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SPECTRAL EXPANSIONS OF LAPLACE-BELTRAMI OPERATOR ON UNIT SPHERE

AHMAD FADLY NURULLAH BIN RASEDEE

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By

AHMAD FADLY NURULLAH BIN RASEDEE

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Doctor of Philosophy

July 2015

DEDICATIONS

I would like to dedicate this thesis to my loving family.
To my parents, Mum and Dad for their support and guidance.

To my LOVELY Wife, Hawa
For her Patience, Love and Support.

And my children, Faresyah, Hana
And also baby Firas,
I Hope this INSPIRES you to work hard in order to achieve your dreams and a better future.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

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Chair: Anvarjon Ahmedov, PhD
Faculty: Institute for Mathematical Research (INSPEM)

The reconstruction of functions from its expansions is a prominent problem in harmonic analysis. These type of problems are not always solvable with the definition of the sum of a Fourier series as the limit of its partial sums. Functions that are not very smooth (such functions are the most interesting and from a pratical point of view have important expansions), have successive strongly oscillating terms of the partial sums which does not correspond with the characterization of the to be reconstructed function. This causes the sequence of partial sums to oscillate around the function rather than approach it. When attempting to overcome this issue, it is interesting to consider some arithmetic means. For purpose of this research, the Riesz means is taken into consideration. Due to the consistent behavior of these oscillations, the Riesz means acts as a regularization of the partial sums to better approximate the function.

In the present research we investigated convergence and summability problems of the spectral expansions of differential operators. The most specific properties of the decompositions are established. The investigations of spectral expansions of the differential operator in modern methods of harmonic analysis, incorporates the wide use of methods from functional analysis, modern operator theory and spectral decomposition. New methods for approximating the functions from different spaces (Nikolskii, Sobolev, Liouville) are constructed using asymptotic behavior of the spectral function of the differential operators.

We consider the summation, conditions and principles for the localization of Riesz means of the Fourier-Laplace series of distributions. When working in the field of spectral theory of differential operators, specifically on localization of spectral decompositions by Riesz means, it is important to consider how the dependence on convergence of spectral decompositions of their Riesz means affects the behavior of the function in a small neighborhood of the given point.

This research focuses on convergence and summability problems of the spectral expansions of differential operators related to the Fourier-Laplace series. The Fourier-Laplace series can be interpreted as eigenfunction expansions of the Laplace-Beltrami operator, which is a symmetric and nonnegative elliptic operator on the unit sphere. Application of Neuman's Theorem allows us to present the Laplace-Beltrami through partial sums of the Fourier-Laplace series, which is referred as spherical expansions related to the Laplace-Beltrami operator.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Doktor Falsafah

PENGEMBANGAN SPEKTRUM OPERATOR LAPLACE-BELTRAMI DI ATAS PERMUKAAN SFERA

Oleh

AHMAD FADLY NURULLAH BIN RASEDEE

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Pengerusi: Anvarjon Ahmedov, PhD
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Pembinaan semula fungsi dari pengembangan adalah satu masalah yang penting dalam analisis harmonik. Jenis masalah sebegini tidak selalunya boleh diselesaikan dengan mentakrif jumlah siri Fourier sebagai had kepada penjumlahan separa bagi siri tersebut. Fungsi-fungsi yang tidak begitu lancar (jenis fungsi tersebut adalah yang paling menarik dan dari sudut pandangan yang lebih praktikal mempunyai pengembangan penting) , mempunyai penjumlahan separa berturut yang kuat berayun jumlah separa yang tidak sepadan dengan pencirian fungsi untuk dibina semula. Ini menyebabkan urutan penjumlahan separa berayun di sekitar penyetempatan itu dan bukannya menghampirinya. Dalam usaha untuk mengatasi isu ini, ia adalah menarik untuk mempertimbangkan penggunaan purata aritmetik. Bagi tujuan kajian ini, purata Riesz diambil kira. Disebabkan ayunan yang konsisten, purata Riesz bertindak mengawal kelakuan penjumlahan separa itu.

Dalam kajian ini, kami telah mengkaji penumpuan dan penjumlahan masalah pengembangan spektrum operator pengkamiran. Ciri-ciri yang paling khusus penghuraian diwujudkan. Siasatan pengembangan spektrum pengendali pengkamiran dalam kaedah moden analisis harmonik, menggabungkan penggunaan pelbagai kaedah analisis fungsi, teori pengendali moden dan penguraian. Kaedah baru untuk menghampiri fungsi dari ruang yang berbeza (Nikolskii, Sobolev, Liouville) yang dibina menggunakan tingkah laku asimptot fungsi spektrum pengendali pengkamiran.

Kami mempertimbangkan penjumlahan, syarat dan prinsip-prinsip untuk penyetempatan menggunakan purata Riesz daripada siri Fourier-Laplace bagi fungsi pengagihan. Apabila bekerja dalam bidang teori spektrum operator pengkamiran, khususnya penyetempatan penghuraian spektrum oleh purata Riesz, ia adalah penting untuk mempertimbangkan bagaimana pergantungan kepada penumpuan penghuraian spektrum purata Riesz mereka mempengaruhi sikap fungsi itu di kawasan kejiranan yang kecil titik yang diberikan.

Siri Fourier-Laplace ini boleh ditafsirkan sebagai fungsi pengembangan eigen bagi pengendali Laplace-Beltrami, yang merupakan operator elips yang simetrik dan bukan negatif pada unit sfera. Aplikasi Teorem Neuman ini membolehkan kita untuk membentangkan Laplace-Beltrami melalui jumlah hasil tambah separa daripada siri Fourier-Laplace, yang disebut sebagai pengembangan sfera yang berkaitan dengan operator Laplace-Beltrami.

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I am blessed with parents like Rasedee and Fatimah. It is because of their sacrifices and dedication that I am here today.

Last yet far from the least, my family. My Loving wife Siti Hawa, through thick and thin shared my struggles. I couldn't have imagine a more caring and supportive better half than her. To my children, Faresyah Hani, Hana Fazeyah and baby Firas, its because of all of you I am inspired to be better person.

I certify that a Thesis Examination Committee has met on 2 July 2015 to conduct the final examination of Ahmad Fadly Nurullah bin Rasedee on his thesis entitled "Spectral Expansions of Laplace-Beltrami Operator on Unit Sphere" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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Declaration by graduate student

I hereby confirm that:

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This is to confirm that:

- the research conducted and the writing of this thesis was under our supervision;
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LIST OF ABBREVIATIONS

Δ	Laplace Operator
Δ_s	Laplace-Beltrami Operator
$\frac{\partial}{\partial x}$	Partial derivative
$Y_j^{(k)}$	Eigenfunctions
λ_k	Eigenvales
(C,k)	Cesaro Means of Order k
(R,k)	Riesz Means of Order k
S^N	N- dimensional sphere
$d\sigma(y)$	Elements on the surface of the sphere
R^{N+1}	$N+1$ - dimensional Euclidean space
S^N	N - dimensional unit sphere
PDEs	Partial Differential Equations
det	Determinant
div	Divergence
grad	Gradient
H_k	Class of all Spherical Harmonics of Degree k
dim	Dimension
$\bigoplus$	Direct Sum of the Set
C^∞	Class of infinitely times differentiable function
W_p^k	Sobolev Space
H_p^r	Nikolskii Space
L_p^k	Liouville Space
Γ	Gamma funtion
A_n^α	Binomial coefficients
$\mathscr{L}_n^\alpha$	Lesbegue Constant
$\mathrm{vol}(S^N)$	Volume of the N- dimensional sphere
mes	The Lebegue measure
$E_\lambda f(x)$	The partial sum of the spectral expansion
$E_n^\alpha f(x)$	The Riesz means of the spectral expansion of order α
$C_n^\alpha f(x)$	The Cesaro means of the spectral expansion of order α
$\Theta^\alpha(x,y,n)$	The kernel of Riesz means of the spectral function of order α
$\Phi^\alpha(x,y,n)$	The kernel of Cesaro means of the spectral function of order α

CHAPTER 1

INTRODUCTION

... a mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street.

-David Hilbert.

In this thesis we deal with convergence and summability problems pertaining to eigenfunction expansions of the Laplace-Beltrami operator on the unit sphere. The Laplace-Beltrami operator is an unbounded differential operator, with many applications in mathematics and physics. Symmetricity and positiveness of the Laplace-Beltrami operator in classes of infinitely times differentiable functions enable us to represent the latter operator by the family of projection operators using spectral theorem. Projection operators are partial sums of the Fourier series in eigenfunctions of the Laplace-Beltrami operator and they play an essential role in the reconstruction of the function from its eigenfunction expansions. When the partial sums are not convergent, the Riesz means of the partial sum will be considered to solve reconstruction problems for the functions from the classes of Sobolev, Nikolskii, Liouville and etc. This chapter will focus on main fundamental ideas and concepts from the spectral theory of differential operators to understand modern status of the investigations in the field of convergence and summability of eigenfunction expansions related to differential operators. Besides background on our subject matter, this chapter will entail descriptions on our research objectives, motivation and a brief outline of this thesis.

1.1 Motivation

Problems in the spectral theory of differential operators arises when attempting to reconsrtuct the function from its eigenfunction expansions. Such occurrence are noticeable under conditions when the partial sums of the eigenfunction expansions are not convergent. Motivated by these convergence issues, we employ solutions with the use of the Laplace-Beltrami operator by arithmetic means. The proposed means in this thesis is by method of Riesz means which is an improvement over the Cesaro means. Investigations conducted are mainly on convergence issues of eigenfunction expansions for the Fourier-Laplace series on the sphere.

Because the mathematical description of the physical processes taking place in real space are based on the spectral theory of differential operators, the most difficult engineering problems concerning heat and mass transfer processes can be modelled by Partial Differential Equations (PDEs), particularly by Laplace-Beltrami operator on the sphere. The eigenfunctions of the Laplace-Beltrami operator on unit sphere which are commonlly known as spherical harmonics have huge applications in science and engineering, varying from the basic vibrations of the drum to the complex solution of 3D image reconstruction and the propagation of electromagnetic waves.

1.2 Spherical Harmonics

Spherical harmonics which are eigenfunctions of the Laplace-Beltrami operator on unit sphere have huge applications in science and engineering. We recall here definitions and fundamental facts from the general theory of spherical harmonics.

Let denote by S^N the N-dimensional unit sphere:

$$S^N = \{(x_1, x_2, \ldots, x_{N+1}) \in R^{N+1} : \sum_{i=1}^{N+1} x_i^2 = 1\}.$$

In order to define the Laplace-Beltrami operator on the sphere S^N and establish its relationship with the Laplace operator on R^{N+1} we need to define the Laplace operator on the Riemannian manifolds (M,g). It can be shown that a Riemannian metric, g denotes a smooth family of inner products, $g = (g_p)$, on a manifold M, where for any $p \in M$, g_p is an inner product on the tangent space, T_pM. We are able to establish a distinct duality between the space of T_pM and T_p^*M, due to g_p. Specifically, we define the isomorphism, $\flat : T_pM \to T_p^*M$ such that for every $u \in T_pM$, the linear form, $u^\flat \in T_p^*M$, is given by,

$$u^\flat(v) = g_p(u,v), \quad v \in T_pM.$$

and the inverse isomorphism, $\# : T_p^*M \to T_pM$ such that for each $\omega \in T_p^*M$, the vector, $\omega^\#$, is the unique vector in T_pM hence

$$g_p(\omega^\#, v) = \omega(v), \quad v \in T_pM.$$

The isomorphism $\flat$ and $\#$ induce isomorphisms between vector fields, $X \in \chi(M)$, and one-forms, $\omega \in A^1(M)$. Particulalrly, the vector field corresponding to the one-form, df, is the *gradient*, grad f, of f for every smooth function, $f \in C^\infty(M)$. The gradient of f is uniquely determined by the following condition

$$g_p((\text{grad} f)_p, v) = df_p(v), \quad v \in T_pM, \ \ p \in M.$$

If ∇_X is the derivatives along the tangent vector or also known as the covariant derivatives in colleration with the Levi-Civita connection by virtue of g, then the *divergence* of a vector field, $X \in \chi(M)$, is the function, div $X : M \to R$ defined so that

$$(\text{div}\, X)(p) = \text{tr}(Y(p) \mapsto (\nabla_Y X)_p);$$

where $(\text{div} X)(p)$ is the trace of the linear map, $Y(p) \mapsto (\nabla_Y X)_p$ for every p. Hence the Laplace-Beltrami operator, known also as the Laplacian, is the linear operator, $\Delta : C^\infty(M) \to C^\infty(M)$, denoted by

$$\Delta f = \text{div} \ \ \text{grad}\, f.$$

Let (U,φ) be a chart of M where $p \in M$ and

$$\Delta f = \text{div } \text{ grad} f.$$

If (U,φ) maps M, with $p \in M$ and if the basis of T_pM induced by φ is denoted by

$$\left(\left(\frac{\partial}{\partial x_1}\right)_p,\ldots,\left(\frac{\partial}{\partial x_n}\right)_p\right),$$

then the $n \times n$ matrix, $(g_{ij})_p$ given by

$$(g_{ij})_p = g_p\left(\left(\frac{\partial}{\partial x_i}\right)_p,\ldots,\left(\frac{\partial}{\partial x_j}\right)_p\right),$$

denotes the local expression of the metric g at p. The matrix $(g_{ij})_p$ is postive definite, symmectric and has an inverse represented as $(g^{ij})_p$ where we consider $|g|_p = \det(g_{ij})_p$. We will oftenly ommit the subscript p to simplify the notation. This allows us to show that every function, $f \in C^\infty(M)$ in the form of local coordinates which is given by the chart (U,φ) as

$$\text{grad} f = \sum_{ij} g^{ij}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i},$$

where

$$\frac{\partial f}{\partial x_j}(p) = \left(\frac{\partial}{\partial x_j}\right)_p f = \frac{\partial(f\circ\varphi^{-1})}{\partial u_j}(\varphi(p)),$$

with $(u_1,\ldots,u_n)$ as the coordinate functions in R^n. For every vector field, $X \in \chi X(M)$, expressed in local coordinates as

$$X = \sum_{i=1}^{n} X_i\frac{\partial}{\partial x_i},$$

we have

$$\text{div}\, X = \frac{1}{\sqrt{|g|}}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}X_i\right)$$

and for every function, $f \in C^\infty(M)$, the Laplacian, δf, is given by

$$\Delta f = \frac{1}{\sqrt{|g|}}\sum_{i,j}\frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial f}{\partial x_j}\right).$$

If we consider N-dimensional Eucliedean space R^N with standard coordinates, then the Laplacian Δf will have the following form

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_N^2}.$$

Here are a few examples.
Example 1: As an application we will derive the three dimensional Laplacian in spherical coordinates. We begin by letting,

$$x = r\sin\theta\cos\varphi,$$
$$y = r\sin\theta\sin\varphi,$$
$$z = r\cos\theta.$$

By differentiating them respectively give us,

$$\frac{\partial}{\partial r} = \sin\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\sin\varphi + \cos\theta\frac{\partial}{\partial z} = \hat{r},$$
$$\frac{\partial}{\partial\theta} = r\left(\cos\theta\cos\varphi\frac{\partial}{\partial x} + \cos\theta\sin\varphi - \sin\theta\frac{\partial}{\partial z}\right) = r\hat{\theta},$$
$$\frac{\partial}{\partial\varphi} = r\left(-\sin\theta\sin\varphi\frac{\partial}{\partial x} + \sin\theta\cos\varphi\right) = r\hat{\varphi}.$$

It becomes apparent that $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ are pairwise orthogonal. Hence the matrix (g_{ij}) is denoted by

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix},$$

where $|g| = r^4\sin^2\theta$. The inverse of (g^{ij}) is represented by

$$(g^{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}\theta \end{bmatrix}.$$

This calculation shows that

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta\,\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}.$$

Because (θ, φ) are coordinates on the sphere S^2 through

$$x = \sin\theta\cos\varphi,$$
$$y = \sin\theta\sin\varphi,$$
$$z = \cos\theta,$$

these coordinates, the metric, $(\tilde{g}_{ij})$, on S^2 is given by the matrix

$$(\tilde{g}_{ij}) = \begin{bmatrix} ,1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

where it is visble that $|\tilde{g}| = \sin^2\theta$, with the inverse

$$\left(\tilde{g}^{ij}\right) = \begin{bmatrix} 1 & 0 \\ 0 & \sin^{-2}\theta \end{bmatrix}.$$

It follows that

$$\Delta_{S^2} f = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2},$$

which validates

$$\Delta_f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^2} f.$$

We now proceed to establish in similar manner, the genearalized formula for the Laplacian, Δ on R^{N+1} and the Laplacian Δ_{S^N} on S^N where

$$S^N = \left\{(x_1,\ldots,x_{N+1}) \in R^{N+1} : x_1^2 + \cdots + x_{N+1}^2 = 1\right\}.$$

By implementing polar coordinates and mapping $R_+ \times S^N$ to $R^{N+1} - \{0\}$ given by

$$(r,\sigma)r \mapsto r\sigma,$$

which is diffeomorphism (defined one to one different values). This verifies that for any system of coordinates, $(u_1,\ldots u_n)$, on S^N, the tuple $(u_1,\ldots,u_N,r)$ is a system of coordinates on $R^{N+1} - \{0\}$ called polar coordinate. We establish the relationship between the Laplacian, Δ on $R^{N+1} - \{0\}$ and the Laplacian, Δ_{S^N} on S^N in local coordinates $(u_1,\ldots u_n)$.

We prove that Laplacian Δf on $R^{N+1} - \{0\}$ and Laplacian Δ_{S^N} on the sphere S^N are related by

$$\Delta f = \frac{1}{r^N}\frac{\partial}{\partial r}\left(r^N\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^N} f. \tag{1.1}$$

To prove the formula (1.1) we compute $(N+1)\times(N+1)$ matrix, $G = (g_{ij})$, which expresses the metric on R^{N+1} is polar coordinates and the $N\times N$ metric, $\tilde{G} = \left(\widetilde{g}_{ij}\right)$which expresses the metric on S^N. Let $\sigma \in S^N$ then $\sigma\cdot\sigma = 1$, hence

$$\frac{\partial\sigma}{\partial u_i}\cdot\sigma = 0,$$

as

$$\frac{\partial\sigma}{\partial u_i}\cdot\sigma = \frac{1}{2}\frac{\partial(\sigma\cdot\sigma)}{\partial u_i} = 0.$$

If $x = r\sigma$ with $\sigma \in S^N$, then we have

$$\frac{\partial x}{\partial u_i} = r\frac{\partial\sigma}{\partial u_i}, \quad 1 \le i \le N,$$

and

$$\frac{\partial x}{\partial r} = \sigma,$$

which follows

$$\begin{aligned} g_{ij} &= \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} = r^2 \frac{\partial \sigma}{\partial u_i} \cdot \frac{\partial \sigma}{\partial u_j} = r^2 \widetilde{g}_{ij}, \\ g_{i(N+1)} &= \frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial r} = r \frac{\partial \sigma}{\partial u_i} \cdot \sigma = 0, \\ g_{i(N+1N+1)} &= \frac{\partial x}{\partial r} \cdot \frac{\partial x}{\partial r} = \sigma \cdot \sigma = 1. \end{aligned}$$

Subsequently, we obtain

$$G = \begin{bmatrix} r^2 \widetilde{G} & 0 \\ 0 & 1 \end{bmatrix},$$

where $|g| = r^{2n} |\widetilde{g}|$ and

$$G^{-1} = \begin{bmatrix} r^{-2} \widetilde{G}^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

Implementing the equations above and

$$\Delta f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left(\sqrt{|g|} g^{ij} \frac{\partial f}{\partial x_j} \right),$$

give us

$$\begin{aligned} \Delta &= \frac{1}{r^N \sqrt{|\widetilde{g}|}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(\sqrt{|\widetilde{g}|} \frac{1}{r^2} \widetilde{g}^{ij} \frac{\partial f}{\partial x_j} \right) + \frac{1}{r^N \sqrt{|\widetilde{g}|}} \frac{\partial}{\partial r} \left(r^N \sqrt{|\widetilde{g}|} \frac{\partial f}{\partial r} \right), \\ &= \frac{1}{r^N \sqrt{|\widetilde{g}|}} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(\sqrt{|\widetilde{g}|} \widetilde{g}^{ij} \frac{\partial f}{\partial x_j} \right) + \frac{1}{r^N} \frac{\partial}{\partial r} \left(r^N \frac{\partial f}{\partial r} \right), \\ &= \frac{1}{r^2} \Delta_{S^N} f + \frac{1}{r^N} \frac{\partial}{\partial r} \left(r^N \frac{\partial f}{\partial r} \right), \end{aligned}$$

hence obtaining the Laplacian for S^N.

$$\mathrm{vol}(S^N) = \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)},$$

indicates the volume of the sphere S^N. The function $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$ for any α greater than 0 is commonly known as the gamma function, or also as Euler integral of the 2nd kind. We denote the spherical surface S^N by $D(x;h)$, where $D(x;h) = \{y : y \in S^N,\ (x,y) > \cos h,\ 0 < h \leq \pi\}$; with the center x and radius h.

We then have mes$(D(x;h))$, the area of its surface;

$$\begin{aligned}\operatorname{mes}(D(x;h)) &= \operatorname{vol}(S^N)\int_0^h (\sin\vartheta)^{N-1}\,d\vartheta,\\ &= \frac{2\pi^{\frac{N+1}{2}}}{\Gamma\left(\frac{N+1}{2}\right)}\int_0^h (\sin\vartheta)^{N-1}\,d\vartheta.\end{aligned}$$

We then represent the angle between the radius by γ, drawn from the center of the spherical surface to points $x=(\vartheta_1,\vartheta_2,\dots,\vartheta_{N-1},\varphi)\in S^N$ and $y=(\vartheta_1^{'},\vartheta_2^{'},\dots,\vartheta_{N-1}^{'},\varphi^{'})\in S^N$. It becomes evident that

$$\begin{aligned}(x,y)=\cos\gamma=\cos\vartheta_1\cos\vartheta_1^{'}&+\sin\vartheta_1\cos\vartheta_2\sin\vartheta_2^{'}\cos\vartheta_2^{'}\dots\\ &+\sin\vartheta_1\sin\vartheta_2\dots\sin\vartheta_{N-1}\cos\varphi\sin\vartheta_1^{'}\sin\vartheta_2^{'}\dots\sin\vartheta_{N-1}^{'}\cos\varphi^{'}\\ &+\sin\vartheta_1\sin\vartheta_2\dots\sin\vartheta_{N-1}\sin\varphi\sin\vartheta_1^{'}\sin\vartheta_2^{'}\dots\sin\vartheta_{N-1}^{'}\sin\varphi^{'}.\end{aligned}$$

If ϑ_1, coincides with the pole, we can deduce that $\cos\gamma=\cos\vartheta_1^{'}$, hence $\gamma=\vartheta_1^{'}$. Therefore, if the coordinate system is converted so that the pole coincides with x in this case $\vartheta_1^{'}=\gamma$.
Let define $L_p\left(S^N\right)$, $1\le p\le\infty$, $\left(L_1\left(S^N\right)=L\left(S^N\right)\right)$ the space of functions $f(x)$ with the norm

$$\|f\|_{L_p(S^N)}=\left(\int_{S^N}|f(x)|^p\,d\sigma(x)\right)^{\frac{1}{p}},\quad 1\le p<\infty.$$

$$d\sigma=\sin^{N-1}\vartheta_1\sin^{N-2}\vartheta_2\dots\sin\vartheta_{N-1}\,d\vartheta_1 d\vartheta_2\dots d\vartheta_{N-1}d\varphi.$$

In the case of $p=\infty$, the space $L_\infty\left(S^N\right)=C\left(S^N\right)$ consists of continuous functions with the norm

$$\|f\|_{L_\infty(S^N)}=\|f\|_{C(S^N)}=\max_{x\in S^N}|f(x)|,$$

where $d\sigma$ represents an element of the surface area S^N: The Cesaro means of order α can be written as follow

$$\begin{aligned}A_\nu^\alpha &= \binom{\nu+\alpha}{\nu},\\ &= \binom{\nu+\alpha}{\alpha},\\ &= \frac{1}{\nu}(\alpha+1)(\alpha+2)\cdots(\alpha+\nu),\\ &\approx \frac{\nu^\alpha}{\Gamma(\alpha+1)},\quad \alpha\ne-1,-2,\dots\quad \text{nonnegative } \alpha\end{aligned}$$

We denote The scalar product of the squared integrable functions f and g on sphere by

$$(f,g) = \int_{S^N} f(x)g(x)\,d\sigma(x).$$

For the function $f(x)$, given on the sphere S^N the spherical shift $S_h(f;x)$ with increment h is defined by

$$S_h(f;x) = \frac{1}{\mathrm{vol}(S^{N-1})(\sin h)^{N-1}} \int_{(x,y)=\cos\, h} f(y)\,dt(y).$$

Lemma 1.2.1 *If we denote the mean value of the function $f \in L\left(S^N\right)$ by*

$$M_h(f;x) = \frac{1}{mes(D(x;h))} \int_{D(x;h)} f(y)\,d\sigma(y) \quad h > 0,$$

then the following assertions are valid:

i.

$$|M_h(f;x)| \le \sup_{0<\gamma\le h} |S_\gamma(f;x)|. \tag{1.2}$$

ii. When $f \in L_p\left(S^N\right)$, $p \ge 1$ we have

$$\|S_h(f;x)\|_{L_p(S^N)} \le \|f(x)\|_{L_p(S^N)}. \tag{1.3}$$

Proof. i. Let fix $x \in S^N$. By definition of the mean value of the function over $D(x,h), h > 0$, we obtain:

$$\begin{aligned}
M_h(f;x) &= \frac{1}{\mathrm{mes}(D(x;h))} \int_{D(x;h)} f(y)\,d\sigma(y),\\
&= \frac{1}{\mathrm{mes}(D(x;h))} \int_0^h d\gamma \int_{(x,y)=\cos\gamma} f(y)\,dt(y),\\
&= \frac{\mathrm{vol}(S^{N-1})}{\mathrm{mes}(D(x;h))} \int_0^h \left\{ \frac{1}{\mathrm{vol}(S^{N-1})(\sin\gamma)^{N-1}} \int_{(x,y)=\cos\gamma} f(y)\,dt(y) \right\} (\sin\gamma)^{N-1}\,d\gamma,\\
&= \frac{\mathrm{vol}(S^{N-1})}{\mathrm{mes}(D(x;h))} \int_0^h S_\gamma(f;x)(\sin\gamma)^{N-1}\,d\gamma.
\end{aligned} \tag{1.4}$$

Hence (1.2).

ii. Let the first $p = 1$. Then

$$|S_h(f;x)| \leq \frac{1}{\mathrm{vol}(S^N)(\sin h)^{N-1}} \int\limits_{(x,y)=\cos\gamma} |f(y)|\, dt(y),$$

hence

$$\int\limits_{S^N} |S_h(f;x)|\, dS(x) \leq \frac{1}{\mathrm{vol}(S^{N-1})\sin^{N-1} h} \int\limits_{S^N} d\sigma(x) \int\limits_{(x,y)=\cos\gamma} f(\vartheta_1', \vartheta_2', \ldots, \vartheta_{N-1}', \varphi')\, dt(y). \tag{1.5}$$

If we take $x(\vartheta_1, \vartheta_2, \ldots, \vartheta_{N-1}, \varphi)$ as the initial point and introduce new spherical coordinates $(\gamma, \overline{\vartheta}_2, \overline{\vartheta}_3, \ldots, \overline{\vartheta}_{N-1}, \overline{\varphi})$ then $\gamma = \vartheta_1'$. We have the function $f(y) = f(\vartheta_1', \vartheta_2', \ldots, \vartheta_{N-1}', \varphi')$ in the new coordinate system is a function of $F_x(\gamma, \overline{\vartheta}_2, \overline{\vartheta}_3, \ldots, \overline{\vartheta}_{N-1}, \overline{\varphi}) = F_x(\gamma, \overline{y})$, which are the new coordinates, where $\overline{y}(\overline{\vartheta}_2, \overline{\vartheta}_3, \ldots, \overline{\vartheta}_{N-1}, \overline{\varphi})$ is the point of the unit sphere in the space R^N. Given that $dS^N(y) = \sin^{N-1}\gamma\, dS^{N-1}(\bar{y})$, (1.5), we obtain

$$\begin{aligned}
\int\limits_{S^N} |S_\gamma(f;x)|\, dS(x) &\leq \frac{1}{\mathrm{vol}(S^{N-1})\sin^{N-1}\gamma} \int\limits_{S^N} dS^N(x) \int\limits_{S^{N-1}} |F_x(\gamma,\overline{y})| \sin^{N-1}\gamma\, dS^{N-1}(\bar{y}),\\
&= \frac{1}{\mathrm{vol}(S^{N-1})} \int\limits_{S^{N-1}} dS^{N-1}(\bar{y}) \int\limits_{S^N} |F_x(\gamma,\overline{y})|\, dS^N(x),\\
&= \frac{1}{\mathrm{vol}(S^{N-1})} \int\limits_{S^{N-1}} dS^{N-1}(\bar{y}) \int\limits_{S^N} |f(x)|\, dS^N(x),\\
&= \int\limits_{S^{N-1}} |f(x)|\, dS(x),
\end{aligned} \tag{1.6}$$

this concludes the proof of Lemma 1.2.1.

Now assume that $p > 1$. Applying Holder's inequality, we obtain

$$\begin{aligned}
|S_h(f;x)|^p &= \left\{ \frac{1}{\mathrm{vol}(S^{N-1})\sin^{N-1} h} \left| \int\limits_{(x,y)=\cos h} f(y)\, dt(y) \right| \right\}^p,\\
&\leq \left(\frac{1}{\mathrm{vol}(S^{N-1})\sin^{N-1} h} \right)^p \left(\int\limits_{(x,y)=\cos h} dt(y) \right)^{\frac{q}{p}} \int\limits_{(x,y)=\cos h} |f(y)|^p\, dt(y),\\
&= \left(\frac{1}{\mathrm{vol}((S^{N-1})\sin^{N-1} h} \right)^p \left(\mathrm{vol}(S^{N-1})\sin^{N-1} h \right)^{p-1} \int\limits_{(x,y)=\cos h} |f(y)|^p\, dt(y),\\
&= \frac{1}{\mathrm{vol}(S^{N-1})\sin^{N-1} h} \int\limits_{(x,y)=\cos h} |f(y)|^p\, dt(y).
\end{aligned} \tag{1.7}$$

Denoting it by $S_h^*(f;x)$, we rewrite inequality (1.7) as

$$|S_h(f;x)|^p \leq S_h^*(f;x).$$

This inequality and (1.6) gives

$$\begin{aligned}\int_{S^N} |S_h(f;x)|^p dS(x) &\leq \int_{S^N} S_h^*(f;x) dS(x), \\ &\leq \int_{S^N} |f(x)|^p dS(x).\end{aligned}$$

Let $f \in L_p\left(S^N\right)$, for any $p \geq 1$ then

$$\|M_h(f;x)\|_{L_p(S^N)} \leq \|f\|_{L_p(S^N)}. \tag{1.8}$$

From (1.4), the generalized Minkowski inequality, we have

$$\begin{aligned}\|M_h(f;x)\|_{L_p(S^N)} &= \frac{\mathrm{vol}(S^{N-1})}{\mathrm{mes}(D(x;h))}\left\{\int_{S^N}\left|\int_0^h S_\gamma(f;x)(\sin\gamma)^{N-1}\,d\gamma\right|^p dS(x)\right\}^{\frac{1}{p}} \\ &\leq \frac{\mathrm{vol}(S^{N-1})}{\mathrm{mes}(D(x;h))}\int_{S^N}\left\{\int_0^h \left|S_\gamma(f;x)\right|^p dS(x)\right\}^{\frac{1}{p}}(\sin\gamma)^{N-1}\,d\gamma \\ &\leq \frac{\mathrm{vol}(S^{N-1})\|f\|_{L_p(S^N)}}{\mathrm{mes}(D(x;h))}\int_0^h (\sin\gamma)^{N-1} d\gamma \\ &= \|f\|_{L_p(S^N)}.\end{aligned}$$

Inequality (1.8) is proven.

We set

$$\begin{aligned}\widetilde{f}(x) &= \sup_{0,\gamma\leq\pi}\frac{1}{\mathrm{vol}(S^{N-1})(\sin\gamma)^{N-1}}\int_{(x,y)=\cos\gamma}|f(y)|\,dt(y), \\ f^*(x) &= \sup_{0,\gamma\leq\pi}\frac{1}{\mathrm{mes}(D(x;h))}\int_{D(x;\gamma)}|f(y)|\,dS(y).\end{aligned}$$

It is easy to check that

$$f^* \leq \widetilde{f}(x). \tag{1.9}$$

The modulus of continuity in $L_p\left(S^N\right)$ in terms of spherical shift $S_h f(x)$ and mean value $M_h f(x)$ are defined as follows

$$\omega_p(f;\delta) = \omega(f;\delta)_{L_p(S^N)} = \sup_{0<h\le\pi} \|f(x) - S_h f(x)\|_{L_p(S^N)},$$
$$\omega_p^*(f;\delta) = \omega^*(f;\delta)_{L_p(S^N)} = \sup_{0<h\le\pi} \|f(x) - M_h f(x)\|_{L_p(S^N)},$$

and if $f \in C\left(S^N\right)$, then the equality

$$\omega_p f(\delta) = \sup_{0<h\le\delta} \|f(x) - S_h f(x)\|_{C(S^N)},$$
$$\omega_p^* f(\delta) = \sup_{0<h\le\delta} \|f(x) - M_h f(x)\|_{C(S^N)}.$$

If $\omega_p(f;\delta) \le M\left(\sin\frac{\delta}{2}\right)^\alpha \left(\omega_p^* f(\delta) \le M\left(\sin\frac{\delta}{2}\right)^\alpha\right)$, $0 < \alpha \le 2$, then $f \in Lip(\alpha,p)_{S^N}$ $\times(f \in Lip^*(\alpha,p)_{S^N})$ and it can be said that $f(x)$ satisfies the Lipschitz condition (generalized Lipschitz condition) in the metric $L_p\left(S^N\right)$. It is apparent that $Lip(\alpha,p)_{S^N} \subset Lip^*(\alpha,p)_{S^N}$ has the following conditions

I. If $\lim_{\delta\to 0} \frac{\omega_p^* f(\delta)}{\delta^2} = 0$ then $f(x) = const$ almost everywhere on S^N for any $1 \le p \le \infty$, $\left(L_\infty(S^N) = C(S^N)\right)$

$$\omega^* f(\delta)_{L_p(S^N)} \le \omega f(\delta)_{L_p(S^N)}, \tag{1.10}$$

where

$$\left| f(x) \frac{1}{\operatorname{mes}(D(x;h))} \int_{D(x;h)} f(y)\, d\sigma(y) \right|,$$
$$= \frac{1}{\operatorname{mes}(D(x;h))} \left| \int_{D(x;h)} [f(x) - f(y)]\, d\sigma(y) \right|,$$
$$= \frac{1}{\operatorname{mes}(D(x;h))} \left| \int_0^h d\gamma \int_{(x,y)=\cos\gamma} [f(x) - f(y)] dt(y) \right|,$$
$$= \frac{\operatorname{vol}(S^{N-1})}{\operatorname{mes}(D(x;h))} \left| \int_0^h (\sin\gamma)^{N-1}\, d\gamma \frac{1}{\operatorname{vol}(S^{N-1})(\sin\gamma)^{N-1}} \int_{(x,y)=\sin\gamma} [f(x) - f(y)]\, dt(y) \right|,$$
$$\le \frac{\operatorname{vol}(S^{N-1})}{\operatorname{mes}(D(x;h))} \int_0^h \left| f(x) - \frac{1}{\operatorname{vol}(S^{N-1})(\sin\gamma)^{N-1}} \int_{(x,y)=\sin\gamma} f(y)\, dt(y) \right| (\sin\gamma)^{N-1}\, d\gamma.$$

Hence, by the generalized Holder inequality, we have

$$\left\| f(x) - \frac{1}{\operatorname{mes}(D(x;h))} \int_{D(x;h)} f(y)\, d\sigma(y) \right\|_{L_p(S^N)}$$

$$= \frac{\operatorname{vol}(S^{N-1})}{\operatorname{mes}(D(x;h))} \left\{ \int_{S^N} \left[\int_0^h \left| f(x) - \frac{1}{\operatorname{vol}(S^{N-1}) \sin^{N-1}\gamma} \right.\right.\right.$$

$$\left.\left.\left. \times \int_{(x,y)=\cos\gamma} f(y)\, dt(y) \right| (\sin\gamma)^{N-1}\, d\gamma \right]^p d\sigma(x) \right\}^{\frac{1}{p}},$$

$$\leq \frac{\operatorname{vol}(S^{N-1})}{\operatorname{mes}(D(x;h))} \int_0^h \left\{ \int_{S^N} \left| f(x) - \frac{1}{\operatorname{vol}(S^{N-1}) \sin^{N-1}\gamma} \right.\right.$$

$$\left.\left. \times \int_{(x,y)=\cos\gamma} f(y)\, dt(y) \right|^p d\sigma(x) \right\}^{\frac{1}{p}} (\sin\gamma)^{N-1}\, d\gamma,$$

$$\leq \frac{\operatorname{vol}(S^{N-1})\, \omega(f;h)_{L_p(S^N)}}{\operatorname{mes}(D(x;h))} \int_0^h (\sin\gamma)^{N-1}\, d\gamma,$$

$$= \omega(f;h)_{L_p(S^N)}.$$

This implies the validity of (1.10).

II. If $f \in L_p\left(S^N\right)$, $p \geq 1$ is

$$\lim_{\delta \to 0} \omega(f;\delta)_{L_p(S^N)} = 0. \tag{1.11}$$

Let $f \in L_p\left(S^N\right)$, $p \geq 1$. For a given $\varepsilon > 0$ there exists a continuous function g then

$$\|f - g\|_{L_p(S^N)} < \varepsilon,$$

therefore

$$\begin{aligned} \|f(x) - S_h f(x)\|_{L_p(S^N)} &\leq \|f(x) - g(x)\|_{L_p(S^N)} + \|g(x) - S_h f(x)\|_{L_p(S^N)} \\ &\quad + \|S_h g(x) - S_h f(x)\|_{L_p(S^N)}, \\ &\leq 2\|f(x) - g(x)\|_{L_p(S^N)} + \|g(x) - S_h g(x)\|_{L_p(S^N)}, \\ &< 2\varepsilon + \|g(x) - S_h g(x)\|_{L_p(S^N)}, \end{aligned} \tag{1.12}$$

by (1.3), we have

$$\|S_h g(x) - S_h f(x)\|_{L_p(S^N)} \leq \|f(x) - g(x)\|_{L_p(S^N)},$$

where as

$$\|g(x) - S_h g(x)\|_{L_p(S^N)} \le \text{vol}(S^N)\omega g(h).$$

Hence, from (1.12) have

$$\|f(x) - S_h f(x)\|_{L_p(S^N)} \le 3\varepsilon + \text{vol}(S^N)\omega g(h).$$

This validates (1.11). From (1.11), by (1.10), as a result we obtain

$$\lim_{\delta \to 0} \omega^* f(\delta)_{L_p(S^N)} = 0. \tag{1.13}$$

Next, we present the unit vector x by the coordinates $(\phi_1, \phi_2, ..., \phi_{N-1}, \phi_N)$ as shown in the following formulas:

$x_1 = \cos\phi_1,$

$x_2 = \sin\phi_1 \cos\phi_2,$

$x_3 = \sin\phi_1 \sin\phi_2 \cos\phi_3,$

..............,

$x_{N-1} = \sin\phi_1 \sin\phi_2 ... \sin\phi_{N-2} \cos\phi_{N-1},$

$x_N = \sin\phi_1 \sin\phi_2 ... \sin\phi_{N-2} \sin\phi_{N-1} \cos\phi_N,$

$x_{N+1} = \sin\phi_1 \sin\phi_2 ... \sin\phi_{N-2} \sin\phi_{N-1} \sin\phi_N.$

The Laplace Beltrami Δ_s will have the following representation

$$\Delta_s = \frac{1}{\sin^{N-1}\phi_1} \frac{\partial}{\partial\phi_1} \sin^{N-1}\phi_1 \frac{\partial}{\partial\phi_1} + \frac{1}{\sin^2\phi_1} \frac{1}{\sin^{N-2}\phi_2} \frac{\partial}{\partial\phi_2} \sin^{N-2}\phi_2 \frac{\partial}{\partial\phi_2} + \dots$$
$$+ \frac{1}{\sin^2\phi_1 \sin^2\phi_2 ... \sin^2\phi_{N-1}} \frac{\partial^2}{\partial{\phi_N}^2}.$$

For any given $n \ge 1$, we define $C^n(S^N)$ as the class of functions $f(x)$ defined as n times differentiable on S^N and with $D^\alpha f(x), |\alpha| = n$ is continuous on S^N.

Lemma 1.2.2 *The Laplace-Beltrami operator Δ_s is symmetric in $C^2(S^N)$:*

$$\int_{S^N} (u(\omega)\Delta_s v(\omega) - v(\omega)\Delta_s u(\omega))\, d\sigma = 0,$$

for all $u, v \in C^2(S^N)$.

Proof. Consider the functions $u(\omega) = u(x/|x|)$ and $v(\omega) = v(x/|x|)$ which are defined on the circular ring with a radius $a \le r \le b$. By applying Green's formula we obtain:

$$\int_{a\le r\le b}(u\Delta v - v\Delta u)dx = \int_{r=a}(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r})d\sigma - \int_{r=b}(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r})d\sigma = 0,$$

where $\partial u/\partial r = \partial v/\partial r = 0$.

Consdering that $\Delta u = r^{-2}\Delta_s u$ and $\Delta v = r^{-2}\Delta_s v$ we are able to show

$$0 = \int_{a\le r\le b}(u\Delta v - v\Delta u)dx = \int_a^b \int_{S^N} r^{N-2}(u(\omega)\Delta_s v(\omega) - v(\omega)\Delta_s u(\omega))drd\omega,$$

hence completing the proof of the Lemma 1.2.2.

Now, consider the homogeneous polynomial of degree k:

$$p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha,$$

where α denotes an $N+1$-tuple $(\alpha_1, \alpha_2, ..., \alpha_{N+1})$ of nonnegative integers,

$$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_{N+1}$$

and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x^{\alpha_{N+1}}.$$

Let P_k denote the class of all homogenous polynomials of degree k. We then introduce the inner product $\langle p, q\rangle$ on P_k by taking $\langle p, q\rangle = p(D)\overline{q}$ for all $p, q \in P_k$, where $p(D)$ is defined as follows:

$$p(D) = \sum_{|\alpha|=k} c_\alpha D^\alpha,$$

here we denote

$$D^\alpha = \frac{\partial^{\alpha_1+\alpha_2+...+\alpha_{N+1}}}{\partial x_1^{\alpha_1} x_2^{\alpha_2} ... x_{N+1}^{\alpha_{N+1}}}.$$

We then proceed to examine whether the defined operation satisfies all conditions of the inner product. We consider $p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha$ and $q(x) = \sum_{|\alpha|=k} d_\alpha x^\alpha$.

1. Conjugate symmetry:

$$\langle p, q\rangle = p(D)\overline{q} = \sum_{|\alpha|=k} \alpha! c_\alpha \overline{d_\alpha} = \overline{\sum_{|\alpha|=k} \alpha! c_\alpha \overline{d_\alpha}} = \overline{q(D)\overline{p}} = \overline{\langle q, p\rangle}.$$

2. Linearity in the first argument:

$$\langle ap, q\rangle = ap(D)\overline{q} = a\langle p, q\rangle.$$

$$\langle p+q, r\rangle = (p(D) + q(D))\overline{r} = p(D)\overline{r} + q(D)\overline{r} = \langle p, r\rangle + \langle q, r\rangle.$$

3. Positive-definiteness:

$$\langle p,p\rangle = p(D)\overline{p} = \sum_{|\alpha|=k} \alpha!|c_\alpha|^2 \geq 0, \forall p \in P_k$$

and

$$\langle p,p\rangle = \sum_{|\alpha|=k} \alpha!|c_\alpha|^2 = 0,$$

if and only if all coefficients $c_\alpha = 0$.

This leads to the definition of spherical harmonic. When we define spherical harmonics of degree k we are refering to the constriction of a homogeneous polynomial of degree k to S^N. Let us denote the class of all spherical harmonics of degree k by H_k .

Lemma 1.2.3 *For the dimension* $\dim(H_k)$ *of* H_k *we have*

$$\dim(H_k) = \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!}, k \geq 2.$$

Proof. Our proof begins with establishing the representation for any homogeneous polynomial $p(x) \in P_k$ as follows:

$$p(x) = p_0(x) + |x|^2 p_1(x) + ... + |x|^{2m} p_{2m}(x),$$

where p_l is a homogeneous harmonic polynomial of degree $k-2l, l = 0,1,..,m$.

With this we continue to prove the latter representation. Since it is commonly known that any polynomial of degree less than 2 is harmonic, we assume that $k \geq 2$. It can be seen easily that the Laplace operator Δ maps P_k onto P_{k-2}. Even if this were not the case, we are still able to find a nonzero polynomial $q \in P_{k-2}$ that is orthogonal to range of the Laplace operator $R(\Delta)$:

$$\overline{(\Delta p, q)} = (q, \Delta p) = 0,$$

for every $p \in P_k$. Let us consider $p(x) = |x|^2 q(x)$, then

$$0 = (q, \Delta p) = q(D)\overline{\Delta p} = \Delta q(D)\overline{p} = p(D)\overline{p} = (p,p).$$

This becomes possible because $p \neq 0$.
The set P_j represents a class of homogeneous polynomials of degree j which is the orthogonal direct sum of the set of all harmonic homogeneous polynomials A_j, from P_j where as $B_j = |x|^2 P_{j-2}$:

$$P_j = A_j \bigoplus B_j, j \geq 2.$$

In the particular case when $j = k$ we have $P_j = A_j \oplus B_j$ which entails that any $p(x) \in P_k$ is represented as follows:

$$p(x) = p_0(x) + |x|^2 q(x), \Delta p_0(x) = 0, q(x) \in P_{k-2}.$$

In a similar manner, we consider the case of $j = k-2$, thus obtaining the following decomposition

$$q(x) = p_1(x) + |x|^2 q_1(x), \Delta p_1(x) = 0, q_1(x) \in P_{k-4}.$$

This leads to

$$p(x) = p_1(x) + |x|^2 p_1(x) + |x|^4 q_1(x).$$

Through mathematical induction it can be shown that

$$p(x) = p_0(x) + |x|^2 p_1(x) + \ldots + |x|^{2m} p_{2m}(x),$$

with p_l as a homogeneous harmonic polynomial of degree $k-2l, l = 0,1,..,m$.

With the implementation of this representation it becomes obvious that

$$\dim P_k = \dim A_k + \dim A_{k-2} + \ldots$$

and

$$\dim P_{k-2} = \dim A_{k-2} + \dim A_{k-4} + \ldots.$$

Hence, the dimension of the A_k may be represented as

$$\dim A_k = \dim P_k - \dim P_{k-2} = \frac{(N+k)!}{N!k!} - \frac{(N+k-2)!}{N!(k-2)!}, k \geq 2.$$

We consider the constriction of the polynomial $p(x)$ from A_k to S^N which is also a representation of a spherical harmonic of degree k. It is evident that the dimension of H_k is denoted by a_k which indubitably proves that the dimensions of H_k and A_k are one and same: $a_k = \dim A_k$.

Lemma 1.2.4 *If Y_k and Y_l are spherical harmonics of degrees k and l, with $k \neq l$, then*

$$\int_{S^N} Y_k(\theta) Y_l(\theta) d\sigma = 0.$$

Proof. For $x \neq 0$ let $r = |x|$ and $\theta = x/r$ where we also define $G_k(x) = r^k Y_k(\theta)$ and $G_l(x) = r^l Y_l(\theta)$. We also assign $G_k(0) = 0 = G_l(0)$. By Greens's theorem we obtain

$$\begin{aligned} 0 &= \int_{|x| \leq 1} (G_k \Delta G_l - G_l \Delta G_k)\, d\sigma = \int_{S^N} \left(G_k \frac{\partial G_l}{\partial r} - G_l \frac{\partial G_k}{\partial r} \right) d\sigma, \\ &= \int_{S^N} (l Y_k(\theta) Y_l(\theta) - k Y_k(\theta) Y_l(\theta))\, d\sigma = (l-k) \int_{S^N} Y_k(\theta) Y_l(\theta) d\sigma. \end{aligned}$$

Finally, dividing the equation above by $(l-k)$ for any $l \neq k$, proves lemma 1.2.4.

Lemma 1.2.5 *If $Y \in H_k$ we then have*

$$\Delta_s Y(\theta) + k(N-1+k)Y(\theta) = 0,$$

on the surface of the unit sphere S^N.

Proof. Consider the function $G_k(x) = r^k Y(\theta), r = |x|, \theta = x/|x|$. We know that this function is a homogeneous polynomial of degree k and harmonic, hence $\Delta G_k(x) = 0$. This can be written as

$$\begin{aligned}\Delta(r^k Y(\theta)) &= r^{-2+k}\left[\Delta_s Y + [k(k-1)+(N+1)k]Y\right],\\ &= r^{-2+k}\left[\Delta_s Y + (k+N)kY\right],\\ &= 0.\end{aligned}$$

Thus it becomes perceivable that the elements of the space H_k are eigenfunctions of the Laplace-Beltrami operator corresponding to the eigenvalue $\lambda_k = k(k+N-1)$. We denote

$$\{Y_1^{(k)}(\theta), Y_2^{(k)}(\theta), \ldots, Y_{a_k}^{(k)}(\theta)\}, a_k = \dim H_k,$$

an orthonormal basis of H_k.

Lemma 1.2.6 *The set of all finite linear combinations of elements in $\bigcup_{k=0}^{\infty} H_k\left(S^N\right)$ is*

i . *dense in $C\left(S^N\right)$ with respect to the L^{∞}-norm;*

ii . *dense in $L_2\left(S^N\right)$.*

i Using the compactness of S^N and applying Stone-Weierstrass approximation theorem, any function g continuous on S^N can uniformly approximated by polynomials, P_j which are restricted to S^N. The restriction of every P_j to S^N is a linear combination of elements in $\bigcup_{k=0}^{\infty} H_k\left(S^N\right)$

ii It is a known that $C\left(S^N\right)$ is dense in $L^2\left(S^N\right)$. Let $f \in L^2\left(S^N\right)$, for every $\varepsilon > 0$, we can choose a continous functions, g, such that $\|f-g\|_2 < \varepsilon/2$. By manner of (i) we are able to obtained a linear combination h, of elements in $\bigcup_{k=0}^{\infty} H_k\left(S^N\right)$ so that $\|g-h\|_{\infty} < \varepsilon/\left(2\sqrt{\mathrm{vol}\left(S^N\right)}\right)$, where $\mathrm{vol}\left(S^N\right)$ is the volume of S^N. Thus, we have

$$\begin{aligned}\|f-h\|_2 &\le \|f-g\|_2 + \|g-h\|_2\\ &< \varepsilon/2 + \sqrt{\mathrm{vol}\left(S^N\right)}\|g-h\|_{\infty}\\ &< \varepsilon/2 + \varepsilon/2\\ &= \varepsilon\end{aligned}$$

this validates (ii).

The following proposition is required to show $H_k\left(S^N\right)$ constitute an orthogonal Hilbert space decomposition of $L_2\left(S^N\right)$.

Thus, if $f \in L_2(S^N)$ then there exists a unique representation.

$$f = \sum_{k=0}^{\infty} Y^{(k)},$$

where $\sum_{k=0}^{\infty} Y^{(k)}$ converges to f in the norm of L_2 and

$$Y^{(k)} = c_1^{(k)} Y_1^{(k)} + c_2^{(k)} Y_2^{(k)} + \ldots + c_{a_k}^{(k)} Y_{a_k}^{(k)},$$

with

$$c_j^{(k)} = \int_{S^N} f(\theta) Y_j^{(k)}(\theta) d\sigma, j = 1, 2, \ldots, a_k, \; k = 0, 1, 2, 3 \ldots.$$

The facts above allows us to conclude that the Laplace-Beltrami operator Δ_s has in $L_2(S^N)$ a complete orthonormal system of eigenfunctions

$$\{Y_1^{(k)}(\theta), Y_2^{(k)}(\theta), \ldots, Y_{a_k}^{(k)}(\theta)\}, k = 0, 1, 2, \ldots,$$

Theorem 1.2.7 *The family of spaces, $H_k\left(S^N\right)$ yields a Hilbert spave direct sum decomposition*

$$L_2\left(S^N\right) = \bigoplus_{k=0}^{\infty} H_k\left(S^N\right),$$

which means that the summands are closed, pairwise orthogonal, and that every $f \in L_2\left(S^N\right)$ is the sum of a converging series

$$f = \sum_{k=0}^{\infty} f_k,$$

in the L_2-norm, where the $f_k \in H_k\left(S^N\right)$ are uniquely determined functions.

We fix a point $\omega \in S^N$ and consider the linear functional Υ on H_k that assigns the value $F(\omega)$ to each $F \in H_k$. By the self duality of the finite dimensional space H_k there exists a unique spherical harmonic $Z_k(\theta, \omega)$ such that

$$\Upsilon(F) = F(\omega) = \int_{S^N} F(\theta) Z_k(\theta, \omega) d\sigma(\theta), \forall F \in H_k.$$

The function $Z_k(\theta, \omega)$ is referred as the zonal harmonic of degree k with the pole ω. If

$$\{Y_1^{(k)}(\theta), Y_2^{(k)}(\theta), \ldots, Y_{a_k}^{(k)}(\theta)\},$$

is an orthonormal basis of H_k then we have

$$Z_k(\theta, \omega) = \sum_{j=1}^{a_k} \left(Z_k(\theta, \omega), Y_j^{(k)}(\theta)\right) Y_j^{(k)}(\omega).$$

By the defining property of zonal harmonics, we have

$$\left(Z_k(\theta,\omega), Y_j^{(k)}(\theta)\right) = \int_{S^N} Y_j^{(k)}(\theta) Z_k(\theta,\omega)\, d\sigma(\theta) = Y_j^{(k)}(\omega).$$

This gives us the following representation for the zonal spherical harmonics:

$$Z_k(\theta,\omega) = \sum_{j=1}^{a_k} Y_j^{(k)}(\theta) Y_j^{(k)}(\omega).$$

1.3 The Spectral Function Of The Laplace-Beltrami Operator On Unit Sphere

Spectral function of the Laplace-Beltrami operator is the kernel of spectral decomposition related to the given operator and plays an important role in investigations of the convergence problems of eigenfunction expansions. In this section we compare the asymptotical behavior of Cesaro and Riesz means of the spectral function of the Laplace-Beltrami operator on the sphere S^N.

The Laplace-Beltrami operator $-\Delta_s$ is known to be symmetrical and nonnegative. Friedrich's theorem (Alimov (1976)) states that exist a nonnegative self-adjoint extension $\hat{A}$, in $L_2(S^N)$. The self-adjoint extension of $-\Delta_s$ is unique and corresponds with the closure $\overline{-\Delta_s}$ of $-\Delta_s$.

It is commonlly known that the operator $\hat{A}$ has in $L_2(S^N)$ a complete orthonormal system of eigenfunctions

$$\{Y_1^{(k)}(x), Y_2^{(k)}(x), \ldots, Y_{a_k}^{(k)}(x)\}, k = 0,1,2,\ldots,$$

where

$$a_k = \begin{cases} 1, & k = 0, \\ N, & k = 1, \\ \dfrac{(N+k)!}{N!k!} - \dfrac{(N+k-2)!}{N!(k-2)!}, & k > 1, \end{cases}$$

is the multiplicity of eigenvalue λ_k. The system of eigenfunctions also corresponds to the eigenvalues $\{\lambda_k = k(k+N-1)\}, k = 0,1,2,\ldots$.

As stated by von Neumann's theorem (Alimov(1976)), the extension $\hat{A}$ is similar to every other self-adjoint operator that has a decomposition of unity $\{E_\lambda\}$ and which it can be written in the form

$$\hat{A} = \int_0^\infty \lambda\, dE_\lambda,$$

where E_λ has the form

$$E_\lambda f(x) = \sum_{\lambda_n < \lambda} Y_n(f,x),$$

and

$$Y_n(f,x) = \sum_{j=1}^{a_n} Y_j^{(n)}(x) \int_{S^N} f(y) Y_j^{(n)}(y) d\sigma(y),$$

which allows us to define $Y_n(f,x)$ as the Fourier-Laplace coefficients of thefunction $f \in L_2(S^N)$.

Among the key problems faced in harmonic analysis is the reconstruction of functions from their expansion:

$$f(x) \sim \sum_{n=0}^{\infty} Y_n(f,x). \tag{1.14}$$

There are many occurence where these types of problems are not solvable by simply defining the sum of Fourier-Laplace series as the limit of partial sums $E_\lambda f(x)$. For those who are unfamiliar with harmonic analysis, it would be interesting to know that not so smooth functions are the most interesting expansions and from a practical perspective are known to be significant. The point in hand is that fore mentioned functions have successive strongly oscillating terms of the partial sums is very influential, yet does not correspond to their insignificant role in the characterisation of the function to be reconstructed. This causes the sequence of the partial sums to merely oscillates around the function rather than approach it. However, due to certain contraints these oscillations have a regular behaviour. As an example, the arithmetic means of the partial sums is expected to better approximate the function under confederation. This is proven to be the mathematical equivalent to taking in the calculation of the partial sum, each term of the series with a weight which decreases in size as we increase the index.

We define the Riesz means of order $\alpha \geq 0$ of the spectral expansions $E_\lambda f$ by the following expression

$$E_\lambda^\alpha f(x) = \sum_{\lambda_n < \lambda} \left(1 - \frac{\lambda_n}{\lambda}\right)^\alpha Y_n(f,x).$$

Definition 1.3.1 *The spectral expansion $E_\lambda f$ is said to be summable to $f(x)$ by the Riesz method of order α if*

$$\lim_{\lambda \longrightarrow \infty} E_\lambda^\alpha f(x) = f(x).$$

If partial sums $E_\lambda f$ converge, the mean $E_\lambda^\alpha f$ will also converge to the same value however the converse is not true. It is known that the Riesz means $E_\lambda^\alpha f$ is viable for any real $\alpha \geq 0$ and complex α with $\mathbb{R}(\alpha) \geq 0$, this allows the use of interpolation theorems in the study of Riesz means. From a perspective, the Riesz means can be consider as a regularisation of the partial sums $E_\lambda f$, and there exists many occurences their asymptotic behaviour gets better when $\mathbb{R}(\alpha)$ is increased.

The Riesz means $E_n^\alpha f$ may be transformed by writing instead of $Y_n(f,x)$ the expression $\int\limits_{S^N} f(y)Z_n(x,y)d\sigma(y)$:

$$\begin{aligned}
E_\lambda^\alpha f(x) &= \sum_{\lambda_k<\lambda}\left(\frac{1-\lambda_k}{\lambda}\right)^\alpha Y_k(f,x),\\
&= \int_{S^N} f(y)\left\{\sum_{\lambda_k<\lambda}\left(\frac{1-\lambda_k}{\lambda}\right)^\alpha Z_k(x,y)\right\}d\sigma(y),\\
&= \int\limits_{S^N} f(y)\Theta^\alpha(x,y,\lambda)d\sigma(y),
\end{aligned}$$

with

$$\Theta^\alpha(x,y,\lambda) = \sum_{\lambda_k<\lambda}\left(1-\frac{\lambda_k}{\lambda}\right)^\alpha Z_k(x,y).$$

The behaviour of the spectral expansion corresponding to the Laplace-Beltrami operator is closely connected with the asymptotical behavior of the kernel $\Theta^a(x,y,\lambda)$.

Let formulate the Theorem on the asymptotical behaviour of the Riesz means of the spectral function of the Laplace-Beltrami operator. First we recall the spherical distance on the sphere. For any two points x and y from S^N spherical distance $\gamma(x,y)$ is a measure of angle between x and y. It is obvious, that $\gamma(x,y) \leq \pi$.

Theorem 1.3.2 *Let $\Theta^\alpha(x,y,n)$ be the kernel of Riesz means of the spectral expansions of the Laplace-Beltrami operator on sphere;*
1) for all $x,y \in S^N$, such that $|\pi/2-\gamma(x,y)| < \frac{\pi n}{2(n+1)}, n \to \infty$

$$\Theta^\alpha(x,y,n) = O(1)\left(\frac{n^{(N-1)/2-\alpha}}{(\sin\gamma)^{(N-1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{n^{(N-3)/2-\alpha}}{(\sin\gamma)^{(N+1)/2}(\sin(\gamma/2))^{1+\alpha}} + \frac{n^{-1}}{(\sin(\gamma)/2)^{1+N}})\right);$$

2) for all $x,y \in S^N : 0 \leq \gamma(x,y) \leq \pi, n > 1$

$$\Theta^\alpha(x,y,n) = O(1)n^N;$$

3) for all $x,y \in S^N : 0 < \gamma_0 \leq \gamma(x,y) \leq \pi, n > 1$

$$\Theta^\alpha(x,y,n) = O(1)n^{N-\alpha};$$

1.4 Stein's Interpolation Theorem

Next, we begin by defining Stein's interpolation theorem in a more suitable form in order to meet our purpose. The function $\varphi(\tau)$, $\tau \in R^1$ is said be of reasonable growth if there exist constants $a < \pi$ and $b > 0$ such that

$$|\varphi| \leq \exp(b \exp s|\tau|). \tag{1.15}$$

Let A_z define a family of operators for simple functions. The family A_z is termed acceptable if for any 2 simple functions f and g the function

$$\varphi(z) = \int_{T^N} f(x) A_z g(x)\, dx$$

is analytic in the strip $0 \leq \text{Re } z \leq 1$ with a permissible growth in IM z and uniformly in Re z.

Theorem 1.4.1 (Stein's Interpolation Theorem) *Let A_z be an admissible family of linear operators such that*

$$\|A_{i\tau} f\|_{L_{p_0}(T^N)} \leq M_0(\tau) \|f\|_{L_{p_0}(T^N)}, \qquad 1 \leq p_0 \leq \infty,$$
$$\|A_{1+i\tau} f\|_{L_{p_1}(T^N)} \leq M_1(\tau) \|f\|_{L_{p_1}(T^N)}, \qquad 1 \leq p_1 \leq \infty,$$

for all simple functions f and with $M_j(\tau)$ independent of τ and of admissible growth. Then there exists for each t, $0 \leq t \leq 1$, a constant M_t such that for every simple function f holds

$$\|A_t f\|_{L_{p_t}(T^N)} \leq M_t(\tau) \|f\|_{L_{p_t}(T^N)}, \qquad \frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}.$$

The application of Stein's Interpolation theorem is best utilized by Riesz means which is dependent on s. For the case of Riesz means, the implementation of admissible growth does not factor in any additional difficulty due to the fact that all functions related in its applications have exponential growth. The imposed restriction of the domain by definition of A_z to simple functions does not undermind the possibility of interpolation, as the simple functions constitue a dense subset in $L_p(T^N)$.

1.5 Functional Spaces

When dealing with functional spaces we can refer to works by Triebel (1986) who dedicated excellent expositions on the theory of function spaces. This theory was vigorously developed by Russsian mathematicians (former Soviet Union): Sobolev (spaces W_p^k), Nikol'skii (classes H_p^r), Lizorkin (Liouville spaces L_p^r), Il'in, Besov and many others. In Chapter 4 of this thesis, our results highlights the distributions of the functional spaces of Sobolev, Nikolskii and Liouville.

1.5.1 Liouville space $L_p^{\ell}\left(S^N\right)$

Definition 1.5.1 *For any real number l, we define a class of Liouville $L_p^l\left(S^N\right)$, which consist of all distributions $f \in D'\left(S^N\right)$, such that*

$$\left\| \sum_{k=0}^{\infty} (1+\lambda_k)^{\frac{l}{2}} Y_k(f,x) \right\|_{L_p} < \infty, \quad p \geq 1.$$

The norm of f in $L_p^l\left(S^N\right)$ is defined by

$$\|f\|_{L_p^l} = \left\| \sum_{k=0}^{\infty} (1+\lambda_k)^{\frac{l}{2}} Y_k(f,x) \right\|_{L_p}.$$

For more properties of $L_p^l\left(S^N\right)$ we refer the reader to Triebel (1986).

1.5.2 Sobelov space $W_p^a\left(S^N\right)$

We define the Sobolev space as follows:

Definition 1.5.2 *For $1 < p < \infty$, $a > 0$, $W_p^a\left(S^N\right)$ is the collection of all $f \in L_p\left(S^N\right)$ such that $f = (I-\Delta)^{-\frac{a}{2}} h$ for some $h \in L_p\left(S^N\right)$ with the norm*

$$\|f\|_{W_p^a(S^N)} = \|h\|_{L_p(S^N)}.$$

If $a < 0$; $W_p^a\left(S^N\right)$ is the collection of all $f \in D'\left(S^N\right)$ of the form $f = (I-\Delta)^k h$ with $h \in W_p^{2k+a}\left(S^N\right)$, where k is a natural number such that $2k+a > 0$, and

$$\|f\|_{W_p^a(S^N)} = \|h\|_{W_p^{2k+a}(S^N)}.$$

If $a = 0$, $W_p^0\left(S^N\right) = L_p\left(S^N\right)$.

If $p = 2$, we have $\|f\|_{W_2^a(S^N)} = \left\| \sum_{k=0}^{\infty} (1+\lambda_k)^{\frac{l}{2}} \sum_{j=1}^{a_k} \left\langle f, y_j^k(x) \right\rangle \right\|_{L_2(S^N)}$.

1.5.3 Nikolskii space $H_p^a(S^N)$

The following expression

$$\omega_p^r(f,\tau) = \sup_{0<t\le\tau} \|(I - S_h(t))^r f\|_{L_p(S^N)}, \quad 0 < t < \pi,$$

allows the definition of the Besov space $B_{p,q}^a\left(S^N\right)$.

Definition 1.5.3 *Let $a > 0,\ 1 \le p,\ q \le \infty$, the Besov space $B_{p,q}^a\left(S^N\right)$ is defined as a set of function $f \in L_p\left(S^N\right)$ such that*

$$\|f\|_{L_p(S^N)} + \left(\int_0^\pi \left[\frac{\omega_p^r(f,t)|^q}{t^a}\frac{dt}{t}\right]\right) < \infty.$$

and the latter expression

$$\|f\|_{B_{p,q}^a(S^N)} = \|f\|_{L_p(S^N)} + \left(\int_0^\pi \left[\frac{\omega_p^r(f,t)|^q}{t^a}\frac{dt}{t}\right]\right).$$

The latter expression is called a norm of f in $B_{p,q}^a\left(S^N\right)$.

We note here that Besov space coinciding with Nikolskii space when $q = \infty$:

$$B_{p,\infty}^a\left(S^N\right) = H_p^a\left(S^N\right).$$

When working with functions from Nikolskii, it is convenient to work with the following equivalent norm:

$$\|f\|_{B_{p,q}^a(S^N)} = \|f\|_{L_p(S^N)} + \left(\int_0^\pi \left[\frac{\|(I - S_h(t))^r f\|_{L_p(S^N)}}{h^a}\right]^q \frac{dh}{h}\right)^{\frac{1}{q}},$$

where $r > a/2$.

Finally, the Nikolskii space can be defined as follows:

Definition 1.5.4 *For $1 < p < \infty$, $a > 0$, $H_p^a\left(S^N\right)$ set a set of function $f \in L_p\left(S^N\right)$ for some $h \in L_p\left(S^N\right)$ with the norm*

$$\|f\|_{H_p^a(S^N)} = \|f\|_{L_p(S^N)} + \sup_{0<h\le\pi} \frac{\|(I - S_h(t))^r f\|_{L_p(S^N)}}{h^a}.$$

Before proceeding to the section on distributions, some important facts will be presented. Each distribution φ on S^N are assigned to its spherical harmonic expansion

$$\sum_{n=0}^{\infty} Y_n(\varphi : x), \tag{1.16}$$

where $Y_n(\varphi) \in H_n(S^N)$ for all $n \geq 0$.

Proposition 1.5.5 *If $f \in C^{\infty}(S^N)$ then $\|Y_n(f)\|_{\infty} = o\left(n^{-A}\right)$ as $n \to \infty$, for every $A > 0$. A formal series $\sum_{n=0}^{\infty} Y_n$ with $Y_n \in \widetilde{H}_n(S^N)$ for all n, is the Fourier series of a distribution on X if and only if $\|Y\|_{\infty} = O\left(n^B\right)$ as $n \to \infty$, for some $B \geq 0$.*

Proposition 1.5.6 *Let $x_0 \in S^N$ and suppose that f is μ-integrable and zonal about x_0. Then f has Fourier series $\sum_{n=0}^{\infty} a_n Z_n(x,x_0)$, where $a_n = d_n^{-1} \int_{S^N} f(x) Z_n(x,x_0)\, d\sigma(x)$.*

Definition 1.5.7 *If φ is a distribution on S^N with Fourier series (1.16), its Riemann-Lebesgue set is*

$$RL(\varphi) = \{x \in S^N : Y_n(\varphi : x) \to 0 \text{ as } n \to \infty.$$

Proposition 1.5.8 *Let S^N be a compact two-point homogeneous space and φ a distribution on S^N. Suppose that $x_0 \in RL(\varphi)$ and that $\varphi \in C^{\infty}(S^N)$ is zonal about $x_0 \in RL(\varphi\varphi)$.*

1.6 The Theory Of Distributions On Unit Sphere

Distributions form a class of objects that contains a subset of the continuous function. Distributions is also commonly referred as "generalized functions" where each distribution is able to be approximated by infinitely differentiable functions. Nevertheless, distributions are not necessarily functions. It can be seen in various of aspects that in comparison to the theory of continuous functions, the calculus of distributions can be more readily developed. As an example, the derivative of any distribution is also a distribution.

1.6.1 Distributions

The conception of distributions was the result of Dirac's research in quantum mechanics. It was in relation to his systematically use of the δ function in quantum mechanics. Sobolev (1936) and Schwartz (1950, 1951) later on paved the way for the ground work of the mathematical theory of distributions which lead to the intensive development of the theory of generalized function by many mathematicians.

Since then, the theory of distributions has further advanced with many applications in mathematics and physics. It has become an imperative tool for physicist, mathematician and engineers.

The distribution is by general definition the generalization of the classical concept of a function which allows us to idealized concept such as, the intensity of a force applied at a point, the density of a simple or double layer and so on, in mathematical form. Yet, the concept of a distribution as a reflection of the fact is simply impossible, i.e to measure the density of a material at a point. In overcoming this obstacle, the generalized function which is a represntation of "average values" to measure a point by the average density in a sufficiently small neighbourhood at the given point.

To make things clearer, we will examine the question of the density created by a material point of mass 1 in more detail. Consider that this point coincides with the originating coordinate and where we will denote the density by $\delta(x)$.

For the purpose of defining the density, we distribute the unit mass uniformly inside the sphere S_ε and should obtain the following average density

$$f_\varepsilon(x) = \begin{cases} \dfrac{3}{4\pi\varepsilon^3}, & |x| < \varepsilon, \\ 0, & |x| > \varepsilon. \end{cases}$$

We begin by taking the point limit of the sequence of average densities f_ε as the density $\delta(x)$, which is

$$\delta(x) = \lim_{\varepsilon\to 0} f_\varepsilon(x) = \begin{cases} \infty, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases} \tag{1.17}$$

Naturally, it is required that the integral of the density δ over any volume V should give the mass of this volume as follows,

$$\int_V \delta(x) = \begin{cases} 1, & \text{if } 0 \in V, \\ 0, & \text{if } 0 \bar{\in} V. \end{cases}$$

Due to (1.17), for the case of an improper integral the left hand side of the equation will always be equal to zero. This contradicts the fact that the limit point of the sequence $f_\varepsilon(x)$ as $\varepsilon \to 0$ can not be taken as the density $\delta(x)$.

We proceed to obtain the weak limit of the sequences $f(x)_\varepsilon(x)$ as $\varepsilon \to 0$. This is done by showing that

$$\lim_{\varepsilon\to 0} \int f_\varepsilon(x)\varphi(x)\, dx = \varphi(0).$$

In regards to the continuity of the function $\phi(x)$ for $\eta > 0$ exists $\varepsilon_0 > 0$ such that

$|\varphi(x)-\varphi(0)|<\eta$ whenever $|x|<\varepsilon_0$.Thus, for all $\varepsilon \le \varepsilon_0$, we have

$$\begin{aligned}\left|\int f_\varepsilon(x)\varphi(x)\,dx-\varphi(0)\right| &= \frac{3}{4\pi\varepsilon^3}\left|\int_{|x|<\varepsilon}[\varphi(x)-\varphi(0)]\,dx\right|,\\ &\le \frac{3}{4\pi\varepsilon^3}\int_{|x|<\varepsilon}|\varphi(x)-\varphi(0)|\,dx,\\ &< \eta\frac{3}{4\pi\varepsilon^3}\int_{|x|<\varepsilon}dx=\eta.\end{aligned}$$

This gives the functional $\phi(0)$ as the weak limit of the sequence of functions f_ε as $\varepsilon \to 0$, assigning to each continuous function $\varphi(x)$ the value of $\varphi(0)$ at the point $x=0$. The functional obtain here is used to define the density $\delta(x)$ and also the same functional referred in the popular Dirac δ function. Hence $f_\varepsilon(x)\to\delta(x)$ as $\varepsilon\to 0$ for any continuous function $\varphi(x)$ is valid for

$$\int f_\varepsilon(x)\varphi(x)\,dx \to (\delta,\varphi), \quad \varepsilon\to 0,$$

where (δ,φ) denotes the value of the functional δ acting on function φ. In recovering the complete mass, it must act with the density (functional) $\delta(x)$ on the function $\varphi(x)=1$, $(\delta,1)=1$.

The density must be considered equal to $m\delta(x)$, if the corresponding mass m is concentrated at the point $x=0$. And also, if mass m is concentrated at x_0, its density is considered equivalent to $m\delta(x-x_0)$, where $(m\delta(x-x_0),\varphi)=m\varphi(x_0)$. If masses m_k are concentrated at different points x_k, $k=1,2,\ldots,N$, the generalized corresponding density is equal to

$$\sum_{k=1}^{N} m_k\delta(x-x_k).$$

1.6.2 Space of test functions $\widetilde{D}$

Test functions for the δ function are defined by continuous functions as a linear continuous functional. This point of view is used as fundamental definition of an arbitary generalized function as a linear continous functional over a space of sufficiently ”good” (test) functions. It can be seen that a more narrow space of test functions will result in more linear continous functional over it. Yet the supply of the test function must be sufficiently large. The following will clarify the importance if test functions $\widetilde{D}$.

The set of test functions $\widetilde{D}=\widetilde{D}(S^N)$ all the infinitely differentiable functions in S^N. We define the convergence in $\widetilde{D}$ as: the sequence of functions $\varphi_1,\varphi,\ldots,$ form $\widetilde{D}$ converges to the function $\varphi\in\widetilde{D}$ if:

For each $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$

$$D^{\alpha}\varphi_k(x) \text{ uniformly converges to } D^{\alpha}\varphi(x), \quad k \to \infty,$$

hence we can write $\varphi \to \varphi$ as $k \to \infty$ in $\widetilde{D}$.

We know that $\widetilde{D}$ is a linear space. The operation of differentation $D^{\beta}\varphi(x)$ is continuous from $\widetilde{D}$ into $\widetilde{D}$. The definition of convergence in $\widetilde{D}$ states that $\varphi_k \to \varphi$ as $k \to \infty$, allowing us to conclude that $D^{\beta}\varphi_k \to D^{\beta}\varphi$ as $k \to \infty$ in $\widetilde{D}$. Similarly, the operations of nonsingular linear change of variable $\varphi(Ay+b)$ and the multiplication by a function $a \in C^{\infty}(S^N)$, $a(x)\varphi(x)$, are continuous from $\widetilde{D}$ into $\widetilde{D}$
The question whether there are test functions distinct from being identically zero. It becomes apparent that such function are not analytic in S^N.

1.6.3 The space of generalized functions $\widetilde{D}'$

Each linear continuous functional over the space of test functions $\widetilde{D}$ is known as generalized function in the Sobolev-Schwartz sense. We will denote the effect of the functional (generalized function) f over the test function φ by (f, φ). We also denote f by $f(x)$ as a representation of the generalized function taking into account that the functional f acts on to x, the argument of the test functions.

Here we will clarify the definition of the generalized function f.

1. The generalized function f is a *f*unctional over $\widetilde{D}$; that is, a (complex) number (f, φ) is associated with each $\varphi \in \widetilde{D}$.

2. The generalized function f is a *l*inear function over $\widetilde{D}$; that is, if $\varphi \in \widetilde{D}$, $\psi \in \widetilde{D}$, and λ, μ are complex numbers, then

$$(f, \lambda\varphi + \mu\psi) = \lambda(f, \varphi) + \mu(f, \psi).$$

3. The generalized function f is a *c*ontinuous functional over $\widetilde{D}$; that is if $\varphi_k \to \varphi$ as $k \to \infty$ in $\widetilde{D}$, then

$$(f, \varphi_k) \to (f, \varphi), k \to \infty.$$

If $\widetilde{D}'$ is linear and the linear combination $\lambda f + \mu g$ of generalized functions f and g is defined as a functional acting in accordance with the formula

$$(\lambda f + \mu g, \varphi) = \lambda(f, \varphi) + \mu(g, \varphi), \quad \varphi \in \widetilde{D}.$$

We will show that the functional $\lambda f + \mu g$ belongs to $\widetilde{D}'$ by showing that it is linear and continous all over $\widetilde{D}$. Let α and β be any complex numbers, and if $\varphi, \psi \in \widetilde{D}$

hence via definition,

$$
\begin{aligned}
(\lambda f+\mu g,\alpha\varphi+\beta\psi) &= \lambda(f,\alpha\varphi+\beta\psi)+\mu(g,\alpha\varphi+\beta\psi),\\
&= \alpha[\lambda(f,\varphi)+\mu(g,\varphi)]+\beta[\lambda(f,\varphi)+\mu(g,\varphi)],\\
&= \alpha(\lambda f+\mu g,\varphi)+\beta(\lambda f+\mu g,\psi),
\end{aligned}
$$

proving this functional is linear. As for its continuity, from the continuity of the functionals f and g: If $\varphi_k \to \varphi$ as $k \to \infty$ in $\widetilde{D}$, then

$$(\lambda f+\mu g,\varphi_k) = \lambda(f,\varphi_k)+\mu(g,\varphi_k) \to \lambda(f,\varphi)+\mu(g,\varphi) = (\lambda f+\mu g,\varphi).$$

For any $\varphi \in \widetilde{D}'$, if there exist $(f_k,\varphi) \to (f,\varphi)$ as $k \to \infty$ we can define the convergence in $\widetilde{D}'$ as the sequence of generalized functions, $f_1, f_2, \ldots$ of a generalized in $\widetilde{D}'$. This type of convergence is known as weak convergence. A linear set $\widetilde{D}'$ with this kind of convergence is referred as the space of generalized functions $\widetilde{D}'$.

The space $\widetilde{D}'$ is complete if, for the sequence $f_1, f_2, \ldots$ in $\widetilde{D}'$ there exist a limits of the numerical sequence $(f_1,\varphi),(f_2,\varphi),\ldots$, for any test function $\varphi \in \widetilde{D}'$ where exists a generalized function that is unique $f \in \widetilde{D}'$, such that

$$\lim_{k\to}(f_k,\varphi) = (f,\varphi), \quad \varphi \in \widetilde{D}',$$

proof of the theorem can be found in Shilov (1965).

1.7 Research Objective(s)

1. To estimate the Riesz means of the spectral function of the Laplace-Beltrami opertor on unit sphere.

2. To prove uniform convergence of the eigenfunction expansions of the Laplace-Beltrami operator by Riesz means in the classes of Nikolskii and Liouville.

3. To establish the conditions for localization of the Fourier-Laplace series of the distributions on unit sphere.

1.8 Thesis Outline

In this thesis, we have organized the chapter as following.

- Chapter 1 - Introduction: In this chapter we provide the background tools used for our subject matter, the objective and methods applied to solve the problems in hand and the factors responsible for our motivation behind the idea.

- Chapter 2 - Literature review: This chapter deals with researches by previous authors that dealt with issues related to our topic.
- Chapter 3 - Here we provide results related to the Riesz means of the spectral function of Laplace-Beltrami operator and divergence theorem on localization in regards to the maximal operator.
- Chapter 4 - Here we address problems related to the summability and localization of the Fourier-Laplace series of Riesz methods.
- Chapter 5 - In this chapter we will address issues of the Localization of Fourier Laplace series of distributions in various classes (Sobolev, Nikolskii, Liouville).
- Chapter 6 - Discussion, Contribution and Future Work: In the last section of the thesis we will present the outcome of this thesis and its contribution to the field of spherical harmonics. We also discuss the possibility for future research and give suggestions on open problems for future researchers of the subject.

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