jigawa state Nigeria, in 1972. He grew up there under parental care enriched with Islamic teachings; norms and culture of Hausa tribe.

He had six years of primary school education at Kudu Central Primary school Kazaure (1977-1983). His three years of junior secondary school education (1983-1986) and three years of secondary school education (1986 -1989) were at Government Secondary School Kazaure and Science Secondary School Dawakin Tofa(all in former Kano state, Nigeria) respectively.

Abbas proceeded to Bayero University Kano for undergraduate studies in the year 1990. He graduates with second class upper degree in Mathematics in the year 1995. During the Bachelor's degree program, he was beneficiary of National Scholarship award given to mathematics students with outstanding performance by the National Mathematical Center Abuja, Nigeria.

His graduation is followed by a Mandatory National Youth service (1996-1997), where he was engaged with teaching service at a secondary school in Adamawa state. Immediately after completion of this program, Jigawa state Polytechnic employed him as assistant lecturer. He remained there until 1st May 2000, when Bayero University employed him as graduate assistant. During the graduate assistantship at Bayero University, Abbas enrolled into Master degree program in Mathematics. Consequently, the senate of Bayero University awarded him Master of Science degree in Mathematics (2003). Subsequently, the author was promoted to the rank of lecturer I, a position that he currently held at Bayero University, Kano.

He enrolled into PhD program in Applied Mathematics at Universiti Putra Malaysia in December, 2008 under the study fellowship scheme of Bayero University, Kano. His research area is Differential games and optimal control. Presently, Abbas is married and has two children.

LIST OF PUBLICATIONS

1. Journals:

1. **A. B. Ja'afaru** and G. I. Ibragimov. 2011. Differential Games Described by Infinite System of Differential Equations of Second Order. The Case of Negative Coefficients. *International Journal of Pure and Applied Mathematics.* 70(7): 927–938.

2. G. I. Ibragimov and **A. B. Ja'afaru**. 2011. On Control Problem Described by Infinite System of First-Order Differential Equations. *Australian Journal of Basic and Applied Sciences.* 5(10): 736–742.

3. G. I. Ibragimov and **A. B. Ja'afaru**. 2011. On Existence-Uniqueness of Solution to Countable Number of First-Order Differential Equations in the Space l_2. *Journal of Applied Sciences Research.* 7(12): 1860-1864.

4. G. I. Ibragimov and **A. B. Ja'afaru**. On Control Problem for Infinite System of Differential Equations of Second Order. Submitted for Publication.

5. **A. B. Ja'afaru** and G. I. Ibragimov.2012. On Some Pursuit and Evasion Differential Game Problems For an Infinite Number of First Order Differential Equations. *Journal of Applied Mathematics,* volume 12, 13 pages, doi:10.1155/2012/717124 (Impact factor 0.63).

2. Award and Conferences:

1. Won Bronze medal in Exhibition of Research and Innovation UPM 2011, On 19–21 July 2011, Paper entitled Differential Games Described by Infinite System of Differential Equations of Second Order. The Case of Negative Coefficients

2. **Abbas Badakaya Ja'afaru** and Gafurjan Ibragimov. 2010. Differential Game Described by infinite System of Differential Equation. In *Prosiding of Seminar Kebangsaan Aplikasi Sains dan Matematik (SKASM 2010)*, 8–10 December 2010, Johor Bahru, Malaysia.

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5. **Abbas Badakaya Ja'afaru** and Gafurjan Ibragimov. 2011. An Optimal Pursuit Problem Described by Infinite System of First Order Differential Equations. In *Foundamental Science Congress 2011*, 5–6 July 2011, UPM, MALAYSIA, pp: 359–360.