IMPROVED RUNGE-KUTTA TYPE METHODS FOR SOLVING ORDINARY AND FUZZY DIFFERENTIAL EQUATIONS

FARANAK RABIEI

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IMPROVED RUNGE-KUTTA TYPE METHODS FOR SOLVING ORDINARY AND FUZZY DIFFERENTIAL EQUATIONS

By

FARANAK RABIEI

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

November 2012

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DEDICATIONS

To

My Dear Mum and Dad

for their encouragement

and

My kind husband, Saeid

For his great support and patience

and

My respected Teachers

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Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy.

IMPROVED RUNGE-KUTTA TYPE METHODS FOR SOLVING ORDINARY AND FUZZY DIFFERENTIAL EQUATIONS

By

FARANAK RABIEI

November 2012

Chairman: Professor Fudziah Ismail, Ph.D.

Faculty: Science

In this study, we constructed the Improved Runge-Kutta (IRK) type of methods for solving first and second order ordinary differential equations as well as fuzzy differential equations. With the aim to increase the computational efficiency of the methods, we obtained the methods of higher order with less number of stages or function evaluations. The methods which arise from the classical Runge-Kutta methods can also be considered as a special class of two-step methods, that is the approximation at the current point is based on the values or information from the two previous points. Hence, the methods contain the current internal stage k_i as well as the previous internal stages k_{-i}. The aim here is to use the available internal stage in the previous step so that the resulting methods are more accurate.

In the first part of the thesis, the order conditions of the methods are obtained using Taylor series expansion. Based on the order conditions, IRK methods of different orders and stages for solving first order ODEs are constructed. The convergence of the method is proven and the stability regions of the methods are

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also presented. Numerical results based on the new methods are compared with the existing methods in the literature showed that they are computationally more efficient.

Next, the order conditions of the methods for solving some special second order ODEs are obtained using Taylor series expansion. Based on the order conditions as well as work done by Dormand (1996), Improved Runge Kutta Nystrom (IRKN) methods of different orders and stages for solving the special second order ODEs $y'' = f(x, y)$ are constructed. The stability polynomial and stability region of of the methods are discussed. Numerical results based on the new methods are compared with the existing methods in the literature and it is showed that the new IRKN methods are computationally more efficient.

We also derived IRKN methods which are specifically designed for the autonomous second order ODEs of the form $y'' = f(y)$ based on the order conditions. These methods are called Accelerated Runge-Kutta Nystrom methods. The stability properties of the methods are discussed and numerical results showed that they are more efficient compared to the existing RKN methods.

Finally, both IRK and IRKN methods are adapted for solving first and second order fuzzy differential Equations (FDEs). The convergence of IRK methods when applied to FDEs is also proven and numerical results proved that the IRK and IRKN methods give accurate results compared to the existing methods in the literature.
In conclusion, the methods derived in this thesis are more efficient than existing methods for solving first and second order ordinary differential equations and fuzzy differential equations.

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Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

PENAMBAHBAIKAN JENIS KAEDAH RUNGE-KUTTA UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA DAN SAMAR

Oleh

FARANAK RABIEI

November 2012

Pengerusi: Profesor Fudziah Ismail, Ph.D.

Fakulti: Sains

Dalam kajian ini, kami telah menerbitkan Kaedah Runge-Kutta Penambahbaikan (RKP) untuk menyelesaikan persamaan pembezaan peringkat pertama dan kedua dan juga persamaan pembezaan kabur. Dengan tujuan untuk memperbaiki kecekapan kaedah tersebut, kami cuba memperolehi kaedah berperingkat tinggi dengan tahap atau pengiraan fungsi yang kurang. Kaedah yang terhasil dari kaedah Runge-Kutta klasik ini boleh di kategorikan sebagai kaedah dua langkah iaitu penghampiran bagi titik semasa bergantung kepada informasi dari dua titik sebelumnya. Maka kaedah ini mengandungi tahap dalaman k_i semasa dan juga tahap dalaman pada titik sebelumnya iaitu k_{-i}. Tujuannya di sini adalah untuk menggunakan tahap yang sedia ada di titik sebelumnya supaya kaedah yang terhasil adalah lebih jitu.

Dalam bahagian pertama tesis ini, syarat peringkat bagi kaedah tersebut diperolehi menggunakan kembangan siri Taylor. Berdasarkan syarat peringkat ini, Kaedah Runge-Kutta Penambahbaikan dengan peringkat dan tahap berbeza untuk

menyelesaikan persamaan pembezaan peringkat pertama diterbitkan. Penumpuan kaedah ini telah dibuktikan dan kestabilannya juga dipersembahkan. Keputusan berangka bagi kaedah yang baru ini dibandingkan dengan kaedah sedia ada dalam literatur telah menunjukkan pengiraannya lebih cekap.

Kemudian syarat peringkat untuk kaedah bagi menyelesaikan persamaan pembezaan (PPB) peringkat kedua yang khas diperolehi melalui kembangan siri Taylor. Berdasarkan syarat peringkat ini dan kerja yang dilakukan oleh Dormand (1996), kaedah Runge-Kutta Nystrom Penambahbaikan (RKNP) yang berbeza peringkat dan tahap bagi menyelesaikan PPB peringkat kedua yang khas $y'' = f(x, y)$ diterbitkan. Polinomial kestabilan dan rantau kestabilan kaedah ini dipersembahkan. Keputusan berangka bagi kaedah yang baru ini dibandingkan dengan kaedah sedia ada dalam literatur yang menunjukkan kaedah (RKNP) yang baru ini adalah lebih cekap dari segi pengiraan.

Berdasarkan syarat peringkat tersebut kami juga menerbitkan kaedah (RKN) yang khas untuk persamaan pembezaan berautonomi peringkat kedua $y'' = f(y)$. Kaedah ini disebut kaedah Runge-Kutta Nystrom dipercepatkan. Ciri kestabilan kaedah ini dibincangkan dan keputusan berangkanya menunjukkan kaedah ini lebih cekap berbanding denga kaedah RKN sedia ada.

Akhir sekali kedua-dua kaedah IRK dan IRKN disesuaikan untuk menyelesaikan persamaan pembezaan kabur (PPK) peringkat pertama dan kedua. Penumpuan keadah (RKP) bila disesuaikan kepado (PPK) is proven dan di mana ianya juga memberikan keputusan berangka yang lebih baik dari kaedah sedia ada.

Kesimpulannya kaedah yang diterbitkan dalam tesis ini adalah lebih cekap dari

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kaedah sedia ada bagi menyelesaikan persaman pembezaan peringkat pertama, kedua dan juga persamaan pembezaan kabur.

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the most merciful, the most beneficent

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I certify that a Thesis Examination Committee has met on **6 November 2012** to conduct the final examination of **Faranak Rabiei** on her thesis entitled "**Improved Runge-Kutta Type Methods for Solving Ordinary and Fuzzy Differential Equations** " in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the **Doctor of Philosophy**.

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This thesis was submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee were as follows:

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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.

Faranak Rabiei

Date: 6 November 2012

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LIST OF ABBREVIATIONS

ODEs	Ordinary Differential Equations
FDEs	Fuzzy Ordinary Differential Equations
RK	Runge-Kutta method
ARK	Accelerated Runge-Kutta method
RKN	Runge-Kutta Nystrom method
IRK	Improved Runge-Kutta method
IRKN	Improved Runge-Kutta Nystrom method
ARKN	Accelerated Runge-Kutta Nystrom method
FIRK	Fuzzy Improved Runge-Kutta method
FRK	Fuzzy Runge-Kutta method
FIRKN	Fuzzy Improved Runge-Kutta Nystrom method
FRKN	Fuzzy Runge-Kutta Nystrom method

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Differential equations are used to model problems in science and engineering. These differential equations basically are classified into two groups, Ordinary Differential Equation (ODEs) and Partial Differential Equation (PDEs). A mathematical formulation of physical phenomena in science and engineering often leads to ODEs such as celestial mechanics, molecular dynamics, semi-discretization of wave equations, electronics, etc.

The most popular ODEs are in class of first order and second order ODEs. In nature, the differential equations that model problems are often extremely difficult or some times impossible to solve analytically. Therefore, numerical methods are used for understanding the behavior of their solutions. The numerical methods of finding solution of initial value problems of ODEs may generally be classified into two classes:

- Single step methods: the approximated solution is evaluated using the information of only one previous point.
- Mulistep methods: the approximated solution is evaluated using the information of k previous points.

One of the most common numerical methods for solving ODEs is the Runge-Kutta method which can be of one step and two step. The two step Runge-Kutta method used the approximated values from the previous step during the current step. Most second order ODEs can be solved by reducing to the first order or it can be solved directly using methods which are specified for them. One of the popular numerical methods for solving second order ODE directly is the Runge-Kutta Nystrom

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method.

Fuzzy Differential Equations (FDEs) are another type of differential equations which are used to analyze the behavior of the problems that are subjected to imprecise or uncertain factors or ranging. Similarly to ODEs, the exact analytical solution of FDEs are often difficult or sometime impossible to obtain, thus constructing the numerical methods with a wide range of accuracy that processes some properties of solution of FDEs are particularly important.

1.2 Objective of the thesis

The main objective of the research is to construct Improved Runge-Kuta type methods for solving first and second order ordinary as well as fuzzy differential equations. This goal can be attained by :

1. Construction of Improved Runge-Kutta method for solving first order ordinary differential equations by using Taylor series expansion. The coefficients of method are determined by using minimization of the error norm.

2. Derivation of the Improved Runge-Kutta Nystrom method for solving second order ordinary differential equations based on Improved Runge-Kutta method given in first part of this thesis and following the approach proposed by Dormand (1996) on derivation of Runge-Kutta Nystrom methods.

3. Construction the Accelerated Runge-Kutta Nystrom method for solving autonomous second order ordinary differential equations $y'' = f(y)$, based on Accelerated Runge-Kutta method derived by Udwadia and Farahani (2008) and following the same approach of Improved Runge-Kutta Nystrom methods given in second part of this thesis.

4. Developing the Fuzzy Improved Runge-Kutta method for solving first order

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fuzzy differential equations by adapting the Improved Runge-Kutta methods derived in first part of thesis for solving fuzzy differential equations.

5. Deriving the Fuzzy Improved Runge-Kutta Nystrom method for solving second order fuzzy differential equations based on Runge-Kutta Nystrom methods derived in second part of thesis.

1.3 Scope of thesis

In this study we aim to construct the Improved Runge-Kuta type of methods for solving first and second order ordinary and fuzzy differential equations. This scheme proposed here arise from the classical Runge-Kutta methods and also can be considered as a special class of two-step methods. The advantage of these constant step size methods is, by using the less number of stages which leads to less number of function evaluations, require less time to approximate the more accurate results compared with the existing methods. Therefore, the new schemes with less number of function evaluations and more accurate results, are computationally more efficient than the existing methods.

1.4 Outline of thesis

In chapter 1, a brief introduction on differential equations and the application of numerical methods for solving different type of differential equations are given.

Chapter 2 consists of earlier researches and related study on Runge-Kutta type of methods for solving ODEs and FDEs. Some basic definitions and theorems on numerical methods for solving ODEs and FDEs will also be given.

Chapter 3 describes the construction of Improved Runge-Kutta method for solving first order ordinary differential equations. The scheme is two step in nature

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and requires less number of function evaluations when compared with the classical Runge-Kutta method. The order conditions of the method using Taylor series expansion are obtained for up to order six and methods of order three, four and five with different stages are derived. The derivation of the methods are based on the order conditions together with minimization of the error norm to find the free parameters. The convergence of the method is proven and the stability region of the methods is also presented. Numerical results show that the method are more efficient compared to the existing well known classical Runge-Kutta methods.

Chapter 4 discusses the derivation of Improved Runge-Kutta Nystrom method for solving second order ordinary differential equations. The order conditions of the method using the Taylor series expansion are obtained for up to order five and methods of order three, four and five with two, three and four stages, respectively, are obtained. The stability properties of the new methods are discussed and to illustrate the efficiency of method a number of tested problem are validated and the numerical results are compared with some existing Runge-Kutta Nystrom methods.

Chapter 5 introduces the Accelerated Runge-Kutta Nystrom method for solving autonomous second order ordinary differential equations $y'' = f(y)$. The order conditions, derivation and stability of method are provided. Numerical examples are given and comparison with existing methods are also presented.

In chapter 6, a brief introduction to fuzzy differential equations are given and Improved Runge-Kutta method of order three, four and five derived in Chapter 3, are adapted for solving first order fuzzy differential equations. Numerical examples are given and numerical results are compared with existing Fuzzy Rung-Kutta method.

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In chapter 7, we adapt the obtained Improved Runge-Kutta Nystrom method of order three, four and five in Chapter 4, for solving second order fuzzy differential equations. Numerical examples are given and the numerical performances of the methods are compared with existing Fuzzy Rung-Kutta Nystrom method.

Finally the summary of the whole thesis, conclusion and future research are presented in Chapter 8.

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APPENDIX A

Derivation of order condition for IRK4 method with three stages using Taylor series expansion for solving first order ODEs.

```
> restart;
> D(y):= x->f(x,y(x)):
> alias(
 F = f(x, y(x)),
 Fx = (D[1](f))(x, y(x)),
 Fy = (D[2](f))(x, y(x)),
 Fxy = (D[1, 2](f))(x, y(x)),
 Fyy = (D[2, 2](f))(x, y(x)),
 Fxx = (D[1, 1](f))(x, y(x)),
 Fxxx = (D[1, 1, 1](f))(x, y(x)),
 Fxxy = (D[1, 1, 2](f))(x, y(x)),
 Fxyx = (D[1, 2, 1](f))(x, y(x)),
 Fxyy = (D[1, 2, 2](f))(x, y(x)),
 Fyyy = (D[2, 2, 2](f))(x, y(x)));
> m := 4:
> taylor(y(x+h), h = 0, m+1);
> TaylorPhi := normal((convert(%, polynom)-y(x))/h);
> k1  := taylor(f(x, y(x)), h = 0, m);
  km1 := taylor(f(x-h, y(x-h)), h = 0, m);
  k2  := taylor(f(x+c[2]*h, y(x)+h*a[2, 1]*k1), h = 0, m);
  km2 := taylor(f(x-h+c[2]*h, y(x-h)+h*a[2, 1]*km1), h = 0, m);
  k3  := taylor(f(x+c[3]*h, y(x)+h*(a[3, 1]*k1+a[3, 2]*k2)), h = 0, m);
  km3 := taylor(f(x-h+c[3]*h, y(x-h)+h*(a[3, 1]*km1+a[3, 2]*km2)),
   h = 0, m);
```


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```
  RungeKuttaPhi := convert(series(b[1]*k1-b[-1]*km1+b[2]*(k2-km2)
  +b[3]*(k3-km3), h, m), polynom);
> d := expand(TaylorPhi-RungeKuttaPhi);
  eqns := {coeffs(d, [h, F, Fx, Fy, Fxx, Fxy, Fyy, Fxxx, Fxxy, Fxyy,
   Fyyy])};
```

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APPENDIX B

Derivation of order condition for y' of IRKN5 method with four stages using Taylor series expansion for solving second order ODEs.

```
> restart;

> D(D(y)) := x->f(x,y(x))

> alias(

yp = (D(y))(x),

F = f(x, y(x)),

Fx = (D[1](f))(x, y(x)),

Fy = (D[2](f))(x, y(x)),

Fxy = (D[1, 2](f))(x, y(x)),

Fyy = (D[2, 2](f))(x, y(x)),

Fxx = (D[1, 1](f))(x, y(x)),

Fxxx = (D[1, 1, 1](f))(x, y(x)),

Fxxy = (D[1, 1, 2](f))(x, y(x)),

Fxyx = (D[1, 2, 1](f))(x, y(x)),

Fxyy = (D[1, 2, 2](f))(x, y(x)),

Fyyy = (D[2, 2, 2](f))(x, y(x)),

Fxxxx = (D[1, 1, 1, 1](f))(x, y(x)),

Fxxxy = (D[1, 1, 1, 2](f))(x, y(x)),

Fxxyx = (D[1, 1, 2, 1](f))(x, y(x)),

Fxyxx = (D[1, 2, 1, 1](f))(x, y(x)),

Fyxxx = (D[2, 1, 1, 1](f))(x, y(x)),

Fxxyy = (D[1, 1, 2, 2](f))(x, y(x)),

Fxyxy = (D[1, 2, 1, 2](f))(x, y(x)),

Fyxxy = (D[2, 1, 1, 2](f))(x, y(x)),

Fxyyx = (D[1, 2, 2, 1](f))(x, y(x)),
```


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```
Fyyxx = (D[2, 2, 1, 1](f))(x, y(x)),
Fxyyy = (D[1, 2, 2, 2](f))(x, y(x)),
Fyyxy = (D[2, 2, 1, 2](f))(x, y(x)),
Fyyyx = (D[2, 2, 2, 1](f))(x, y(x)),
Fyyyy = (D[2, 2, 2, 2](f))(x, y(x)));
> m := 5;
> taylor((D(y))(x+h), h = 0, m+1);
> TaylorPhi := normal((convert(%, polynom)-yp)/h);
>  T := taylor((D(y))(x), h = 0, m);
  Tm := taylor((D(y))(x-h), h = 0, m)
  k1 := taylor(f(x, y(x)), h = 0, m);
 km1 := taylor(f(x-h, y(x-h)), h = 0, m);
  k2 := taylor(f(x+c[2]*h, y(x)+h*c[2]*yp+h^2*a[2, 1]*k1), h = 0, m);
 km2 := taylor(f(x-h+c[2]*h, y(x-h)+h*c[2]*Tm+h^2*a[2, 1]*km1),
        h = 0, m);
  k3 := taylor(f(x+c[3]*h, y(x)+h*c[3]*yp+h^2*(a[3, 1]*k1+a[3, 2]*k2)),
        h = 0, m);
 km3 := taylor(f(x-h+c[3]*h, y(x-h)+h*c[3]*Tm+h^2*(a[3, 1]*km1
       +a[3, 2]*km2)), h = 0, m);
  k4 := taylor(f(x+c[4]*h, y(x)+h*c[4]*yp+h^2*(a[4, 1]*k1+a[4, 2]*k2
       +a[43]*k3)), h = 0, m);
 km4 := taylor(f(x-h+c[4]*h, y(x-h)+h*c[4]*Tm+h^2*(a[4, 1]*km1
       +a[4, 2]*km2+a[43]*km3)), h = 0, m);

RungeKuttaPhi := convert(series(b[1]*k1-b[-1]*km1+b[2]*(k2-km2)
+b[3]*(k3-km3)+b[4]*(k4-km4), h, m), polynom);
> d := expand(TaylorPhi-RungeKuttaPhi);
```


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```
eqns := {coeffs(d, [yp, h, F, Fx, Fy, Fxx, Fxy, Fyy, Fxxx, Fxxy,
Fxyy, Fyyy, Fxxxx, Fxxxy, Fxxyx, Fxyxx, Fyxxx, Fxxyy, Fxyxy,Fyxxy,
Fxyyx, Fyyxx, Fxyyy, Fyyxy, Fyyyx, Fyyyy]))};
```

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APPENDIX C

Maple program code for IRKN4 with three stages for solving second order ODEs.

```
> restart;
> Digits := 20;
> f(t,y):=-y+t:
  g(t):=evalf(sin(t)+cos(t)+t):
  gp(t):=evalf(cos(t)-sin(t)+1):
> h := 0.005;
n := evalf((b-a)/h);
x[0] := 0;
x[1] := x[0]+h;
y[0] := g(x[0]);
y[1] := g(x[1]);
yp[0] := gp(x[0]);
yp[1] := gp(x[1]);
> for i to n do
 k[1]  := f(x[i], y[i]);
 k[-1] := f(x[i-1], y[i-1]);
 k[2]  := f(x[i]+c[2]*h, y[i]+yp[i]*c[2]*h+h^2*a[21]*k[1]);
 k[-2] := f(x[i-1]+c[2]*h, y[i-1]+yp[i-1]*c[2]*h+h^2*a[21]*k[-1]);
 k[3]  := f(x[i]+c[3]*h, y[i]+yp[i]*c[3]*h+h^2*(a[31]*k[1]+a[32]*k[2]));
 k[-3] := f(x[i-1]+c[3]*h, y[i-1]+yp[i-1]*c[3]*h+h^2*(a[31]*k[-1]
          +a[32]*k[-2]));
 y[i+1]:= y[i]+(3/2)*h*yp[i]-(1/2)*h*yp[i-1]+h^2*(b[2]*(k[2]-k[-2])
          +b[3]*(k[3]-k[-3]));
 yp[i+1]:= yp[i]+h*(bp[1]*k[1]-bp[-1]*k[-1]+bp[2]*(k[2]-k[-2])
```


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```
          +bp[3]*(k[3]-k[-3]));
 x[i+1]:= x[i]+h;
 error[i+1] := abs(g(x[i+1])-y[i+1]);
 errorp[i+1] := abs(gp(x[i+1])-yp[i+1]);
 if error[i+1] >= maxerror then
     maxeror := eror[i+1]
 end if;
 if errorp[i+1] >= maxerrorp then
    maxerrorp := errorp[i+1]
 end if;
 maxError := max(maxerror, maxerrorp)
end do;
maxError
```

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BIODATA OF THE STUDENT

The author was born on the 10 of July 1982 in Esfahan, Iran. She started her primary school at Nabovat School, Malekshahr, Esfahan, Iran and completed her secondary education at Molana Secondry school, Malekshahr, Esfahan, Iran. She finished her High school education level on Maktabi High school, Malekshahr, Esfahan, Iran.

In 2000 she arrived in Islamic Azad University Branch of Khorasgan, Esfahan, Iran and she received her Bachelor degree of Applied Mathematics in 2004. After that she joined as a lecturer assistant in Islamic Azad University Branch of Khomainishahr, Esfahan, Iran. In 2006 she started her Master degree on Islamic Azad University Branch of Tehran Center, Tehran, Iran and she finished her Master degree in field of Numerical Analysis in Applied Mathematics, in 2008.

In January 2009 she received the offer letter for PhD program in Applied Mathematics from Universiti Putra Malaysia. Currently, she is attending a PhD program in field of Numerical Methods at Department of Mathematics, Universiti Putra Malaysia and working on Improved Runge-Kutta type of methods for solving first and second order ODEs and FDEs.

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