UNIVERSITI PUTRA MALAYSIA

NUMERICAL SOLUTION OF SPECIAL SECOND ORDER INITIAL VALUE PROBLEMS BY HYBRID TYPE METHODS

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By

DAUDA YUSUF JIKANTORO

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

March, 2014

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DEDICATIONS

To my late sister, Salamatu Dauda.

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

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March, 2014

Chair: Professor Fudziah Bint Ismail, PhD

Faculty: Science

We derived in this thesis new highly dispersive and highly dissipative two-step explicit hybrid methods for solving oscillatory problems. Dispersion conditions up to order ten and dissipation conditions up to order eleven for five stage hybrid methods are presented. The derivation of the methods was largely based on maximization of order of dispersion and dissipation while minimizing the principal error norm. Stability of the methods was investigated and their intervals of stability presented. The methods, which can be applied using constant step size, were tested on model problems. Numerical results revealed the superiority of the methods over several existing methods in the literature.

In order to achieve higher accuracy and efficiency, trigonometrically fitted hybrid methods based on existing zero-dissipative methods were derived. Their ability to approximate the solution of problems with large fitted frequency using large step size proved their accuracy and efficiency for solving highly oscillatory problems compared to phase-fitted methods and trigonometrically fitted methods which are based on dissipative hybrid methods.

Finally, semi-implicit hybrid methods based on explicit hybrid methods were derived. Dispersion and dissipation conditions for the methods were presented. Stability analysis along side stability or periodicity intervals of the methods was presented. Results obtained from numerical experiment showed the accuracy and efficiency of the new methods compared to the existing methods.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

PENYELESAIAN BERANGKA BAGI MASALAH KHAS NILAI AWAL PERINGKAT KEDUA DENGAN MENGGUNAKAN KAEDAH JENIS HIBRID

Oleh

DAUDA YUSUF JIKANTORO

Mac, 2014

Pengerusi: Professor Fudziah Bint Ismail, PhD

Fakulti: Sains

Dalam tesis ini kami menerbitkan kaedah dua langkah tak tersirat dengan serakan dan lesapan yang tinggi untuk menyelesaikan masalah berayun. Syarat serakan sehingga peringkat kesepuluh dan syarat lesapan sehingga peringkat kesebelas untuk kaedah hibrid tahap lima dipersembahkan. Penerbitan kaedah tersebut adalah dengan memaksimumkan peringkat serakan dan lesapan dan meminimumkan norma ralat prinsipal. Kestabilan kaedah tersebut dikaji dan selang kestabilannya dipersembahkan. Kaedah tersebut yang boleh digunakan dengan panjang langkah malar, diuji ke atas model masalah. Keputusan berangka menunjukkan kelebihan kaedah berbanding kaedah sedia ada dalam literatur.

Untuk mencapai kejituan dan kecekapan yang tinggi, kaedah hibrid suaisecara trigonometri berasaskan kaedah lesapan sifar sedia ada diterbitkan. Kebolehan kaedah ini menghampiri penyelesaian masalah dengan penyuaian frekuensi yang besar menggunakan panjang langkah besar membuktikan kejituan dan kecekapannya menyelesaikan masalah yang sangat berayun berbanding kaedah penyuaian fasa dan trigonometri yang berasaskan kaedah hibrid lesapan.

Akhir sekali, kaedah semi-tersirat berasaskan kaedah hibrid tak terserirat diterbitkan. Syarat serakan dan lesapan bagi kaedah tersebut dipersembahkan. Analisis kestabilan, kestabilan atau selang kalaan bagi kaedah dipersembahkan. Keputusan yang diperolehi dari eksperimen berangka menunjukkan kejituan dan kecekapan kaedah baharu ini berbanding kaedah sedia ada.

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Finally, I appreciate the understanding, support and prayers given to me by the entire members of my family, especially, my late sister (Salamatu Dauda) who lost her life recently and my mother (Zainab Muhammad Sani).

I certify that a Thesis Examination Committee has met on **26th March, 2014** to conduct the final examination of **Yusuf Dauda Jikantoro** on his thesis entitled "**Numerical solution of special second order initial value problems by hybrid type methods**" in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the **Master of science**.

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Declaration by Graduating Student

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LIST OF ABBREVIATIONS

PDEs	Partial Differential Equations
ODEs	Ordinary Differential Equations
IVPs	Initial Value Problems
LTE	Local Truncation Error
E_rN	Principal Error Norm
HM5(5,6,∞)	Zero-dissipative fifth order five stage sixth order dispersive hybrid method derived in chapter 3
HM5(5,10,7)	Fifth order five stage hybrid method with dispersion order ten and dissipative of order seven derived in chapter 3
HM5(4,8,5)	Fifth order four stage hybrid method with dispersion order eight and dissipation order five in Franco (2006)
BRKN5(6)FSAL	Fifth order six stage Runge-Kutta-Nyström method with first same as last property presented in Bettis (1973)
TFZDHM5	Trigonometrically fitted zero-dissipative fifth order hybrid method derived in chapter 4
TFHM	Trigonometrically fitted fifth order hybrid method derived in Dizicheh et al. (2012)
MRKN5(8,7)	Fifth order five stage Runge-Kutta-Nyström method with dispersion and dissipation order eight and seven respectively presented in Mohamad et al. (2011)
TFZDHM6	Trigonometrically fitted zero-dissipative sixth order hybrid method derived in chapter 4
HM6(5,6,∞)	Zero-dissipative sixth order five stage sixth order dispersive hybrid method presented in Franco (2006)
ZDSIHM5(4,6,∞)	Zero-dissipative fifth order four stage semi-implicit hybrid method derived in Chapter 5
SIHM5(4,8,5)	Fifth order four stage semi-implicit hybrid method with dispersion and dissipation order eight and five respectively, derived in Ahmad et al. (2013a)

CHAPTER 1

INTRODUCTION

1.1 Differential equations

System of differential equations, in general, is an essential tool for modeling in science, engineering and economics to mention a few. Solution to this system of differential equations translates the property of physical system that the equations represent. Some of the physical problems represented by these equations are motion of wave particles, chemical kinetics problems, motion of planets around the sun. Some of the equations are partial differential equations (PDEs) while some are ordinary differential equations (ODEs) depending on the phenomena under consideration. PDEs have two or more independent variables while ODEs have one independent variable. Each of these classes of differential equations occur in different orders. Second order ordinary differential equation is an equation with highest order of derivative of dependent variable as two. The general form of second order ordinary differential equations considered is

$$y''(x) = f(x, y), \tag{1.1}$$

which is the special case of

$$y''(x) = f(x, y, y'). \tag{1.2}$$

The specialty associated with (1.1) being that $f(x, y)$ does not depend on y' explicitly.

1.1.1 Initial value problems

The ODE given in (1.1) cannot stand alone to give the equation a unique solution, thus, there is a need of additional conditions to be added for a unique solution. Therefore, an equation of the type (1.1) is said to be initial value problem (IVP) if an initial condition is imposed on it, in other words, if the solution $y(x)$ is known at some initial points of x. That is,

$$y''(x) = f(x, y), y(a) = \alpha \tag{1.3}$$

where $x \in R, y(x) \in R^r, f(x, y) \in R^{r+1}, x \in [a, b]$ and $\alpha \in R$. This type of problems arise in different fields of science and engineering, for example, astrophysics, celestial mechanics, quantum mechanics, electronics, quantum chemistry.

1.1.2 Existence and Uniqueness of Solution

Initial value problems describe a problem together with the behavior of it's trajectory at some initial points of independent variable x. The question is how reliable are they in predicting the future behavior of the same trajectory? Some of the attributes of initial value problems that answer this question, as given in Butcher

(2008), are existence of solution, uniqueness of the solution if it exists and the sensitivity of the solution to a small perturbation to the initial information. One of the famous conditions that guarantees these attributes is the Lipschitz condition.

Definition 1.1 *A function $f : R \times R^r \to R^r$ is said to satisfy Lipschitz condition in its second variable if there exist a constant L such that for any $x \in [a, b]$ and $y_1, y_2 \in R^r$,*

$$\|f(x, y_1) - f(x, y_2)\| \leq L\|y_1 - y_2\|, \tag{1.4}$$

where L is called Lipschitz constant.

Theorem 1.1 :(Existence and Uniqueness)
Suppose $f(x, y(x))$ is defined and continuous $\forall$ points $(x, y(x))$ in a domain D made up of $x \in [a, b], y \in (-\infty, \infty)$, a and b are finite, and that $f(x, y(x))$ satisfies Lipschitz condition. Then for any given number §, $\exists$ a unique solution $y(x)$ of the IVP (1.3), where $\forall$ $(x, y(x)) \in D$, $y(x)$ is continuous and differentiable. Butcher (2008).

In this thesis, we assume that $f(x, y(x))$ of the IVP (1.3) satisfies Lipschitz condition so that a unique solution is guaranteed.

1.2 Hybrid Method

Runge-Kutta method and linear multi-step method are two different but related methods for approximating the solutions of differential equations. While the former is non-linear and self starting, the latter is linear and requires additional starting information from the former in order to start numerical integration. The idea of combining the ideas of these methods to obtain a single method possessing the characteristics of the methods dates back to six decades, Butcher (2008). The nomenclature of the method varies with authors, it is called Pseudo Runge-Kutta method or modified linear multi-step method. The most commonly used name for this method is 'hybrid method', Butcher (2008). The idea was extended to combining Runge-Kutta Nyström method with linear multi-step method so that second order ODEs can be solved directly.

1.2.1 Two-step Hybrid Methods

The general form of an m-stage two-step hybrid method, as given in Coleman (2003), for numerical integration of the initial value problems (IVPs) (1.3) is defined by

$$y_{n+1} = 2y_n - y_{n-1} + h^2 \sum_{i=1}^{m} b_i f(x_n + c_i h, Y_i), \tag{1.5}$$

$$Y_i = (1 + c_i)y_n - c_i y_{n-1} + h^2 \sum_{j=1}^{i-1} a_{ij} f(x_n + c_j h, Y_j), i = 1, 2, 3, .., m. \tag{1.6}$$

The parameters c_i, b_i, a_{ij} appearing in the method are assumed to be real numbers. For simplicity the two-step hybrid method can be summarized in Butcher tableau as given in Table 1.1.

Two-step hybrids methods, like other numerical methods for solving (1.3), can be divided into two major classes namely; explicit two-step hybrid method and implicit two-step hybrid method. It is said to be explicit if $a_{ij} = 0$ for $j \geq i$ and it is implicit if $a_{ij} \neq 0$ for $j \geq i$. Implicit hybrid method contains subclass, which is semi-implicit. It is called semi-implicit if $a_{ij} = 0$ for $j > i$ and the semi-implicit is in turn called singularly diagonally implicit if all the diagonal elements are equal.

Table 1.1: General coefficients of two-step hybrid method

$$
\begin{array}{c|cccccc}
c_1 & a_{11} & a_{12} & a_{13} & \dots & a_{1m} \\
c_2 & a_{21} & a_{22} & a_{23} & \dots & a_{2m} \\
c_3 & a_{31} & a_{32} & a_{33} & \dots & a_{3m} \\
c_4 & a_{41} & a_{42} & a_{43} & \dots & a_{4m} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
c_m & a_{m1} & a_{m2} & a_{m3} & \dots & a_{mm} \\
\hline
 & b_1 & b_2 & b_3 & \dots & b_m
\end{array}
$$

1.3 Algebraic Order Conditions of Two-step Hybrid Methods

Order conditions of a numerical method are said to be "relationships between the coefficients of the method that cause the successive terms in a Taylor expansion of the local truncation error to vanish", see Coleman (2003). In most cases they determine the order of convergent of the method. The Taylor expansion technique is so famous in the derivation of order conditions of most numerical methods especially families of Runge-Kutta methods and linear multistep methods. The Taylor series techniques require excessive use of machine especially when higher order conditions are required. However, other approach like the rooted tree techniques are also used in the derivation of order conditions of numerical methods. For example, see Butcher (2008). The order conditions of two-step hybrid method have been derived by Coleman (2003) using rooted tree approach. Hence, we adopt the outcome that produces the order conditions in this thesis.

1.3.1 Concept of Rooted Trees

For simplicity and convenience, the following form of (1.1) is considered

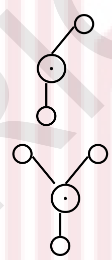
$$y'' = f(y), \tag{1.7}$$

where y is a solution vector of (1.7). Differentiating y repeatedly with respect to independent variable x gives

$$
\begin{aligned}
&y' = y', \\
&y'' = f, \\
&y''' = fy', \\
&y^{iv} = f_{yy}(y')^2 + f_y f, \\
&y^{v} = f_{yyy}(y')^3 + 3f_{yy}(fy') + f_y f_y y'.
\end{aligned}
$$

The expressions of the derivatives of y can be associated with a rooted tree. Rooted tree is a simple combinatorial graph, which have the properties of being connected, having no cycles, and having specific vertex designated as root, Butcher (2008). Two types of vertices are identified for a rooted tree namely; big and small vertices, which respectively correspond to y' and f. Small vertex is represented by a small circle while big vertex with a small circle containing a dot. The line connecting vertices, which can also be seen as 'branch' of the tree, represents differentiation with respect to the component of y' if it runs from small vertex to a big vertex and a differentiation with respect to y component if otherwise.

Illustration:



It can be seen from the illustration that small vertex has at most a son, which is the big vertex and the big vertex has all its sons as small vertices. This is because y' has only a derivative with respect to it self and f is a function of y only. Now, let $t = [t_1, t_2, t_3, ..., t_m]$ be a tree, which is obtained through the connection of the roots of all the trees $t_1, t_2, t_3, ..., t_m$ to a new big vertex and finally to a small root, which basically form the root of the resultant tree.

1.3.2 Generating Trees and their Corresponding Order Conditions

For every rooted tree associated with (1.3) there exist a corresponding summation of parameters of (1.5)-(1.6), which is equal to a numerical quantity. This is referred to as order condition generated by that particular tree. The illustration of how derivatives of y convert to tree t (given in subsection 1.3.1) and how t converts to summation of the parameters of the method (which is the order condition) is given in this subsection.

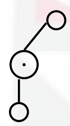
Let N be order of any tree then, according to Coleman (2003),

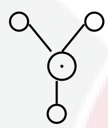
$$\sum_{i=1}^{m} b_i c_i^{N-2} = \frac{1+(-1)^N}{N(N-1)}, \tag{1.8}$$

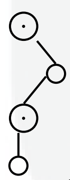
gives order condition for tree of the form t_{N1}. Furthermore, if all lower order conditions are satisfied then, for any natural numbers l, w

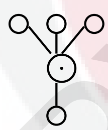
$$\sum_{ij} b_i c_i^{w-1} a_{ij} c_j^{l-1} = \frac{1}{l(l+1)} \left(\frac{1+(-1)^{w+l}}{(w+l+2)(w+l+1))} \right) + (-1)^{l-1} \left(\frac{1+(-1)^{w}}{(w+2)(w+1))} \right), \tag{1.9}$$

gives the order condition for any other tree of the form $t_{N\Gamma+1}$, where $\Gamma = 1, 2, 3, ...$

$f_y y' \Rightarrow\Rightarrow$  $\Rightarrow\Rightarrow [\phi]_2 \Rightarrow\Rightarrow \sum b_i c_i = 1$

$f_{yy}(y')^2 \Rightarrow\Rightarrow$  $\Rightarrow\Rightarrow [\tau', \tau']_2 \Rightarrow\Rightarrow \sum b_i c_i^2 = \frac{1}{6}$

$f_y f \Rightarrow\Rightarrow$  $\Rightarrow\Rightarrow [\tau]_2 \Rightarrow\Rightarrow \sum b_i a_{ij} = \frac{1}{12}$

$f_{yyy}(y')^3 \Rightarrow\Rightarrow$  $\Rightarrow\Rightarrow [\tau', \tau', \tau']_2 \Rightarrow\Rightarrow \sum b_i c_i^3 = 0$

⋮

⋮

⋮

1.3.3 Order Conditions

Following the strategy given in the immediate previous subsection, order conditions of two-step hybrid method up to order eight are given below, Coleman (2003).

Order 1: $$\sum b_i = 1, \tag{1.10}$$

Order 2: $$\sum b_i c_i = 0, \tag{1.11}$$

Order 3: $$\sum b_i c_i^2 = \frac{1}{6}, \sum b_i a_{ij} = \frac{1}{12}, \tag{1.12}$$

Order 4: $$\sum b_i c_i^3 = 0, \sum b_i c_i a_{ij} = \frac{1}{12}, \sum b_i a_{ij} c_j = 0, \tag{1.13}$$

Order 5: $$\sum b_i c_i^4 = \frac{1}{15}, \sum b_i c_i^2 a_{ij} = \frac{1}{30}, \sum b_i c_i a_{ij} c_j = -\frac{1}{60}, \tag{1.14}$$

$$\sum b_i a_{ij} a_{ik} = \frac{7}{120}, \sum b_i a_{ij} c_j^2 = \frac{1}{180}, \tag{1.15}$$

$$\sum b_i a_{ij} a_{jk} c_j = \frac{1}{360}, \tag{1.16}$$

Order 6: $$\sum b_i c_i^5 = 0, \sum b_i c_i^3 a_{ij} = \frac{1}{30}, \sum b_i c_i^2 a_{ij} c_j = 0, \tag{1.17}$$

$$\sum b_i c_i a_{ij} a_{ik} = \frac{1}{30}, \sum b_i c_i a_{ij} c_j^2 = \frac{1}{72}, \tag{1.18}$$

$$\sum b_i c_i a_{ij} a_{jk} = -\frac{1}{720}, \sum b_i a_{ij} a_{ik} c_k = -\frac{1}{120}, \tag{1.19}$$

$$\sum b_i a_{ij} c_j^3 = 0, \sum b_i a_{ij} c_j a_{ik} = \frac{1}{360}, \tag{1.20}$$

$$\sum b_i a_{ij} a_{jk} c_k = 0, \tag{1.21}$$

Order 7: $$\sum b_i c_i^6 = \frac{1}{28}, \sum b_i c_i^3 a_{ij} c_j^2 = -\frac{13}{2530}, \tag{1.22}$$

$$\sum b_i c_i^2 a_{ij} c_j^2 = \frac{1}{336}, \sum b_i c_i a_{ij} c_j^3 = -\frac{11}{1680}, \tag{1.23}$$

$$\sum b_i c_i a_{ij} a_{jk} c_k = \frac{17}{10080}, \sum b_i a_{ij} c_j^4 = \frac{1}{840}, \tag{1.24}$$

$$\sum b_i a_{ij} c_j a_{jk} c_k = -\frac{11}{15120}, \sum b_i a_{ij} a_{jk} c_k^2 = \frac{1}{10080}, \tag{1.25}$$

$$\sum b_i a_{ij} c_j a_{ik} c_k = \frac{29}{15120}. \tag{1.26}$$

All subscripts i, j, k run to m or less. It is obvious from the order conditions that the equations of order conditions to be satisfied by a two-step hybrid method increase more rapidly as the order of the method increases. As a remedial measure, Coleman (2003) proposed the following simplifying condition, which is capable of reducing independent order conditions

$$\sum_{j=1}^{m} a_{ij} c_j^{\Omega} = \frac{c_i^{\Omega+2} + (-1)^{\Omega}}{(\Omega+1)(\Omega+2)}, \Omega \geq 0. \tag{1.27}$$

1.4 Stability Analysis of Two-step Hybrid Method

To analyze the stability of two-step hybrid method, (1.5)-(1.6) is applied to homogeneous test equation

$$y''(x) = -\theta^2 y(x),\, \theta > 0,\, \theta \in R, \tag{1.28}$$

which yields

$$Y_i = (1+c_i)y_n - c_i y_{n-1} - \theta^2 h^2 \sum_{j=1}^{m} a_{ij} Y_j, y_{n+1} = 2y_n - y_{n-1} - \theta^2 h^2 \sum_{i=1}^{m} b_i Y_i,$$

$$Y = (I + z^2 A)^{-1}(e+c)y_n - (I + z^2 A)^{-1} c y_{n-1}, \tag{1.29}$$

where $z = \theta h$, $Y = [Y_1, Y_2, Y_3, ..., Y_m]^T$ and $e = [1, 1, 1, ..., 1]^T$.
Substituting (1.29) in update equation gives

$$y_{n+1} = T(z^2)y_n - D(z^2)y_{n-1}. \tag{1.30}$$

By re-writing and rearranging (1.30) we get

$$\varpi^2 - T(z^2)\varpi + D(z^2) = 0. \tag{1.31}$$

Equation (1.31) is the stability polynomial of the two-step hybrid method, where

$$T(z^2) = 2 - z^2 b^T (I + z^2 A)^{-1}(e+c), D(z^2) = 1 + z^2 b^T (I + z^2 A)^{-1} c.$$

Definition 1.2 *Two-step hybrid method is said to be absolutely stable if* $\forall\ z \in (0, z_s)$, $D(z^2) < 1$, $|T(z^2)| < 1 + D(z^2)$, *where* $(0, z_s)$ *is the interval of absolute stability. On the other hand, the method is said to be periodic with interval of periodicity* $(0, z_p)$ *if* $\forall\ z \in (0, z_p)$; $D(z^2) \equiv 1$ *and* $|T(z^2)| < 2$.

1.4.1 Determination of stability region/interval

The idea that brought about Definition 1.2 above is that the roots $(\varpi_{1,2})$ of the equation (1.31) must satisfy $|\varpi_{1,2}| < 1$ for the hybrid method to be absolutely stable or $|\varpi_{1,2}| = 1$ to be periodic, see Van der Houwen and Sommeijer (1987a). Therefore, the stability region of two-step hybrid method is a region enclosed by the set of points for which $|\varpi| = 1$.

To obtain the region we follow these few steps:

- substitute $\varpi = \exp(I\theta)$ into (1.31) for values $\theta \in [0,\, 2\pi]$,
- solve for z in the equation,
- plot all the z to trace out the boundary of the region,

- then trace the absolute stability or periodicity intervals on the graph, see Norazak Senu (2010).

Maple program developed for this task can be found in Appendix B.

1.5 Dispersion and Dissipation Analysis of Two-step Hybrid Methods

To study dispersion and dissipation of two-step hybrid method, the test equation (1.28) needs to be solved by (1.5)-(1.6). This task has been accomplished in Section 1.4. The solution of difference equation (1.30), (see, Van der Houwen and Sommeijer, 1987b; Ahmad et al., 2013a), is given by

$$y_n = 2|c||\rho|^n \cos(\omega + n\phi), \tag{1.32}$$

and the test problem (1.28) is satisfied by

$$y(x_n) = 2|\delta| \cos(\psi + nz), \tag{1.33}$$

where ρ is amplification factor, ϕ is the phase, ω, δ, ψ are real constants.

Definition 1.3 *From (1.32) and (1.33), the quantity $R(z) = z - \phi$ is called the dispersion or phase-lag of the two-step hybrid method. The method is said to have phase-lag or dispersion error of order q if $R(z) = O(z^{q+1})$. Furthermore, the quantity $S(z) = 1 - |\rho|$ is called dissipation or amplification error of the method. And the method is said to be dissipative of order r if $S(z) = O(z^{r+1})$.*

From the definition above, dispersion and dissipation errors of two-step hybrid method can be written respectively as

$$R(z) = z - \cos^{-1}\left(\frac{T(z^2)}{2\sqrt{D(z^2)}}\right), \; S(z) = 1 - \sqrt{D(z^2)}. \tag{1.34}$$

1.6 Construction of Exponentially Fitted and Trigonometrically Fitted Two-step Explicit Hybrid Methods

Exponentially and trigonometrically fitted methods are basically modifications of existing numerical methods, for example, Runge-Kutta method, Runge-Kutta Nyström method, multi-step method, collocation method, hybrid method and so on, to integrate exactly differential equations whose solutions are linear combination of functions of the form $\{x^j e^{\theta x}, x^j e^{-\theta x}\}$, where $\theta \in R/\imath R$, see Vanden Berghe et al. (1999), Berghe et al. (2000) and Simos (2002). In this section, we discuss the construction of recursive relations for exponentially and trigonometrically fitted methods based on two-step hybrid methods.

It is known that a numerical method with approximation y_n for true solution of a differential equation at a point n is said to integrate exactly $G(x)$ if $y_n = G(x_n)$, where $G(x)$ is the true solution of the given differential equation. Furthermore, for some internal stages Y_i of the numerical method (for example, equation (1.6)), $Y_i = G(x_n + c_i h)$ is true, see Berghe et al. (2000).

1.6.1 Exponentially Fitted Two-step Hybrid Method

Following the assertion given above, the construction goes thus; from equations (1.5)-(1.6) where $G(x) = e^{\pm\theta x}$;

$$e^{(x_n+c_ih)\theta} = (1+c_i)e^{x_n\theta} - e^{(x_n-h)\theta} + \theta^2 h^2 \sum_{j=1}^{m} a_{ij} e^{(x_n+c_jh)\theta},$$

$$e^{c_iz} = (1+c_i) - c_ie^{-z} + z^2 \sum_{j=1}^{m} a_{ij} e^{c_jz}. \tag{1.35}$$

Similarly,

$$e^{-c_iz} = (1+c_i) - c_ie^{z} + v^2 \sum_{j=1}^{m} a_{ij} e^{-c_jz}. \tag{1.36}$$

Also for the update stage:

$$e^{(x_n+h)\theta} = 2e^{x_n\theta} - e^{(x_n-h)\theta} + \theta^2 h^2 \sum_{j=1}^{m} b_i e^{(x_n+c_ih)\theta},$$

$$e^{z} = 2 - e^{-z} + z^2 \sum_{i=1}^{m} b_i e^{c_iz}, \tag{1.37}$$

similarly;

$$e^{-z} = 2 - e^{z} + z^2 \sum_{i=1}^{m} b_i e^{-c_iz}, \tag{1.38}$$

where $z = h\theta$.

Equations (1.35)-(1.38) are the recursive relations that guarantee an exact integration of (1.3) (whose solutions are exponential functions) by two-step hybrid method of the form (1.5)-(1.6).

1.6.2 Trigonometrically Fitted Two-step Hybrid Method

For trigonometrically fitted hybrid method $G(x) = e^{\imath\theta x}$, where $\imath = \sqrt{-1}$ is imaginary unit. Following the assertion given in the earlier part of this section we get the relations for internal and update stages as follows:

$$e^{\imath(c_iz)} = (1+c_i) - c_ie^{-\imath z} - z^2 \sum_{j=1}^{m} a_{ij} e^{\imath c_jz},$$

$$\cos(c_i z) = (1 + c_i) - c_i \cos(z) - z^2 \sum_{i=1}^{m} a_{ij} \cos(c_j z), \sin(c_i z) = c_i \sin(z)$$
$$- z^2 \sum_{i=1}^{m} a_{ij} \sin(c_j z). \tag{1.39}$$

Similarly;

$$\cos(z) = 1 - \frac{1}{2} z^2 \sum_{i=1}^{m} b_i \cos(c_i z), \sum_{i=1}^{m} b_i \sin(c_i z) = 0. \tag{1.40}$$

1.7 Objectives of the Thesis

The objective of this thesis is to develop an improved numerical methods that can integrate non-stiff second order IVPs of the form (1.3) (whose solution is oscillatory in nature) accurately and efficiently. To accomplish this, we propose the following methods:

1. Improved explicit two-step hybrid methods for solving special second order IVPs with oscillatory solutions.
2. Trigonometrically fitted two-step hybrid methods for the solution of special second order IVPs with oscillatory solutions.
3. Semi-implicit two-step hybrid methods for solving special second order IVPs with oscillatory solutions.

1.8 Organization of the Thesis

In Chapter 1 of this thesis, introductory background of second order initial value problems and theory of existence of their solution is presented. Two-step hybrid method is briefly discussed and its algebraic order conditions. Stability as well as dispersion and dissipation analysis of two-step hybrid method is presented. Overview of the modification of two-step hybrid methods to solve problems whose solutions are linear combination of trigonometric functions is presented. In Chapter 2, literature review is given. In Chapter 3, zero-dissipative and higher order dispersive fifth order five stage hybrid methods are derived. Stability of the methods is analyzed and numerical results are presented. Trigonometrically fitted based on zero-dissipative hybrid methods are derived in Chapter 4. In Chapter 5, we derive zero-dissipative semi-implicit hybrid methods. Conclusion of the thesis is given in Chapter 6.

REFERENCES

Ahmad, S. Z., Ismail, F., Senu, N. and Suleiman, M. 2013a. Semi implicit hybrid methods with higher order dispersion for solving oscillatory problems. *Abstract and Applied Analysis* 2013.

Ahmad, S. Z., Ismail, F., Senu, N. and Suleiman, M. 2013b. Zero-dissipative phase-fitted hybrid methods for solving oscillatory second order ordinary differential equations. *Applied Mathematics and Computation* 219 (19): 10096–10104.

Al-Khasawneh, R. A., Ismail, F. and Suleiman, M. 2007. Embedded diagonally implicit Runge-Kutta-Nyström 4 (3) pair for solving special second-order IVPs. *Applied Mathematics and Computation* 190 (2): 1803–1814.

Berghe, G. V., Meyer, H. D., Daele, M. V. and Hecke, T. V. 2000. Exponentially fitted Runge-Kutta methods. *Journal of Computational and Applied Mathematics* 125 (1): 107–115.

Bettis, D. G. 1973. A Runge-Kutta Nyström algorithm. *Celestial Mechanics* 8 (2): 229–233.

Butcher, J. C. 2008. *Numerical methods for ordinary differential equations*. John Wiley & Sons.

Cash, J. R. 1981, High order P-stable formulae for the numerical integration of periodic initial value problems.

Chawla, M. M. 1981. Two-step fourth order P-stable methods for second order differential equations. *BIT Numerical Mathematics* 21 (2): 190–193.

Chawla, M. M., Al-Zanaidi, M. A. and Boabbas, W. M. 1996. Extended two-step P-stable methods for periodic initial-value problems. *Neural, Parallel & Scientific Computations* 4 (4): 505–521.

Coleman, J. P. 1989. Numerical methods for $y'' = f(x, y)$ via rational approximations for the cosine. *IMA Journal of Numerical Analysis* 9 (2): 145–165.

Coleman, J. P. 2003. Order conditions for a class of two-step methods for $y'' = f(x, y)$. *IMA Journal of Numerical Analysis* 23 (2): 197–220.

De Vogelaere, R. 1955. A method for the numerical integration of differential equations of second order without explicit first derivatives. *Journal of Research of the National Bureau of Standard* 54: 119–125.

Dizicheh, A., Ismail, F., Md. Arifin, N. and Abu Hassan, M. 2012. A class of two-step hybrid trigonometrically fitted explicit Numerov-type method for second-order IVPs, 182–192. Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran: International Scientific Publication and Consulting Services.

Dormand, J. R., El-Mikkawy, M. E. A. and Prince, P. J. 1987. Families of Runge-Kutta-Nyström formulae. *IMA Journal of Numerical Analysis* 7 (2): 235–250.

Fang, Y. and Wu, X. 2007. A trigonometrically fitted explicit hybrid method for the numerical integration of orbital problems. *Applied Mathematics and Computation* 189 (1): 178–185.

Fang, Y. and Wu, X. 2008. A trigonometrically fitted explicit Numerov-type method for second-order initial value problems with oscillating solutions. *Applied Numerical Mathematics* 58 (3): 341–351.

Franco, J. M. 2003. A 5 (3) pair of explicit ARKN methods for the numerical integration of perturbed oscillators. *Journal of Computational and Applied Mathematics* 161 (2): 283–293.

Franco, J. M. 2004. Exponentially fitted explicit Runge-Kutta-Nyström methods. *Journal of Computational and Applied Mathematics* 167 (1): 1–19.

Franco, J. M. 2006. A class of explicit two-step hybrid methods for second-order IVPs. *Journal of Computational and Applied Mathematics* 187 (1): 41–57.

Huu Cong, N. 1993. A-stable diagonally implicit Runge-Kutta-Nyström methods for parallel computers. *Numerical Algorithms* 4 (2): 263–281.

Mohamad, M., Senu, N., Suleiman, M. and Ismail, F. 2011. High Order Explicit Runge-Kutta-Nyström Methods For Periodic Initial Value Problems, 78–93. Malaysia.

Norazak Senu. 2010. *Runge-Kutta-Nyström methods for solving oscillatory problems*. PhD, Universiti Putra Malaysia. Malaysia.

Nyström, E. J. 1925. *ber die numerische Integration von Differentialgleichungen:(Mitgeteilt am 23. Sept. 1925 von E. Lindelf und KF Sundman)*. Societas scientiarum Fennica.

Rabiei, F., Ismail, F., Norazak, S. and Seddighi, S. 2012. Accelerated Runge-Kutta-Nyström Method for Solving Autonomous Second-Order Ordinary Differential Equations $y'' = f(y)$. *World Applied Sciences Journal* 17 (12): 1549–1555.

Samat, F., I. F. and Suleiman, M. 2012, High Order Explicit Hybrid Methods for Solving Second-Order Ordinary Differential Equations.

Senu, N., Suleiman, M. and Ismail, F. 2009. An embedded explicit Runge-Kutta-Nyström method for solving oscillatory problems. *Physica Scripta* 80 (1): 015005.

Senu, N., Suleiman, M., Ismail, F. and Othman, M. 2010. Kaedah Pasangan 4 (3) Runge-Kutta-Nyström untuk Masalah Nilai Awal Berkala. *Sains Malaysiana* 39 (4): 639–646.

Simos, T. E. 1999. Explicit eighth order methods for the numerical integration of initial-value problems with periodic or oscillating solutions. *Computer Physics Communications* 119 (1): 32–44.

Simos, T. E. 2002. Exponentially-fitted Runge-Kutta-Nyström method for the numerical solution of initial-value problems with oscillating solutions. *Applied Mathematics Letters* 15 (2): 217–225.

Simos, T. E., Dimas, E. and Sideridis, A. B. 1994. A Runge Kutta Nyström method for the numerical integration of special second-order periodic initial-value problems. *Journal of Computational and Applied Mathematics* 51 (3): 317–326.

Tsitouras, C. 2002. Explicit two-step methods for second-order linear IVPs. *Computers & Mathematics with Applications* 43 (8): 943–949.

Van de Vyver, H. 2005. A Runge–Kutta–Nyström pair for the numerical integration of perturbed oscillators. *Computer Physics Communications* 167 (2): 129–142.

Van der Houwen, P. J. and Sommeijer, B. P. 1987a. Explicit Runge-Kutta-Nyström methods with reduced phase errors for computing oscillating solutions. *SIAM Journal on Numerical Analysis* 24 (3): 595–617.

Van der Houwen, P. J. and Sommeijer, B. P. 1987b. Explicit Runge-Kutta Nyström methods with reduced phase errors for computing oscillating solutions. *SIAM Journal on Numerical Analysis* 24 (3): 595–617.

Vanden Berghe, G., De Meyer, H., Van Daele, M. and Van Hecke, T. 1999. Exponentially-fitted explicit Runge-Kutta methods. *Computer Physics Communications* 123 (1): 7–15.