



**UNIVERSITI PUTRA MALAYSIA**

**STATISTICAL INFERENCE ON THE MODIFIED GUMBEL  
DISTRIBUTION PARAMETERS**

**AHMED ALI OMAR HURAIRAH.**

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**DOCTOR OF PHILOSOPHY  
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**By**

**AHMED ALI OMAR HURAIRAH**

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***TO  
Salman  
Suha and Malak***



Abstract of the thesis presented to the Senate of Universiti Putra Malaysia in  
fulfilment of requirements for the degree of Doctor of Philosophy

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**August 2005**

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**Faculty : Science**

The work in this thesis is concerned with the progress and development of the Gumbel distribution by the introduction of a new parameter namely, the shape parameter. Generalization of the Gumbel distribution is established. The work is also concerned with the investigation of the finite sample performance of asymptotic inference procedures using the likelihood function based on the modified distributions. The study includes investigating the adequacy of asymptotic inferential procedures in small samples. The maximum likelihood estimator of the parameters of modified distributions is not available in closed form. Thus a simulation study is conducted to investigate the bias, asymptotic variance (ASV), finite sample variance (FSV), and the mean square error (MSE) of the maximum likelihood estimator of the parameters of the modified distribution. Exact testing hypothesis procedures for the modified distribution are intractable. Therefore three standard large sample statistics based on maximum likelihood estimator were considered, which are the likelihood ratio, the Wald, and the Rao statistics. Their performances in finite samples in terms of their sizes and powers are investigated and compared. Confidence intervals based



on the likelihood ratio, the Wald, and the Rao statistics were studied. The performances in terms of the attainment of the nominal error probability and symmetry of lower and upper probabilities were investigated and compared.

The main findings of the simulation studies of the inference procedures for the parameters of the modified Gumbel distribution indicate that the estimate of the shape parameter is nearly unbiased, while estimates of the location and scale parameters tend to be slightly biased for small sample size of the univariate distribution, while for bivariate models, estimate of the scale and shape parameters performance are satisfactory in terms of bias and variance in all the situations considered.

In the hypothesis testing of the modified distribution, the likelihood ratio statistic appears to perform better than the Wald and the Rao statistics. Interval estimates for the scale parameter based on Wald and Rao statistics are highly symmetric and tend to be slightly anticonservative, while intervals based on the likelihood ratio statistics are in general symmetric and attain the nominal error probability. For the shape parameter, all intervals tend to be symmetric in the lower and upper error probabilities.

Results of the simulations also indicate that the modified extreme value models can contribute meaningfully in solving several problems of the environmental data, particularly the air pollution data.



Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia  
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

**PENTAKBIRAN STATISTIK BAGI PARAMETER BERTABURAN  
GUMBEL TERUBAHSUAI**

Oleh

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Kajian di dalam tesis ini adalah mengenai perkembangan dan lanjutan bagi taburan Gumbel. Ubahsuaian kepada taburan ini dilakukan dengan memperkenalkan suatu parameter baru iaitu parameter bentuk. Pengitlakan taburan dibina. Penyelidikan keatas perlakuan asimptotik sampel terhingga bertaburan nilai ekstrem terubahsuai dengan menggunakan kaedah fungsi kebolehjadian diterokai. Kajian juga meliputi penyelidikan keatas kecukupan prosedur pentakbiran asimptotik bagi sampel kecil. Anggaran kebolehjadian maksimum bagi parameter taburan terubahsuai ini tidak boleh diperoleh secara tertutup. Yang demikian kajian simulasi dilaksanakan untuk mengkaji kepincangan, ralat piawai asimptotik, varians sampel terhingga dan ralat kuasa dua min keatas penganggar kebolehjadian maksimum ini. Prosedur yang tepat untuk menguji hipotesis bagi taburan terubahsuai ini adalah rumit dan sukar. Oleh itu tiga statistik piawai sampel besar berdasarkan penganggar kebolehjadian maksimum telah dipertimbangkan iaitu ujian nisbah kebolehjadian, ujian Wald dan statistik Rao. Kemampuan tiga ujian ini diselidiki dan dibanding berdasarkan kuasa dan saiz ujian. Selang keyakinan berlandaskan nisbah kebolehjadian, ujian Wald dan statistic Rao

dikaji seterusnya. Perlakuannya dari segi kemampuan mencapai ralat kebarangkalian nominal dan memperoleh kebarangkalian yang simetri di sebelah bawah dan atas diselidiki dan dibanding.

Penemuan utama daripada kajian simulasi keatas prosedur pentakbiran bagi parameter bertaburan ekstrem terubahsuai menunjukkan penganggar parameter bentuk adalah saksama manakala penganggar parameter lokasi dan skala sedikit pincang pada sampel saiz kecil yang bertaburan univariat. Bagi model bivariat prestasi penganggar skala dan bentuk adalah memuaskan terhadap kepincangan dan varians bagi semua keadaan.

Dalam menguji hipotesis taburan terubahsuai ini, perlakuan statistik kebolehjadian nisbah adalah lebih baik berbanding ujian Wald dan Rao. Anggaran selang bagi parameter skala berlandaskan statistik Wald dan Rao adalah simetri dan bersifat antikonservatif secara tidak keterlaluan manakala selang berlandaskan statistik nisbah kebolehjadian secara amnya adalah simetri dan mencapai ralat kebarangkalian yang nominal. Bagi parameter bentuk, kesemua selang menghampiri simetri disebelah bawah dan atas ralat kebarangkalian.

Keputusan daripada kajian simulasi juga menunjukkan model nilai ekstrem terubahsuai boleh menyumbang secara berkesan dalam menyelesaikan masalah yang melibatkan data persekitaran terutamanya data pencemaran udara.



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