SOME SOLUTIONS OF DIOPHANTINE EQUATIO $x^3 + y^3 = p^k z^3$

SEE KOK LEONG

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SOME SOLUTIONS OF DIOPHANTINE EQUATION $x^3 + y^3 = p^k z^3$

By

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Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirement for the degree of Master of Science.

SOME SOLUTIONS OF DIOPHANTINE EQUATION $x^3 + y^3 = p^k z^3$

By

SEE KOK LEONG

October 2011

Chair : **Prof. Dato' Hj. Kamel Ariffin bin Mohd. Atan, PhD**

Institute : **Institute for Mathematical Research**

Diophantine equation is an algebraic equation in two or more variables in which solutions to it are from some predetermined classes and it is one of the oldest branches of number theory. They are studied and investigated to determine the values of the variables that satisfy the equation. This activity began since time immemorial and has developed into branches according to types of the equations.

This research concentrates on investigating the solvability and the determination of representations of the integral solutions of the diophantine equation $x^3 + y^3 = p^k z^3$ where k is a positive integer, $k \not\equiv 0 (\text{mod} 3)$ and p = 2, 3, 5, 13.

The approach used to determine the solutions of the equation is first to look for mathematical patterns of the integral solutions to the diophantine equation

$x^3 + y^3 = p^k z^3$. Once the patterns are obtained on the forms of the solutions, general formulae are then obtained from which solutions can be acquired. Proofs are constructed to show the validity of the formulae.

For this study we fix the values of k where $k \not\equiv 0(\mathrm{mod}3)$. This is because if $k \not\equiv 0(\mathrm{mod}3)$ then the equation $x^3 + y^3 = p^k z^3$ will acquire the form $x^3 + y^3 = (p^t z)^3$. This type of equation has been shown to have no solution as in Fermat Last Theorem. Hence we focus on the values $k \equiv 1, 2(\mathrm{mod}3)$ in this research.

In our investigation, we determine the solutions to the equation for selected values of primes p under certain conditions. We also determine the condition under which as equation do not have solutions. We also show that the number of solutions, if there exist, are either finite or infinite depending on the values of the associated parameter . Our work opens up other possibilities of future work that can be carried out based on the results obtained in this research.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains.

SEBAHAGIAN PENYELESAIAN BAGI PERSAMAAN DIOFANTUS

$$x^3 + y^3 = p^k z^3$$

Oleh

SEE KOK LEONG

Oktober 2011

Pengerusi : Prof. Dato' Hj. Kamel Ariffin bin Mohd. Atan, PhD

Institut : Institut Penyelidikan Matematik

Persamaan Diofantus merupakan persamaan aljabar yang penyelesaiannya terdiri daripada kelas-kelas yang telah ditentukan dan ia merupakan salah satu cabang yang paling lama dalam teori nombor. Ia dikaji bagi menentukan nilai-nilai pembolehubah yang memenuhi suatu persamaan. Aktiviti ini bermula semenjak zaman dahulu dan telah berkembang menjadi cabang-cabang yang bersesuaian bergantung kepada jenis persamaan.

Penyelidikan ini ditumpukan kepada penyelitian kebolehselesaian dan penentuan ungkapan penyelesaian integer kepada persamaan Diofantus $x^3 + y^3 = p^k z^3$ dengan k integer positif , $k \not\equiv 0 (\mathrm{mod}\, 3)$ dan $p = 2, 3, 5, 7$.

Pendekatan yang digunakan untuk menentukan penyelesaian kepada sesuatu persamaan ialah dengan pertama sekali mencari pola bermatematik bagi penyelesaian integer kepada persamaan Diofantus $x^3 + y^3 = p^k z^3$. Apabila bentuk pola penyelesaian diperolehi, formula umum kemudiannya ditentukan daripada penyelesaian yang didapati. Bukti dibina bagi menunjukkan kesahihan formula.

Kami menentukan nilai-nilai k dengan $k \not\equiv 0(\text{mod}3)$ dalam penyelidikan ini. Ini disebabkan oleh jika $k \equiv 0(\text{mod}3)$ maka persamaan $x^3 + y^3 = p^k z^3$ akan berbentuk $x^3 + y^3 = (p^t z)^3$. Bentuk persamaan ini telah dibuktikan bahawa tidak mempunyai penyelesaian seperti yang ditunjukkan dalam Teorem terakhir Fermat. Maka, kami memberi tumpuan ke atas nlai-nilai $k \equiv 1, 2(\text{mod}3)$ dalam kajian ini.

Dalam Kajian ini, kami menentukan penyelesaian kepada persamaan bagi nombor perdana p yang terpilih di bawah syarat-syarat tertentu. Kami juga menentukan syarat-syarat bagi persamaan-persamaan yang tidak mempunyai penyelesaian. Kami juga menunjukkan bahawa penyelesaian-penyelesaian yang wujud, samaada terhingga atau tak terhingga, adalah bergantung kepada nilai-nilai pembolehubah yang berkaitan. Kajian kami membuka kemungkinan-kemungkinan yang lain bagi penyelidikan bidang ini yang boleh dilanjutkan pada masa hadapan berdasarkan keputusan-keputusan dalam kajian ini.

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Not forgetting, my deepest gratitude to my family for giving me the courage and ability that have enabled to complete this master course.

I certify that a Thesis Examination Committee has met on 28 Oct 2011 to conduct the final examination of See Kok Leong on his thesis entitled "Some Solutions of Diophantine Equation $x^3 + y^3 = p^k z^3$" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the University Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Master of Science.

Members of the Thesis Examination Committee were as follows:

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

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SEE KOK LEONG

Date: 28 Octorber 2011

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LIST OF ABBREVIATIONS

$\gcd(a, b, c)$	Greatest common divisor of a, b and c
mod	modulus
$\mathbb{Z}$	Integer
$\mathbb{Q}$	Rational
$a \mid b$	a divide b
$\equiv$	congruent to

CHAPTER 1

INTRODUCTION

1.1 Preliminary

This chapter provides a background on the problem of finding solutions to diophantine equation, the way the term is derived, definition of diophantine equation, some examples of the problem and methods of determining its solutions. Subsequently objective and methodology of our research in general are described. Literature review section gives some examples of methods of finding solutions to diophantine equation carried out by previous researchers. The thesis outline is given at the end of this chapter.

1.2 Background

Diophantus was among the first authors who investigated exact solutions rather than approximate solutions to these types of equations. He was a Hellenistic mathematician of the third century and one of the first mathematicians to introduce symbolism into algebra. Diophantus's interest was to find solutions to polynomial equations in one or more variables in either integers or rational numbers. Since his time such polynomial equations are called diophantine equation.

Diophantine equation is an algebraic equation in two or more variables in which solutions to it are from some predetermined classes. Let us begin with an elementary problem familiar to anyone who has studied elementary algebra. Suppose a, b, c integers such that $ax+by=c$. The equation has integer solutions if $\gcd(a,b) \mid c$. And the equation has no integer solution if $\gcd(a,b) \nmid c$. The equation in two variables discussed above fall into the category of linear diophantine equation. It can be extended to a single equation into n variables where $n \geq 3$. Suppose a, b, c, d are integers such that $ax+by+cz=d$, then $\gcd(a,b,c) \mid d$. Similarly the equation $a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_kx_k = b$ has solutions where a_i and b are integers, $i = 1,2,\ldots,k$, then $\gcd(a_1, a_2, a_3, \ldots, a_k) \mid b$. Solutions to linear diophantine equations in several variables can be formed by extending the method discussed above, or by using congruence and Chinese remainder theorem. Other classes of diophantine equations include homogeneous equations of degree $d \geq 2$, and exponential diophantine equation. One of the particular forms of exponential diophantine equations is Ramanujan-Nagell equation of the form $2^n - 7 = x^2$.

One other class of the diophantine equation is the non-linear Diophantine equation of the form $ax^r + by^s = cz^t$ where a, b, c integers, r, s, t positive integers, and x, y, z variables. An example of such equation is the diophantine equation $x^2 + y^2 = z^2$, the Pythagoras equation.

Pell's equation is another type of diophantine equation of the form $x^2 - Ny^2 = 1$ where N is non square integer and x, y are integers. Trivially, $x = 1$ and $y = 0$ is always the solution to the Pell's equation. An elementary problem is that of finding integral Pythagorean triangles whose non-hypotenuse sides differ by 1. These sides are given by $m^2 - n^2$ and $2mn$ for integral values of m and n where $\gcd(m, n) = 1$ and $m \not\equiv n (\mathrm{mod} 2)$. Thus we must find such m and n for which $m^2 - n^2 - 2mn = \pm 1$. This can be rewritten as $(m-n)^2 - 2n^2 = \pm 1$. This is an example of Pell's equation and analogous to the equation with $x^2 - Ny^2 = -1$. Suppose $N = 2$, $m - n = 3$ and $n = 2$, we get the triangle with side 21, 20 and 29.

The most well-known diophantine equation is the equation of the Fermat's last theorem. This theorem states that there is no non-trivial solution to the equation $x^n + y^n = z^n$ if $n \geq 3$. This theorem was proved about three hundred years after Fermat. Andrew Wiles gave the proof in Isaac Newton Institute of Mathematics, at Cambridge in 1993.

1.3 Objective and Methodology.

The quest for finding solutions to higher degree diophantine equations continues to this day. The equations studied have evolved from the equations of Fermat's last theorem to different forms. These include the diophantine equation $x^3 + y^3 = cz^3$ for certain value of c. In our thesis, we concern with the existence of integral solutions of this type of diophantine equation in which $c = p^k$ for $p = 2, 3, 5, 7$ and positive integer k where $k \not\equiv 0(\text{mod}3)$.

We fixing $k \not\equiv 0(\text{mod}3)$ is because of if $k \equiv 0(\text{mod}3)$ will yield the equation of the form $x^3 + y^3 = (p^t z)^3$ which shows that there are no non-zero integer solutions by Fermat's Last Theorem. Besides that we are not interesting on trivial solution since it is easy, thus we are fixing x, y, z as non-zero integer or $xyz \neq 0$.

We first begin by finding integral solutions to this diophantine equation in which $p = 2$ and $k = 1$. That is the equation $x^3 + y^3 = 2z^3$. We further extend our discussion to the equation to $x^3 + y^3 = 2^k z^3$ where $k \not\equiv 0(\text{mod}3)$.

In the subsequent chapter we continue by investigating the solvability of the diophantine equation $x^3 + y^3 = 3z^3$ as well as the equation $x^3 + y^3 = 3^k z^3$. The representations of the solutions to the equation are found as soon as the solvability of the equation is determined. The discussion is then followed by finding integer solutions to the diophantine equation $x^3 + y^3 = 5z^3$. This is then generalized to the equation $x^3 + y^3 = 5^k z^3$ where $k \not\equiv 0(\text{mod}3)$ and z is a prime.

The case $p = 7$ is not considered in this study since it has been study by Amirah et al. (2011). She proved that $(-1\cdot 7^t, 2\cdot 7^t, 1)$ and $(4\cdot 7^t, 5\cdot 7^t, 3)$ are only solutions to the equation $x^3 + y^3 = 7^{3t+1} z^3$ for non negative integer t. She also shows the only solution $(-2\cdot 7^t, 11\cdot 7^t, 3)$ to the equation $x^3 + y^3 = 7^{3t+2} z^3$.

Next, the solvability of the equation $x^3 + y^3 = 13z^3$ and $x^3 + y^3 = 13^2 z^3$ is determined. This is followed by finding integral solutions to the more general equation $x^3 + y^3 = 13^k z^3$ in which $k \equiv 1 (\text{mod} 3)$ and $k \equiv 2 (\text{mod} 3)$.

The methodology applied in our research is first to find samples of integer's solution that satisfy the diophantine equations. The objective is to look for mathematical patterns for the solutions for each diophantine equation. Lemmas and theorems based on the patterns of solution sets are then proved. This is followed by determination of a general formula for finding the integer solutions of diophantine equations that are studied.

1.4 Literature Review.

Mathematicians in ancient times are very interested in finding solutions to simple algebraic equations involving integers or rational numbers only. Nowadays the field of study of diophantine equations is an active field of study in pure mathematics. It finds applications in the field of cryptography and coding theory. The study of diophantine equations have evolved into study of varied forms of diophantine equations.

Among the researchers who study the problem of diophantine equation is Cohn (1992). He considered the diophantine equation $x^2 + 2^k = y^n$ where $n \geq 3$ and k is an odd integer. He proved that there are exactly three families of solutions to this equation, which are

$$(k, x, y, n) = (6\alpha + 1, 5 \cdot 2^{3\alpha}, 3 \cdot 2^{2\alpha}, 3);$$

$$(k, x, y, n) = (4\alpha + 5, 7 \cdot 2^{\alpha}, 3 \cdot 2^{\alpha}, 4);$$

$$(k, x, y, n) = (10\alpha + 5, 11 \cdot 2^{5\alpha+3}, 3 \cdot 2^{2\alpha+1}, 5)$$

where $\alpha \geq 0$. Arif and Abu Muriefah (1997) studied the same diophantine equation with $n \geq 3$ and k an even integer. Subsequently, they gave conjecture that the equation $x^2 + 2^{2m} = y^n$ where $n \geq 3, m \geq 1$ has only two families of solutions. Cohn (1999) revisited the equation $x^2 + 2^k = y^n$, and proved the result to support the conjecture provided by Arif and Abu Muriefah.

Many special cases of the diophantine equation of the type $x^2 + q^k = y^n$ where q is a prime and m, n, x, y are positive integers have been studied in the last ten years. Arif and Abu Muriefah (1998) proved that this equation has only positive solution $(m, n, x, y) = (5 + 6M, 10 \cdot 3^{3M}, 7 \cdot 3^{2M}, 3)$ when $q = 3$ and k is odd integer, and recently Luca (2000) showed the existence of exactly one family of solutions when q=3 and k is an even integer. Arif and Abu Muriefah (2002) also proved that when q is an odd prime, $q \not\equiv 7 (\mathrm{mod} 8)$, n is an odd integer greater than 5, n is not multiple of 3 and $\gcd(n, h) = 1$ where h is class number of the field $Q(\sqrt{-q})$, then the equation $x^2 + q^{2k+1} = y^n$ has exactly two families of solution (q, n, k, x, y) which are:

$$(q, n, k, x, y) = (19, 5, 5M, 22434 \cdot 19^{5M}, 55 \cdot 19^{2M}) \text{ and}$$

$$(q, n, k, x, y) = (341, 5, 5M, 2759646 \cdot 341^{5M}, 377 \cdot 19^{2M}).$$

Abu Muriefah (2008) improved his results of previous studies and provided a complete solution to the diophantine equation $px^2 + q^{2m} = y^p$, where $m \geq 0$, $p \geq 3$ and q is prime integer. In early 2009, Demirpolat (2009) proved that if $n \geq 3$ where n is an odd integer and $h = 1$ is the class number of field $Q(\sqrt{-11})$, then the ring of integers $Q(\sqrt{-11})$ is a unique factorization domain and the equation $x^2 + 11^{2k+1} = y^n$ has only one family of solutions (n, k, x, y).

The diophantine equation $x^2 + C = y^n$, in integers x, y and $n \geq 3$ has a long and distinguished history. The first case to be solved appears to be in which $C = 1$ in 1850 by Lebesgue. He showed by using elementary factorization argument that the only solution is $x = 0$, $y = 1$. Over the next 140 years many equations of the form $x^2 + C = y^n$ have been solved using Lebesgue's elementary method. In 1993, Cohn (1993) published an exhaustive historical survey of this equation which completes the solution for all but 23 values of C in the range $1 \leq C \leq 100$. After that he successfully solved 77 cases for the various values of C. The next major breakthrough came in 2006 when Bugeaud et al. (2006) applied a combination of Baker's theory and the modular approach to the diophantine equation $x^2 + C = y^n$ and found its solutions for $1 \leq C \leq 100$.

By comparison, the diophantine equation $x^2 + C = 2y^n$ with the same restrictions has received little attention. Cohn showed that the only solutions to this equation for $C = 1$ are $x = y = 1$ and $x = 239$, $y = 13$, $n = 4$. Abu Muriefah showed that the only solutions to the equation $x^2 + C = 2y^n$ with x, y coprime integers, $n \geq 3$ and

$C \equiv 1 (\text{mod} 4)$, $1 < C \leq 100$ are

$79^2 + 9 = 2 \cdot 5^5$, $5^2 + 29 = 2 \cdot 3^3$, $117^2 + 29 = 2 \cdot 19^3$,

$993^2 + 29 = 2 \cdot 79^3$, $11^2 + 41 = 2 \cdot 3^4$, $69^2 + 41 = 2 \cdot 7^4$,

$171^2 + 41 = 2 \cdot 11^4$, $1^2 + 53 = 2 \cdot 3^3$, $25^2 + 61 = 2 \cdot 7^3$,

$51^2 + 61 = 2 \cdot 11^3$ and $37^2 + 89 = 2 \cdot 9^3$.

Chen and Vautier (1997) used a method of Thue and Siegel, based on explicit Pade approximations to algebraic functions, to completely solve a family of quartic Thue equation. They proved that the equation $x^2 + 1 = dy^4$ has at most one solution in positive integers when $d \geq 3$. Moreover, when such a solution exists, it is of the form $(u, \sqrt{v})$ where (u, v) is the fundamental solution of $x^2 + 1 = dy^2$.

Broughan (2003) considered the equation $n = x^3 + y^3$ modulo m to study the intrinsic characteristics of positive integers that can represent the sum of two cubic numbers. He stated that an integer can be written as the sum of two cubic numbers modulo m if and only if m is not divisible by 7 or 9.

Scott and Styer (2006) studied the equation $\pm a^x \pm b^y = c$ where a, b, x, y are positive integers. On determining the number of solutions (x, y, u, v) to the equation $(-1)^u a^x + (-1)^v b^y = c$, he proved that this equation has at most two solutions for given integers a and b both greater than one and c greater than zero, except when (a, b, c) or (b, a, c) is (3,2,5) which gives four solutions, and

$(3,2,1)$, $(3,2,7)$, $(3,2,11)$, $(3,2,13)$, $(4,3,13)$, $(5,2,3)$, $(4,2,3\cdot 4^k)$ each of which give three solutions. This equation is a generalized form of the familiar Pillai equation $a^x - b^y = c$ which has been treated by many researchers from the standpoint of considering the number of solutions (x, y) for a given (a,b,c).

Terai (1993) gave conjectures that if the equation $a^2 + b^2 = c^2$ holds with $\gcd(a,b,c) = 1$ and a is an even number, then the equation $x^2 + b^m = c^n$ has only one positive integral solution $(x,m,n) = (a,2,2)$. Subsequently he considered the equation $x^2 + q^m = p^n$ and attempted to answer the question whether the equation has positive integer solutions or not other than $(x,m,n) = (p-1,2,2)$. Since the time of Terai's conjectures, many researchers begin to examine these conjectures in their study. In 1995, Maohua Le proved Terai's conjectures when $b > 8\cdot 10^6$, $b \equiv \pm 5 (\text{mod} 8)$ and c is a prime power. This is followed by Yuan (1998) who proved Terai's conjectures by a different method if $a^2 + b^2 = c^2$, $(a,b,c) = 1$, $b \equiv \pm 5 (\text{mod} 8)$ and c is a prime power.

Among the countless results relating to Fermat's last theorem is a theorem due to Lebegue that states the following: For any natural number *n*, if $x^n + y^n = z^n$ has no non-trivial integral solutions, then neither does $x^{2n} + y^{2n} = z^2$. Since Fermat's last theorem is now known to hold for all *n* such that $n \geq 3$, the equation $x^{2n} + y^{2n} = z^2$ also has only trivial solutions for this range of n. Actually $x^4 + y^4 = z^2$ has only a trivial solution, the proof of which can be found in any elementary number theory book and which implies Fermat's last theorem for exponent 4.

Darmon (1993) proposed two conjectures on the solutions of the equations $x^n + y^n = z^2$ and $x^n + y^n = z^3$. These conjectures state that there are no non-trivial solutions to the equation $x^n + y^n = z^2$ for $n \geq 4$ and $x^n + y^n = z^3$ for $n \geq 3$. In 1997, Darmon and Merel (1997) proved three main theorems which state that there are no non-trivial primitive solutions to the equation $x^n + y^n = 2z^n$ for $n \geq 3$, $x^n + y^n = z^2$ for $n \geq 4$, and $x^n + y^n = z^3$ for $n \geq 3$. Gopalan (2006) gave

$$(x, y, z) = \{(0,0,0), (0, 2^{3\alpha-1}, 2^{2\alpha-1}), (2^{3\alpha-1}, 0, 2^{2\alpha-1}), (\alpha^3, \alpha^3, \alpha^2)\}$$

as trivial solutions to the Diophantine equation $x^2 + y^2 = 2z^3$.

In 2005, Andrica (2005) considered the diophantine equation of the form $ax^2 + pxy + by^2 = z^{k^n}$. They gave integral solutions to the equations for two special cases in which $ax^2 + by^2 = z^{3^n}$ and $x^2 + pxy + y^2 = z^{2^n}$ where x and y are relatively prime.

Cenberci Senay (2009) considered the diophantine equation $x^2 + B^2 = y^4$. They proposed a new conjecture analogous to Terai's conjecture which is that if the equation $a^2 + B^2 = y^4$ with $(a, B, y) = 1$, a is an even integer and (a, B, y^2) is a Pythagoras triangle, then the diophantine equation $x^2 + B^m = y^n$ has only one positive integer solution $(x, m, n) = (a, 2, 4)$. Subsequently they proved that the conjectures are valid if $y \equiv 5 \pmod 8$ is a prime power. Conversely if B is a prime power, $y^2 = Y \equiv 1 (\text{mod} 8)$, so that both conjectures of theirs and Terai are valid.

1.5 Thesis Outline.

We start with the next chapter which discusses the integral solution to the diophantine equations $x^3 + y^3 = 2z^3$ and $x^3 + y^3 = 2^2 z^3$. This is followed by finding integral solution to the equation $x^3 + y^3 = 2^{3t-2} z^3$ and $x^3 + y^3 = 2^{3t-1} z^3$ where t>0.

In chapter III, integer solutions to the diophantine equation $x^3 + y^3 = 3^2 z^3$ are determined. This is followed by the equation $x^3 + y^3 = 3^{3t+2} z^3$ where t is a natural number.

In chapter IV, a different methods are used to obtain integer solutions to the diophantine equation $x^3 + y^3 = 5^k z^3$ where $k \not\equiv 0 (\text{mod} 3)$. A number of examples are given to illustrate how the integer solutions are found. They are $x^3 + y^3 = 5z^3$, $x^3 + y^3 = 5^4 z^3$, $x^3 + y^3 = 5^k z^3$ where $k \equiv 1 (\text{mod} 3)$ and $x^3 + y^3 = 5^2 z^3$, $x^3 + y^3 = 5^5 z^3$, $x^3 + y^3 = 5^k z^3$ where $k \equiv 2 (\text{mod} 3)$.

In the subsequent chapter, the solvability of the diophantine equation $x^3 + y^3 = p^k z^3$ for the special case $p = 13$ where $k \not\equiv 0 (\text{mod} 3)$ are discussed. The results of the study, conclusions based on these results and some suggestions for future research are given in the last chapter.

BIBLIOGRAPHY

Abu Muriefah F.S. (2008). On the Diophantine Equation $px^2 + q^{2m} = y^n$. *Journal of Number Theory 128*. 1670-1675.

Abu Muriefah F.S., Luca F., Siksek S. and Tengely S. (2007). *On the Diophantine Equation* $x^2 + C = 2y^n$ from http://www.warwick.ac.uk/~maseap/papers/lmst6. pdf .

Amirah Zainal Abidin S.N., Mohd Atan K.A & Sapar S.H.(2011). *On the Diophantine Equation* $x^3 + y^3 = 7^k z^3$. Proceeding Regional Fundemental Science Congress 2011. The 3rd Fundemental Science Congress. 373-374.

Andrica D. and Tudor G.M. (2005). Some Diophantine Equations of the Form $ax^2 + pxy + by^2 = z^{k^n}$. *Genaral Mathematics*. 13[2]. 121-130.

Arif S.A. and Abu Muriefah F.S. (1997). On The Diophantine Equation $x^2 + 2^k = y^n$. *Internat. J.Math. & Math. Sci.* 20[2]. 299-304.

Arif S.A. and Abu Muriefah F.S.(1998). On The Diophantine Equation $x^2 + 3^m = y^n$. *International Journal Math. & Math. Sci.* 21[3]. 619-620.

Arif S.A. and Abu Muriefah F.S.(2002). On The Diophantine Equation $x^2 + q^{2k+1} = y^n$. *Journal of Number Theory*. 95. 95-100.

Broughan K.A. (2003). Characterizing The Sum of Two Cubes. *Journal of Integer Sequences.* 6. 1-7.

Bugeaud Y., Mignotte M. and Siksek S. (2006). Classical and Modular Approaches to Exponential Diophantine Equations II. The Lebesgue-Nagell equation, *Compositio Math*. 142. 31-62.

Cenberci S. and Senay H. (2009). The Diophantine Equation $x^2 + B^m = y^n$. *International Journal of Algebra.* 3[13]. 657-662.

Chen J.H. and Vautier P. (1997). Complete Solution Of The Diophantine Equation $X^2 + 1 = dY^4$And A Related Family Of Quartic Thue Equation. *Journal of Number Theory 62. 71-99.*

Cohen M. (2000). *A course in Computational Algebraic Number Theory*. Springer Verlag.

Cohen H. (2007). *Graduate Texts in Mathematics Number Theory Analytic and Modern Tools*. Volume 2, Springer. 467-472.

Cohn J.H.E. (1992). The Diophantine Equation $x^2 + 2^k = y^n$. *Arch. Math.* (Basel). 59[4]. 341-344.

Cohn J.H.E (1993). The Diophantine Equation $x^2 + C = y^n$. *Acta Arithmetica*. 65[4]. 367-381.

Cohn J.H.E (1998). Perfect Pell Powers, *Glasgow Math*. J.38, 19-20.

Cohn J.H.E. (1999). The Diophantine Equation $x^2 + 2^k = y^n$, II. *Internat. J. Math. & Math. Sci.* 22[3]. 459-462.

Darmon H. (1993). The Equations $x^2 + y^n = z^2$and $x^2 + y^n = z^3$. *International Mathematics Research Notices* 10. 263-274.

Darmon H. and Merel L. (1997). Winding Quotients and Some Variants of Fermat's Last Theorem. *J. Reine Angew. Math.* 490. 81-100.

Demirpolat E., Cenbersi S. and Senay H. (2009). On The Diophantine Equation $x^2 + 11^{2k+1} = y^n$. *International Mathematical Forum*. 4[6]. 277-280.

Frohlich A. and Taylor M.J.(1993). *Algebraic Number Theory*. Cambridge University Press.

Gopalan M.A, Somanath M. and Vanitha N. (2006). On ternary Cubic Diophantine Equation $x^2 + y^2 = 2z^3$. *Advances in Theoretical and Applied Mathematics (ATAM)*. 1[3]. 227-231.

Levesque (2003). On A Few Diophantine Equation of The Form $x^n + y^n = z^n$. *Acta Arithmetica*. 86[3]. 193-205.

Luca F. (2000). On a Diophantine Equation, Bull. *Austral. Math.* Soc 61. 241-246.

Niven and Zuckerman (1991). *An Introduction to The Theory of Numbers*. John Wiley and Sons INC. 5th edition.

Pohst M. and Zassehaus H. (1989). *Algorithmic Algebraic Number Theory*. Cambridge University Press.

Scott R. and Styer R. (2006). On the Generalized Pillai Equation $\pm a^x \pm b^y = c$. *Journal of Number Theory 118*. 236-265.

Terai N. (1993). The Diophantine Equation $x^2 + q^m = p^n$. *Acta Arith*. 63[4]. 351-358.

Wiles A. (1995). Modular Elliptic Curves and Fermat's Last Theorem, *Annals of Mathematics*, 141:443-551.

Yuan P.Z. and Wang J.B. (1998). On the Diophantine Equation $x^2 + q^m = p^n$. *Acta Arith*. 63[4]. 351-358.

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LIST OF PUBLICATIONS

See Kok Leong, Kamel Ariffin Mohd Atan, Siti Hasana Sapar. *On the Positive Solutions of the Diophantine Equation* $x^3 + y^3 = 2^k z^3$. Prosiding Seminar Kebangsaan Aplikasi Sains & Matematik 2010 sempena Simposium Kebangsaan Sains Matematik Ke-18. 23-27.