UNIVERSITI PUTRA MALAYSIA

MODIFICATION OF INTERVAL SYMMETRIC SINGLE -STEP PROCEDURE FOR SIMULTANEOUS BOUNDING POLYNOMIAL ZEROS

NORAINI BINTI JAMALUDIN

FS 2014 33

MODIFICATION OF INTERVAL SYMMETRIC SINGLE -STEP PROCEDURE FOR SIMULTANEOUS BOUNDING POLYNOMIAL ZEROS

By

NORAINI BINTI JAMALUDIN

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirements for the Degree of Master of Science

January 2014

DEDICATIONS

to

Azizah bt Yahya

Jamaludin bin Idris

Mohd Khairul bin Jamaludin

Norazrin bt Jamaludin

Zarina bt Sahri

Naim Hakimi bin Mohd Khairul

Fathul Rahman bin Idrus

and

all my family members

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

MODIFICATION OF INTERVAL SYMMETRIC SINGLE- STEP PROCEDURE FOR SIMULTANEOUS BOUNDING POLYNOMIAL ZEROS

By

NORAINI BINTI JAMALUDIN

January 2014

Chairman: Mansor bin Monsi, PhD

Faculty: Science

The focus of this research is on the bounding of simple and real polynomial zeros simultaneously, focusing on the interval analysis approaches. This procedure started with some disjoint intervals $X_i^{(0)}$ for $i = 1, \ldots, n$ each of which contains a zero of the polynomial and finally produced successively smaller closed bounded intervals, which always converge to the zeros x_i^* for $i = 1, \ldots, n$ respectively. In relation to that, the previous work on Interval Symmetric Single-step (ISS2) procedure is investigated to ensure this procedure is useful for solving polynomials. Thus, this procedure is extended to some modifications in order to improve the efficiency of the procedure.

Starting from the authentic ISS2 procedure, four modified procedures are developed. The procedures are Interval Symmetric Single-Step (ISS2-5D) procedure, Interval Zoro-Symmetric Single-Step (IZSS2-5D) procedure, Interval Midpoint Symmetric Single-Step (IMSS2-5D) procedure and Interval Midpoint Zoro-Symmetric Single-Step (IMZSS2-5D) procedure. The programming language Intlab toolbox for Matlab is used to record the numerical results, whereby the stopping criterion used is $w_i^{(k)} \leq 10^{-10}$. The results are numerically compared to the original ISS2 procedure to supervise the improvements and efficiencies of the modified procedures.

In order to assure that the outcomes of the procedures are promising, convergence rate for each modified procedures is analyzed for comparing purposes. Other than that, the analysis of inclusion to certify the convergence of the modified procedures is included. All the modifications

are proven to have better rate of convergences and these are well-supported on the reduction of CPU times, number of iterations and the value of the interval width of the procedures. In a nutshell, this study reveals that the new modified procedures are capable and efficient for bounding the simple and real polynomial zeros simultaneously.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Master Sains

PENGUBAHSUAIAN PADA PROSEDUR SELANG LANGKAH-TUNGGAL DALAM MEMERANGKAP SUATU PENSIFAR POLINOMIAL SECARA SERENTAK

Oleh

NORAINI BINTI JAMALUDIN

Januari 2014

Pengerusi : Mansor bin Monsi, PhD

Fakluti: Sains

Fokus penyelidikan kami adalah menghadkan punca nyata dan ringkas polinomial secara serentak yang memfokuskan kepada pendekatan analisis selang. Prosedur ini bermula dengan beberapa selang permulaan yang tidak bercantum $X_i^{(0)}$ bagi $i = 1, \ldots, n$ yang mana setiap satunya mengandungi punca polinomial dan akhirnya menghasilkan selang rapat yang lebih kecil secara berturutan yang mana sentiasa menumpu kepada punca-punca polinomial x_i^* bagi $i = 1, \ldots, n$. Selanjutnya, hasil kerja terdahulu terhadap prosedur selang langkah-tunggal bersimmetri ISS2 dikaji bagi memastikan prosedur ini dapat digunakan untuk menyelesaikan polinomial. Prosedur ini kemudiannya diperluaskan dengan beberapa pengubahsuaian bagi tujuan meningkatkan kecekapan prosedur.

Bermula dengan prosedur asal ISS2, kami menghasilkan empat prosedur-prosedur baru terubahsuai yang mana dibentangkan sebagai sumbangan utama kami dalam tesis ini. Prosedur ini adalah prosedur ISS2-5D, prosedur IZSS2-5D, prosedur IMSS2-5D and prosedur IMZSS2-5D. Keputusan-keputusan berangka direkod dengan menggunakan perisian Matlab dan dibantu oleh perisian Intlab dimana syarat berhenti program yang dikenakan adalah $w_i^{(k)} \leq 10^{-10}$. Keputusan-keputusan secara berangka dibandingkan dengan prosedur asal ISS2 untuk melihat peningkatan dan kecekapan prosedur-prosedur terubahsuai.

Bagi menyakinkan bahawa prosedur-prosedur ini berjaya, kami juga menganalisa kadar penumpuan bagi setiap prosedur terubahsuai untuk dibandingkan. Kami juga menyertakan analisa rangkuman bagi menjamin penumpuan prosedur tersebut. Kesemua prosedur-prosedur terubahsuai telah terbukti mempunyai kadar penumpuan yang lebih baik dan disokong

dengan pengurangan masa pemprosesan, bilangan lelaran dan nilai lebar lelaran bagi prosedur-prosedur. Pada kesimpulannya, kajian ini menunjukkan bahawa prosedur-prosedur baru terubahsuai berkebolehan dan cekap untuk menghad punca nyata dan ringkas polinomial secara serentak.

ACKNOWLEDGEMENTS

In the name of Allah, the most Compassionate and the most Merciful. I would like to extend my gratitude to those involved in helping me complete this research in any way possible.

Alhamdulillah, praises to The Al-Mighty who has blessed me with strength and health to make it through this laborious yet amazing and knowlegable journey and peace be upon the Prophet Muhammad S.A.W.

First and foremost, I wish to express my sincere and deepest gratitude to the chairman of the Supervisory Committee, Dr Mansor Monsi for his invaluable insights and guidance throughout the duration of the study.

I'm also utterly grateful to the members of Supervisory Committee whose contributions of ideas helped in improving my research work immensely. Special acknowledgement is also extended to Dr Fakhrul Razman from Uitm Shah Alam for his assistance in developing algorithms and its implementations in programming language (Intlab). Special thanks are also expressed to the first person who brought me to closer insight of the Matlab programming, theTea Boo Chan.

My deepest appreciation goes to my humble family, especially my parents, Jamaludin bin Idris and Azizah binti Yahya, my siblings Mohd Khairul, Zarina and Norazrin, my fiancé Fathul Rahman bin Idrus and the rest of the family members. Thank you for being supportive, giving me encouragement and endless love, especially in enduring the hard phases of my studies. May Allah beautify their lives, here and hereafter.

Finally, a special thanks to my best friend Atiyah bt Wan Mohd Sham. It has been of a great fortune to be able to work alongside her. I sincerely believe that the outcome of our cooperation has lead to more than either one of us could have reached by owned selves. I wish her the best in her future career as a mathematician and hope that the opportunity to work with her will surface again. I would also wish to thank all my friends especially Nur Shakila, Iskandar Shah, Yusra, Nurzeehan, Iszdihar Ilzam, Syahidatussyakirah and Shazana, for their help, advices and motivation in the completion of this research.

Special thanks to the staffs of Department of Mathematics, Faculty of Science, staff of Sultan Abdul Samad Library for their excellent facilities and services. In terms of financial support, I am also thankful to School Graduate of Studies (SGS) for the Grant Research Funds (GRF) and Ministry of Higher Education (MOHE) for the MyBrain15.

TABLE OF CONTENTS

		Page
DEDICATIONS		ii
ABSTRACT		iii
ABSTRAK		v
ACKNOWLEDGEMENTS		vii
APPROVAL		ix
DECLARATION		xi
LIST OF TABLES		xvi
LIST OF FIGURES		xvii
LIST OF ABBREVIATIONS		xix

CHAPTER

1	**INTRODUCTION**	
1.1	Background	1
1.2	Fundamental Definitions and Properties of Interval Analysis	1
1.3	Interval Evaluation of Real Function	9
1.4	The Concept of the *R*-order of Convergence	10
1.5	Research Objectives	12
1.6	Scope of the Problem	12
1.7	Thesis Outline	14
2	**LITERATURE REVIEW**	
2.1	Simultaneous Methods in Solving Zeros of Polynomials	16
2.2	Interval Iterative Procedures	20
3	**THE INTERVAL SINGLE-STEP (ISS2) PROCEDURE** (The original procedure)	
3.1	Introduction	27
3.2	Algorithm of ISS2 Procedure	29

3.3 Analysis of *R*-order of Convergence of ISS2 Procedure 30

4 **THE INTERVAL SYMMETRIC SINGLE-STEP (ISS2-5D) PROCEDURE**
(The first modification)
4.1 Introduction 33
4.2 Algorithm of ISS2-5D Procedure 33
4.3 Analysis of *R*-order of Convergence of ISS2-5D Procedure 34
4.4 Numerical Results and Discussion 53

5 **THE INTERVAL ZORO SYMMETRI SINGLE-STEP (IZSS2-5D) PROCEDURE**
(The second modification)
5.1 Introduction 58
5.2 Algorithm of IZSS2-5D Procedure 59
5.3 Analysis of *R*-order of Convergence of IZSS2-5D Procedure 60
5.4 Numerical Results and Discussion 68

6 **THE INTERVAL MIDPOINT-SYMMETRIC SINGLE-STEP (IMSS2-5D)PROCEDURE**
(The third modification)
6.1 Introduction 73
6.2 Algorithm of IMSS2-5D Procedure 73
6.3 Analysis of *R*-order of Convergence of IMSS2-5D Procedure 74
6.4 Numerical Results and Discussion 77

7 **THE INTERVAL MIDPOINT ZORO-SYMMETRIC SINGLE-STEP (IMZSS2-5D) PROCEDURE**
(The fourth modification)
7.1 Introduction 81
7.2 Algorithm of IMZSS2-5D Procedure 81

7.3	Analysis of *R*-order of Convergence of IMZSS2-5D Procedure	83
7.4	Numerical Results and Discussion	86

8 GENERAL CONCLUSION AND RECOMMENDATIONS FOR FUTURE RESEARCH

8.1	Conclusion	92
8.2	Summary of the Numerical Results and Discussion	93
8.3	Recommendations for Future Research	98

REFERENCES	103
APPENDIX	107
BIODATA OF STUDENT	109
LIST OF PUBLICATIONS	110

LIST OF TABLES

Table		Page
4.1	Number of iteration and CPU times for all the test polynomials.	54
4.2	Values of intervals for each component i for test polynomial 2	55
	4.2(a) Component $i = 1$	55
	4.2(b) Component $i = 2$	55
	4.2(c) Component $i = 3$	55
	4.2(d) Component $i = 4$	54
	4.2(e) Component $i = 5$	54
4.3	Widths of the intervals $w_i^{(k)}$ for each step at iteration k for test polynomial 2	57
5.1	Number of iteration and CPU times for all test polynomials	68
5.2	Values of intervals for each component i for test polynomial 2	69
	5.2(a) Component $i = 1$	69
	5.2(b) Component $i = 2$	70
	5.2(c) Component $i = 3$	70
	5.2(d) Component $i = 4$	70
	5.2(e) Component $i = 5$	70
5.3	Widths of the intervals $w_i^{(k)}$ for each step at iteration k for test polynomial 2	71
6.1	Number of iteration and CPU times for all test polynomials	77
6.2	Values of intervals for each component i for test polynomial 2	79
	6.2(a) Component $i = 1$	79
	6.2(b) Component $i = 2$	79
	6.2(c) Component $i = 3$	79
	6.2(d) Component $i = 4$	79
	6.2(e) Component $i = 5$	79
6.3	Widths of the intervals $w_i^{(k)}$ for each step at iteration k for test polynomial 2	80
7.1	Number of iteration and CPU times for all test polynomials	87

7.2 Values of intervals for each component i for test polynomial 2 88
7.2(a) Component $i = 1$ 88
7.2(b) Component $i = 2$ 88
7.2(c) Component $i = 3$ 88
7.2(d) Component $i = 4$ 89
7.2(e) Component $i = 5$ 89

7.3 Widths of the intervals $w_i^{(k)}$ for each step at iteration k for test polynomial 2 90

8.1 Number of iterations(k) for all procedures 94

8.2 CPU times(s) for all procedures 94

8.3 Widths of the intervals $w_i^{(k)}$ for each step at iteration k for all procedures for test polynomial 2 97

LIST OF FIGURES

Figure		Page
4.1(a)	The selection of new bounds of $X_i^{(k,1)}$ if $(p(x_i^{(k)}) \geq 0)$	41
4.1(b)	The selection of new bounds of $X_i^{(k,1)}$ if $(p(x_i^{(k)}) \geq 0)$	42
4.2(a)	The selection of new bounds of $X_i^{(k,1)}$ if $(p(x_i^{(k)}) < 0)$	43
4.2(b)	The selection of new bounds of $X_i^{(k,1)}$ if $(p(x_i^{(k)}) < 0)$	44
4.3(a)	The selection of new bounds of $X_i^{(k,1)}$ if $(p(x_i^{(k)}) > 0)$	46
4.3(b)	The selection of new bounds of $X_i^{(k,1)}$ if $(p(x_i^{(k)}) > 0)$	47
4.4(a)	The selection of new bounds of $X_i^{(k,1)}$ if $(p\left(x_i^{(k)}\right) < 0)$	48
4.4(b)	The selection of new bounds of $X_i^{(k,1)}$ if $(p\left(x_i^{(k)}\right) < 0)$	49
4.5	Bar chart of number of iterations against test polynomials	54
4.6	Bar chart of CPU times against test polynomials	54
5.1	Bar chart of number of iterations against test polynomials	68
5.2	Bar chart of CPU times against test polynomials	69
6.1	Bar chart of number of iterations against test polynomials	78
6.2	Bar chart of CPU times against test polynomials	78
7.1	Bar chart of number of iterations against test polynomials	87
7.2	Bar chart of CPU times against test polynomials	87
8.1	Bar chart of number of iterations against test polynomials for all procedures	95
8.2	Bar chart of CPU times against test polynomials for all procedures	95

LIST OF ABBREVIATIONS

R	Real numbers
$I(R)$	Set of real interval numbers
x^*	Zeros
$x_i^{(k)}$	Point x of component i at iteration k
$X_i^{(k)}$	Interval X of component i at iteration k
$X_i^{(0)}$	Initial interval X of component i
x_{iI}	Infimum of interval X of component
x_{iS}	Supremum of interval X of component i
$X \cap Y$	Intersection of interval $X \cap Y$
$X \subseteq Y$	Inclusion of intervals.
$w_i^{(k)}$	Width of interval of component i at iteration k
$\text{midpoint}(X)$	Midpoint of interval X
$p(x)$	Polynomial of x
$p'(x_i^{(k)})$	Interval gradients of the polynomials of $x_i^{(0)}$
$O_R(I, x^*)$	R-order of procedure I which converge to x^*
$R_P(w^{(k)})$	R-factor of a null sequence $w^{(k)}$
max	Maximum
min	Minimum

CHAPTER 1

INTRODUCTION

1.1 Background

Interval analysis is a branch of applied mathematics. Referring to Neumaier (1990), interval analysis is considered to be an elegant tool for practical work with inequalities, approximate numbers, error bounds and more generally with certain convex and bounded sets. By using interval numbers we can define another number system where an interval number consists of a pair of real numbers representing the lower and upper bound of the parameter range.

In depth discussion of topics related to interval analysis could be found in books by Moore (1966), Alefeld and Herzberger (1983), Neumaier (1990), Hansen (1992), and more recently with Jaulin et al. (2001) and Hansen and Walster (2004). Interval analysis has the advantage of providing rigorous bounds for the exact solutions.

Recently, interval analysis has been widely applied in various kinds of problem such as finding the bounds on the value of a function, finding zeros of polynomial, solving equation or a system of equation, optimization, differential equation as well as integral equation.

It is of evidence that significant improvements are possible in interval analysis. With regards to contributing to the development of this field, research is conducted, coming up with this thesis entitled "*Modifications on the Interval Symmetric Single -Step Procedure for Simultaneous Bounding of Real Polynomial Zeros*".

1.2 Fundamental Definitions and Properties of Interval Analysis

Description on some of the definitions and properties on interval analysis to be used throughout this research are included in this section of this thesis. The following definitions and properties can be found in Alefeld and Herzberger (1983).

An interval number is defined as an ordered pair of real numbers

$$A = [a_1, a_2] = \{x : a_1 \le x \le a_2,\ a_1, a_2 \in \mathbb{R}\} \quad (1.1)$$

An interval parameter is written with brackets where $a_1 \in \mathbb{R}$ is the left endpoint or the infimum of A or $\inf(A) = a_1$, and $a_2 \in \mathbb{R}$ is the right endpoint or the suprimum of A or $\sup(A) = a_2$.

In interval analysis the real number is denoted by $\mathbb{R}$ and member of $\mathbb{R}$ are denoted by lowercase letters $a, \ldots, x, y, z$. A subset of $\mathbb{R}$ of the form $[a, b]$ is called a bounded closed real interval. The set of all bounded closed real interval is denoted by $I(\mathbb{R})$ and the member of $I(\mathbb{R})$ by uppercase letter $A, B, \ldots, Z$. Real number $x \in \mathbb{R}$ may be considered special members $[x, x]$ from $I(\mathbb{R})$, and they will generally be called point interval.

Definition 1.2.1:

Two intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are said to be equal that is $A = B$ if they are in the sense of the theoretical set. From Definition 1.2.1 it follows that

$$A = B \Leftrightarrow a_1 = b_1,\ a_2 = b_2.$$

The relation "=" between two elements in $I(\mathbb{R})$ is reflexive, symmetric, and transitive.

Definition 1.2.2:

Let $* \in \{+, -, \cdot, /\}$ be a binary operation on the set of real numbers $\mathbb{R}$. If $A, B \in I(\mathbb{R})$, we define

$$A * B = \{a * b : a \in A,\ b \in B\} \quad (1.2)$$

a binary operation on the real interval $I(\mathbb{R})$.

From Definition 1.2.2 the operations on intervals $A = [a_1, a_2]$ and $B = [b_1, b_2]$ can be calculated explicitly as

$$A + B = [a_1 + b_1, a_2 + b_2]$$

$$A - B = [a_1 - b_2, a_2 - b_1] = A + (-1)B$$

$$A \cdot B = [\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$$

and if [illegible], then

[illegible]

[illegible]

otherwise, [illegible] undefined if [illegible].

The following definitions and propositions can be found in Monsi (1988).

Definition 1.2.3:

The set [illegible] is a degenerate interval (or the set of point interval) if and only if [illegible]. The set [illegible] and the set [illegible] of real numbers are isomorphic. This permits a meaning to be given to [illegible].

Definition 1.2.4:

If [illegible] and [illegible] then

[illegible]

[illegible]

[illegible]

and if [illegible], then

[illegible]

[illegible]

Proposition 1.2.1:

Interval arithmetic is *inclusion monotonic* that is to say, if [illegible] then for all [illegible],

[illegible]

Proof: By Definition 1.2.2

[illegible]

[illegible]

[illegible]

Definition 1.2.5:

Let [illegible] be given. Then the intersection [illegible] of [illegible] and [illegible] is defined

[illegible] (1.2.5)

The intersection between the two intervals will yield an interval with rigorously narrow width which guaranteed to contain at least a zero.

Proposition 1.2.2:

[illegible]

[illegible]

[illegible]

Proof of [illegible]:

By Definition 1.2.5

[illegible]

[illegible]

[illegible].

Proof of [illegible]:

By definition 1.2.5

[illegible]

So,

[illegible]

that is

[illegible]

and

[illegible]

thus

[illegible]

Proof of [illegible]:

By [illegible] so

[illegible]

Conversely, if [illegible] then by Definition 1.2.5

$$[illegible]$$

Therefore

$$[illegible]$$

Interchanging [illegible] and [illegible] and using [illegible], it follow that

$$[illegible]$$

Proposition 1.2.3:

If [illegible] and [illegible] are members of the real interval [illegible]. Then it follows that

[illegible] (associativity of addition);

[illegible] (associativity of multiplication);

[illegible] (commutativity of addition);

[illegible] (commutativity of multiplication).

The proof of [illegible] follow from Definition 1.2.2 .

Proposition 1.2.4:

If [illegible] and [illegible], then [illegible] and [illegible] are the unique element with respect to addition and multiplication, that is

[illegible] for all [illegible],

[illegible] for all [illegible],

Proof of [illegible]:

$$[illegible]$$

Conversely, suppose that

$$[illegible]$$

Then, setting [illegible],

$$[illegible]$$

that is [illegible], So

4 o _o

Proof of :

Suppose that ; L s _. Then k :4 ; o

; L [U = U s _

L = =

L

So,

k; L s_o : ; L ;

Conversely, suppose that

; L k :4 ; o

Then, in particular, L holds with L s _ ; whence ; L s _ ; So

; L k :4 ; o k; L s _o

Proposition 1.2.5:

k :4 o : L r k L r _o N k L r _o

Proof:

k L r_o : L r

k L r o

k L r N L r o

L r N L r

Proposition 1.2.6:

Interval arithmetic is subdistributive; that is k :4 ; o,

: E ; C E

Proof:

: E L : E

L E

C *E*

L E

Proposition 1.2.7:

Let [illegible] where [illegible] and [illegible] are given. Then

[illegible]

[illegible]

[illegible]).

Proof of [illegible]:

[illegible]

Also,

[illegible]

Proof of [illegible]:

If [illegible] is defined then [illegible]. So, either [illegible]. If [illegible] then

[illegible]

So,

(i) [illegible]

(ii) [illegible]

(iii) [illegible]

The case [illegible] is similar. So [illegible]

Example 1.2.1:

Let L su where r

If L sy where O = 5. Then

L sysu L ssu yyussu yyu=

L suy.

If L Fy Fs , where = 6 Q r. Then

L Fy Fssu

L Fy Fyu Fs Fsu Fy Fyu Fs Fsu=

L Fy Fsu.

If L Fys , where = 5 Q r Q = 6. Then

L Fyssu

L Fy Fyussu Fy Fyussu=

L Fys.

Definition 1.2.6:

The distance between two intervals L = 5 = 6 L 5 6 :4; is defined as

: ; L = 5 F 5 = 6 F 6 stw;

The following properties is hold

; R r and M: ; L r; L ,

: ; Q : ; E ; (triangle inequality).

Definition 1.2.7:

The absolute value of an interval L = 5 = 6 :4; is defined as

L : rr ; L = 5 = 6 stx;

can also be written

L sty;

Clearly, if :4; then

C Q

Definition 1.2.8:

The width $d(X)$ of an interval $X = [x_1, x_2]$ where $X \in I(\mathbb{R})$ is defined by

$$d(X) = x_2 - x_1 \geq 0$$

Definition 1.2.9:

The midpoint $m(X)$ of $X \in I(\mathbb{R})$ is defined by

$$m(X) = \frac{1}{2}(x_1 + x_2)$$

Theorem 1.2.1:

Every sequence of intervals $\{X^{(k)}\}_{k=1}^{\infty}$ for which

$$X^{(1)} \supseteq X^{(2)} \supseteq X^{(3)} \supseteq \cdots$$

is valid converges to the interval $\bigcap_{k=1}^{\infty} X^{(k)}$.

1.3 Interval Evaluation of Real Function

In this section, we assume f is a continuous real function. An expression $f(x)$ belonging to f is a calculating procedure that will determine a value of the function f for every argument x. If an expression belonging to f also contains constants $a^{(1)}, \ldots, a^{(n)}$ then this can be clarifies by writing $f(x; a^{(1)}, \ldots, a^{(n)})$. Assume that each constant $a^{(i)}, i = 1, \ldots, n$, occurs only once in an expressions. Otherwise, introduce new constant that made equivalent for the multiple occurrences of these new constants, one may transform the expression into the required form.

The following expression referred to Alefeld and Herzberger (1983)

$$W(f; X, A^{(1)}, \ldots, A^{(n)})$$

$$= \{ f(x; a^{(1)}, \ldots, a^{(n)}) \mid x \in X, a^{(i)} \in A^{(i)}, i = 1, \ldots, n \}$$

$$= \left[\min_{x \in X,\, a^{(i)} \in A^{(i)}} f(x; a^{(1)}, \ldots, a^{(n)}), \max_{x \in X,\, a^{(i)} \in A^{(i)}} f(x; a^{(1)}, \ldots, a^{(n)}) \right]$$

denote the interval of all values of the function f when $x \in X$ and $a^{(i)} \in A^{(i)}, i = 1, \ldots, n$ are considered independent of each other. This definition is independent of the expression for f.

Let an expression be given for the function f. In this expression all operands are replaced by intervals and all operations by interval operations resulting in the expression $f(X_1, \dots, X_n)$. Then, this is called *interval evaluation or interval arithmetic evaluation* if all operands within the domain of definition of f. The constants $A_1, \dots, A_n$ as well as the variable x are replaced by intervals.

The following example illustrates to well-defined interval evaluation when all operands by interval operations leads to interval expression.

Example 1.3.1:

Let $f(x) = 5x^2 - x$ and $X = [0,1]$, then we get

$$W(f,[0,1]) = \{5x^2 - x : 0 \le x \le 1\} = \left[-\frac{1}{20}, 4\right].$$

The example for equivalent expressions in multiple occurrences of x:

Expression 1 (Triple occurrences of x):

$$f_1(x) = 5x^2 - x, \quad f_1([0,1]) = 5[0,1]^2 - [0,1] = [-1,5].$$

Expression 2 (Double occurrences of x):

$$f_2(x) = x(5x - 1), \quad f_2([0,1]) = [0,1](5[0,1] - 1) = [-1,4].$$

Expression 3 (Single occurrences of x):

$$f_3(x) = 5\left[\left(x - \frac{1}{10}\right)^2 - \frac{1}{100}\right], \quad f_3([0,1]) = 5\left\{\left([0,1] - \frac{1}{10}\right)^2 - \frac{1}{100}\right\}$$

$$= \left[-\frac{1}{20}, 4\right],$$

where $f_1 \supseteq f_2 \supseteq f_3 = W(f,[0,1])$.

Clearly the interval evaluation of a function f is dependent on the choice of expression for f. This theorem is proven in Alefeld and Herzberger (1983).

1.4 The Concept of the *R*-order of Convergence

The *R*-order of convergence of an iterative procedure is used in this thesis as a measure of the asymptotic convergence rate of the procedure. The concept of *R*-order of convergence is discussed in detail in Ortega and Rheinboldt (1970), Alefeld and Herzberger (1983) and Monsi and Wolfe (1988).

The proof of following theorem is in Ortega and Rheinboldt (1970).

Theorem 1.4.1

Let I be an iteration procedure with the limit x^*, and let $C(I, x^*)$ be the set of all sequences generated by I having the properties that $\lim_{k\to\infty} x^{(k)} = x^*$ and $x^{(k)} \in D, k \geq 0$. If there exist a $p \geq 1$ and a constant c such that for all $\{x^{(k)}\} \in C(I, x^*)$ and for a norm $\|\cdot\|$ it holds that

$$\|x^{(k+1)} - x^*\| \leq c\|x^{(k)} - x^*\|^p, \quad k \geq k_0$$

then follow that R-order of I satisfies the inequality $O_R(I, x^*) \geq p$. The R-order of convergence of procedure I which converges to x^* is denoted by $O_R(I, x^*)$.

Definition 1.4.1

If the exists a $p \geq 1$ such that for any null sequence $\{w^{(k)}\}$ generated from $\{x^{(k)}\}$, then the

R-factor of the sequence is define to be,

$$R_p\{w^{(k)}\} = \begin{cases} \limsup_{k\to\infty} \|w^{(k)}\|^{1/k}, & p = 1 \\ \limsup_{k\to\infty} \|w^{(k)}\|^{1/p^k}, & p > 1 \end{cases}$$

where R_p is independent of the norm $\|\cdot\|$. Suppose that $R_p\{w^{(k)}\} < 1$ then it follow from Orthega and Rheinboldt (1970) that the R-order of I satisfied the inequality $O_R(I, x^*) \geq p$. The R-factor of a null sequence $\{w^{(k)}\}$ generated from the procedure I is denoted by $R_p\{w^{(k)}\}$.

Definition 1.4.2

Let I be an iteration procedure converging to x^*. Then, we may now define the R-order of the procedure I in term of R-factor as

$$O_R(I, x^*) = \begin{cases} \infty, & R_p(I, x^*) = 0,\ p \geq 1 \\ \inf\{p \mid p \in [1, \infty),\ R_p(I, x^*) = 1\} \end{cases}$$

Suppose that $R_p\{w^{(k)}\} < 1$ then it follow from Orthega and Rheinboldt (1970) that the R-order of convergence of procedure I which convergence to x^* satisfies the inequality $O_R(I, x^*) \geq p$. We used this result in order to calculate the rate of convergence of all the modified procedures in the subsequent chapters.

1.5 Research Objectives

The main objective of the study is to find the real and simple zeros of the polynomial simultaneously by using the interval analysis approach.

In particular, the objectives of this thesis are:

1. to modify the Interval Symmetric Single-step (ISS2) procedure introduced by Salim et al. (2011) and developed with four modified procedures namely; Interval Symmetric Single-Step (ISS2-5D) procedure, Interval Zoro-Symmetric Single-Step (IZSS2-5D) procedure, Interval Midpoint Symmetric Single-Step (IMSS2-5D) procedure and Interval Midpoint Zoro-Symmetric Single-Step (IMZSS2-5D) procedure on bounding simple and real polynomial zeros simultaneously.

2. to developed numerical algorithm to solve the problems in objective 1 using a program Matlab associate with Intlab toolbox.
3. to compare the efficiencies for both original and modified procedures in terms of CPU times, number of iterations and the value of the intervals width of the procedures by collecting numerical data.

4. to analyze the *R*-order of convergence for each modified procedures for comparisons.

1.6 Scope of the Problem

The scope of this study is on polynomial. In order to narrow down the scope of the problem, this thesis focuses on finding the inclusion of real and simple polynomial zeros simultaneously.

Furthermore, one must apply some numerical methods to find the zeros of polynomials of higher degree (Petkovic, 1989), thus numerical (iterative) interval analysis approach is applied throughout this thesis. According to (McNamee et al., 2007) iterative procedure is the process of reiterating the procedure in obtaining an outcome considered to be closed enough to the required number.
A polynomial is an expression of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \quad (1.1)$$

where $a_n \neq 0$. If the highest power of x is x^n, the polynomial is said to have degree n. It was proved by Gauss in the early 19th century that every polynomial has at least one zero. It follows that a polynomial of degree n has n zeros and they are not necessarily distinct. Often we used x for a real variable, and z for a complex. A zero of a polynomial is equivalent to a "root" such that a value x^* which makes $p(x)$ equal to zero of the equation $p(x) = 0$ (McNamee et al., 2007).

Let consider Newton's method which is start with a single initial guess x_0, preferably fairly close to a true root x^* and apply the iteration:

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)}$$

and stop when

$$\frac{|x_{k+1} - x_k|}{|x_{k+1}|} < \epsilon$$

or $|p(x_k)| < \epsilon$.

Next, let consider simultaneous methods, such as

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{\substack{j=1,\\ j\neq i}}^{n} (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, \ldots, n),$$

starting with initial guesses $x_i^{(0)} \ (i = 1, \ldots, n)$, where $x_i^{(k)}$ is the k-th approximation to the i-th zero $x_i^* \ (i = 1, \ldots, n)$ (McNamee et al., 2007).

The interval iterative procedure is in need of some pre-conditions for initial interval $X_i^{(0)} \ (i = 1, \ldots, n)$ to be converged to the zeros $x_i^* \ (i = 1, \ldots, n)$ respectively, starting with some disjoint intervals $X_1^{(0)}, X_2^{(0)}, \ldots, X_n^{(0)}$ each of which contains a polynomial zero. It will produce a set of intervals of smallest possible width such that each interval includes one or more zeros of $p(x)$ from a given interval $X_i^{(0)} \subseteq X^{(0)}$. In the other words, the interval sequence generated by the procedures are always converging to the zeros, which is

$$X_i^{(0)} \supseteq X_i^{(1)} \supseteq X_i^{(2)} \supseteq \cdots \quad \lim_{k\to\infty} X_i^{(k)} = x_i^*$$

or the sequence comes to the rest at $X_i^{(k)} = x_i^*$ after a finite of steps (Rusli et al., 2011).

In the matter of finding zeros of polynomial it is necessary to have reliable bounds on the errors in the estimated solutions (McNamee et al., 2007). The interval iterative procedures which are used in this research are the very significant steps in meeting the needs of these criteria.

Thus, in this research, it is suppose that p has n real zeros $x_1, x_2, \ldots, x_n$ and they are distinct. It is then assumed henceforth that $a_n = 1$ in the sequel (1.6.1). The including intervals $X_i^{(0)} = [\underline{x}_i^{(0)}, \overline{x}_i^{(0)}]$, $(i = 1, \ldots, n)$ are the initial intervals and are pairwise disjoints, that is

$$X_i^{(0)} \cap X_j^{(0)} = \emptyset \quad (1 \le i \ne j \le n).$$

The polynomial $p_n(x)$ will be written as

$$p_n(x) = (x - x_1)(x - x_2) \cdots (x - x_n),$$

or equivalent

$$p_n(x) = \prod_{j=1}^{n} (x - x_j),$$

thus let $p(x) = p_n(x)$. These principal and necessary tools are to be applied in the algorithm, which will be discussed in detail in chapters 3, 4, 5, 6 and 7.

1.7 Thesis Outline

The thesis comprises of the following:

In chapter 1 the background of interval analysis is introduced. Brief yet concise descriptions on some of the definitions and properties of interval analysis will be provided. In addition, the *R*-order of convergence concept is included in this section as well.

Chapter 2 includes brief discussions on the previous work on simultaneous procedure as well as the point and interval iterative procedure of finding the zero of polynomials.

Chapter 3 comprehends the description on the original Interval Symmetric Single-step (ISS2) procedure by Salim et al. (2011). The *R*-order of convergence of ISS2 procedure is analyzed, thus utilizing its full potential as a stepping stone for the new modified procedures generated in the next four chapters.

Chapter 4, 5, 6 and 7 consist of detail discussions on all four modified procedures, elaborating how the new procedures are generated, as well as elements involved. These modified procedures are called Interval Symmetric Single-step (ISS2-5D) procedure, Interval Zoro-Symmetric Single-step (IZSS2-5D) procedure, Interval Midpoint Symmetric Single-step (IMSS2-5D) procedure and lastly Interval Midpoint Zoro-Symmetric Single-step (IMZSS2-5D) procedure. The mentioned chapters include the algorithms, theoretical analysis of *R*-order of convergence, with numerical results for each modification in their respective chapters to support findings. The efficiencies for both the modified procedures and the original procedure are compared in terms of CPU time, number of iterations and the value of the width of the intervals. Serving the purpose of providing readers with clearer view of the overall outcome, the numerical results for each test polynomials will be displayed in the forms of tables and bar charts in each chapter.

Finally, Chapter 8 summarizes the conclusion of the research. Future works, which are made to relate to the research findings will also be recommended towards the end of the said section.

REFERENCES

Aberth, O. (1973). Iteration Methods for Finding All Zeros of Polynomial Simultaneously. *Maths. Of Comput*. 27: 339-344.

Alefeld, G. (1984). On the Convergence of Some Interval-Arithmetic Modification of Newton's Method. *SIAM J. Numer.Anal*. 21: 363-372.

Alefeld, G. and Herzberger, J. (1974). On the Convergence Speed of Some Algorithms for The Simultaneous Approximation of Polynomial Roots. *SIAM J. Numer.Anal*. 11: 237-243.

Alefeld, G. and Herzberger, J. (1983). *Introduction to Interval Computations.* Translated by Jon Rokne, New York: Academic Press.

Aitken, A.C. (1950). On the Iterative Solution of Linear Equation. *Proc. Roy. Soc. Edinburgh.* 63: 52-60.

Bakar, N.A., Monsi, M., Hassan, M.A. and Leong, W.J. (2011). On the Modification of the p-RF Method for Inclusion of a Zero of a Function. *Applied Mathematical Sciences.* 5: 1347-1361.

Bakar, N.A., Monsi, M. and Hassan, N. (2012). An Improved Parameter Regula Falsi Method for Enclosing a Zero of a Function. *Applied Mathematical Sciences.* 6: 1347-1361.

Borsch-Supan, W. (1963). A Posterior Error Bounds for the Zeros of Polynomials. *Numerische Mathematik*. 5: 380-398.

Butt, R. (2008). *Introduction to Numerical Analysis Using Matlab*. Infinity Science Press.

Docev, K. (1962). An alternative method of Newton for simultaneous calculation of all the roots of a given algebraic equation. *Phys. Math. J., Bulg. Acad. Sci.* 5: 136-139.

Ehrlich, L.W. (1967). *A modified Newton method for polynomials*. The Johns Hopkins Applied Physics Laboratory. Maryland, Silver Spring.

Farmer, M.R. and Loizou, G. (1975). A Class of Iteration Function for Improving, Simultaneously, Approximations to the Zeros of a Polynomial. *BIT* 15: 250-258.

Gargantini, I. (1978). Further application circular arithmetic: Schroeder-like algorithm with error bounds for finding zeros of polynomial. *SIAM Journal of Numerical Analysis*. 19: 149-154.

Gargantini, I. and Henrici, P. (1972). Circular Arithmetic and the Determination of Polynomial Zeros. *Numer. Math*. 18**:** 305-320.

Hansen, E. (1992). *Global Optimization Using Interval Analysis*. Marcel Decker N.Y.

Hansen, E.R. and Walster, G.W. (2004). *Global Optimization Using Interval Analysis*. Massachusetts MIT Press, Cambridge.

Hopkins, M. (1994). On a method of Weierstrass for the simultaneous calculation of the roots of a polynomial. *Z. angew. Math. Mech*. 74: 295-306.

Hull, T.E. and Mathon, R. (1996). The Mathematical Basis and a Prototype Implementation of a New Polynomial Rootfinder with Quadratic Convergence. *ACM Trans. Math. Softw*. 22: 261-280.

Ilief, L. (1948). On the approximation of Newton. *Annual Sofia Univ*. 46: 167-171.

Jaulin, L., Kieffer, M., Didrit, O. and Walter, E. (2001). *Applied Interval Analysis*. Springer.

Kerner, O. (1966). Total Step Procedure for the Calculation of the Zeros of Polynomials. *Numer, Math*. 8:290-294.

McNamee, J.M., Chui, C.K. and Wuytack, L. (2007). *Numerical Methods for Roots of Polynomials.*United Kingdom: Elsevier Publishing Company.

Milovanovic, G.V. and Petkovic, M.S. (1983). On The Convergence of a Modified Method for Simultaneous Finding of Polynomial Zeros. *Computing* 30: 171-178.

Moore, R. (1966). *Interval Analysis*. Prentice-Hall, Englewood Cliffs New Jersey, USA.

Moore , R.E., Kearfott, R.B. and Cloud, M.J. (2009). *Introduction to Interval Analysis. SIAM*, Philadelphia.

Monsi, M. (1988). *Some Application of Computer Algebra and Interval Mathematics.* University of St. Andrews, United Kingdom.

Monsi, M. (2011). The Interval Symmetric Single-Step ISS1 Procedure for Simultaneously Bounding Simple Polynomial Zeros. *Malaysian Journal of Mathematical Science*. 5(2):211-227.

Monsi, M. (2012). The Point Symmetric Single Step Procedure for Simultaneous Approximation of Polynomial zeros. *Malaysian Journal of Mathematical Science*. 6(1):29-46.

Monsi, M., Hassan,N. and Rusli, S.F. (2012). The Point Zoro Symmetric Single-Step Procedure for Simultaneous Estimation of Polynomial Zeros.*Journal of AppliedMathematics*. 1-11.

Monsi, M. and Wolfe, M.A. (1988). An Algorithm for the Simultaneous Inclusion of Real Polynomial Zeros. *Applied Mathematics and Computation*. 25:333-346.

Niell, A.M. (2001). The simultaneous approximation of polynomial roots. *Comput. Math. Appl*. 41: 1-14.

Neumaier, A. (1990). *Interval Methods for Systems of Equations*. Cambridge University Press, Cambridge, England.

Neumaier, A. (2001). *Introduction to Numerical Analysis*. Cambridge University Press.

Nourein, A.W. (1975). An iteration formula for the simultaneous determination of the zeros of a polynomial. *Journal of Computational and Applied Mathematics.* 1(4):251-254.

Nourein, A.W. (1977a). An improvement on Nourien's method for the simultaneous determination of the zeroes of a polynomial (an algorithm). *Journal of Computational and Applied Mathematics.* 3(2):109-110.

Nourein, A.W. (1977b). An improvement on two iteration methods for simultaneous determination of the zeros of a polynomial. *Intern. J. Computer Math*. 6: 241-252.

Ortega, J.M. and Rheinboldt, W.C. (1970). *Iterative solution of nonlinear equations in several variables*, New York: Academic Press.

Pan, V.Y. (2005). Coefficient-free adaptations of polynomial root-finders. *Computers and Math with Application*. 50: 263-269.

Petkovic, M.S. (1982). On an Iterative Method for Simultaneous Inclusion of Polynomial Complex Zeros. *Journal of Computational and Applied Mathematics.* 8: 51-56.

Petkovic, M.S. (1989). *Iterative Methods for Simultaneous Inclusion of Polynomial Zeros*. Springer.

Petkovic, M.S. (2008). *Point Estimations of Root Finding Methods*. Springer.

Petkovic, M.S., Ilic, S. and Rancic, L. (2004). The convergence of a family of parallel zero-finding methods. *Comp. And Math. With Applications*. 48: 455-467.

Petkovic, M.S., Milosevic, M.R. and Milosevic, D.M. (2011). *New Higher-Order methods for the simultaneous inclusion of polynomial zeros*. Springer.

Petkovic, M.S., Petkovic, L.D. and Ilic, S.M. (2007). On the convergence of the third order root-solver. *Ser. Math. Inform*. 22(1): 91-103.

Petkovic, M.S. and Stefanovic, L.V. (1986). On Second Order Method for the Simultaneous Inclusion of Polynomial Complex Zeros in Rectangular Arithmetic. *Computing.* 36:249-261.

Rahmin, N.A.A., Monsi, M., Hassan, M.A. and Ismail, F. (2009). A modification of inclusion of a zero of a function using interval method. *Malaysian Journal of Mathematical Sciences*. 3(1): 67-82.

Rump, S.M. (1999). *INTLAB- Interval Laboratory*.

Rusli, S.F.M., Monsi, M., Hassan, M.A. and Leong, W.J. (2011). On the Interval Zoro Symmetric Single –step Procedure for Simultaneous Finding of Polynomial Zeros. *Applied Mathematics Sciences*. 5: 3693-3706.

Salim, N.R., Monsi, M.,Hassan, M.A. and Leong, W.J. (2011). On the Convergence Rate of Symmetric Single-Step Method ISS for Simultaneous Bounding Polynomial Zeros. *Applied Mathematics Sciences.* 5**:** 3731-3741.

Semerdzhiev, K. (1994). Iterative method for simultaneous finding all roots of generalized polynomial equations. *Math. Balk*. 8: 311-335.

Simeunovic, D.M. (1989). On the convergence of an iterative procedure for the simultaneous determination of all zeros of a polynomial. *Z. angew. Math. Mech*. 69: 108-110.

Sun, F. and Li, X. (1999). On an accelerating quasi-Newton circular iteration. *Appl. Math. Comp*. 106: 17-29.

Terui, A. and Sasaki, R. (2002). Durand-Kerner method for the real roots. *Japan J. Indust. Appl. Math.* 19: 19-38.