UNIVERSITI PUTRA MALAYSIA

EXTENDING BIANCHI'S CLASSIFICATION OF HOMOGENEOUS THREE-MANIFOLDS

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By

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EXTENDING BIANCHI'S CLASSIFICATION OF HOMOGENEOUS THREE-MANIFOLDS

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December 2006

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A few of the Bianchi types of homogeneous cosmologies described by three-dimensional groups of motions G_3 with transitive actions on three-manifolds were extended to those involving transitive groups of motions G_r with additional dimensions of four-manifolds. In order to extend a three-manifold metric which admits the Bianchi type N (N=I, II, V, IX) to that of four-manifold, the group G_r was chosen.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

MELANJUTKAN PENGELASAN BIANCHI TERHADAP MANIFOLD-MANIFOLD HOMOGEN TIGA DIMENSI

Oleh

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Pengerusi: **Profesor Madya Hishamuddin Zainuddin, PhD**

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Beberapa pengelasan *Bianchi* bagi kosmologi-kosmologi homogen yang dijelaskan oleh kumpulan-kumpulan gerakan tiga dimensi G_3 dengan tindakan transitif ke atas manifold-manifold tiga dimensi telah dilanjutkan kepada pengelasan *Bianchi* yang melibatkan kumpulan-kumpulan gerakan transitif G_r dengan dimensi tambahan bagi manifold-manifold empat dimensi. Dalam usaha untuk melanjutkan metrik suatu manifold tiga dimensi yang membenarkan pengelasan *Bianchi* N (N=I, II, V, IX) kepada metrik manifold empat dimensi, kumpulan G_r telah dipilih.

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I certify that an Examination committee met on 1 December 2006 to conduct the final examination of Aslam Bin Abdullah on his Master of Science thesis entitled "Extending Bianchi's Classification of Homogeneous Three-Manifolds" in accordance with Universiti Putra Malaysia (Higher Degree) Act 1980 and Universiti Putra Malaysia (Higher Degree) Regulations 1981. The committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

ASLAM ABDULLAH

Date: 24 January 2007

TABLE OF CONTENTS

	Page
ABSTRACT	iii
ABSTRAK	iv
ACKNOWLEDGEMENTS	v
APPROVAL	vi
DECLARATION	viii
LIST OF TABLES	xi
LIST OF NOTATIONS	xii
CHAPTER	
1 **INTRODUCTION**	1
1.1 Homogeneous Manifolds with Extra Dimensions	1
1.2 Objectives	2
1.3 The Research Approach	3
2 **LITERATURE REVIEW**	5
2.1 Symmetry	5
2.2 String Theory	8
2.2.1 The Accelerating Vacuum Universe	8
2.2.2 The Accelerating Universe with Matters	9
2.3 Kaluza-Klein Cosmologies	10
2.4 Brane Worlds	12
2.5 Conclusion	15
3 **THEORY**	17
4 **METHODOLOGY**	24
4.1 The System (A)	24
4.2 Manifolds which Admit as a Subgroup of Motions the Bianchi Type N	27
4.3 Case 1 – The Bianchi Type N Four-Manifolds (N=I, II, V)	31
4.4 Case 2 – The Bianchi Type IX Four-Manifold	32
5 **RESULTS AND DISCUSSION**	34
5.1 Four-dimensional Bianchi Type I Manifold	35
5.1.1 The System $(A)_{I}$	35
5.1.2 Discussion of the System $(A)_{I}$	36
5.1.3 The System $(B)_{I}$	37
5.2 Four-dimensional Bianchi Type II Manifold	41
5.2.1 System $(A)_{II}$	41
5.2.2 Discussion of the System $(A)_{II}$	42
5.2.3 System $(B)_{II}$	44

5.3 Four-dimensional Bianchi Type V Manifold 48
5.3.1 System $(A)_V$ 48
5.3.2 Discussion of the System $(A)_V$ 49
5.3.3 System $(B)_V$ 51
5.4 Four-dimensional Bianchi Type IX Manifold 55
5.4.1 System $(A)_{IX}$ 55
5.4.2 Discussion of the system $(A)_{IX}$ 59
5.4.3 The System $(B)_{IX}$ 65

6 **CONCLUSION** 69

SUMMARY 72
REFERENCES 74
APPENDICES 76
BIODATA OF THE AUTHOR 93

LIST OF TABLES

	Page
Table 1 List of Bianchi Types	18
Table 2 Table 2: List of Bianchi Type N Four-manifolds (N=I,II,V,IX)	72
Table 3 The Bianchi-Behr Groups Classification	92

LIST OF NOTATIONS

A^3	- Three-manifolds admitting the Bianchi type N (N=I, II, V, IX).
B^4	- Four-manifolds admitting as a subgroup the Bianchi type N (N=I, II, V, IX).
$\{X_i\}_{i=1,2,3}$	- Set of Killing vectors, a basis of infinitesimal generators.
C_{ij}^k	- Structure constants of the Lie group.
$\{\partial_i\}$	- Coordinated basis, where $\partial_i = \partial / \partial x^i$.
$\{x^i\}$	- The coordinate system used to express the Killing vectors X_i with coordinated basis $\{\partial_i\}$.
$\{dx^i\}$	- Coordinated basis, dual to $\{\partial_i\}$.
$\{\zeta_i\}$	- Invariant basis such that $[X_i, \zeta_j]$=0.
$\{\omega^i\}$	- Invariant basis of one-forms, the dual basis to $\{\zeta_i\}$.
X	- Killing vector which generates the r-dimensional group of motions G_r of type N four-manifold (N=I, II, V).
Y	- Killing vector which generates the r-dimensional group of motions G_r of type IX four-manifold.
ξ^i	- Components of the Killing vector X (or Y).
ξ_i	- Components of the dual vector to X (or Y).
R^n	- Additive group of n-tuple with real entries.
R_i^n	- Additive group of n-tuple with real entries which transforms a point in a manifold along x^i coordinate lines.
T_3	- Three-dimensional translation symmetry.
$SO(3,R)$	- Special (unit determinant) orthogonal 3×3 matrices with real coefficients.
r	- Number of group's dimensions.
ξ_i^α	- Components of the Killing vector X_i (or Y_i).
n_A	- Number of equations in the system (A).
G_r	- r-dimensional group of motions.
n	- Number of manifold's dimensions.
$\boldsymbol{g}$	- Lie algebra of the group of motions.

$O(n)$	- Orthogonal group.
$SL(n,\mathrm{R})$	- Special linear group.
$SO(n)$	- Special orthogonal group.
δ_i^k	- Kronecker delta, a quantity which is either 0 or 1 according to $$\delta_i^k = \begin{cases} 1, k = i, \\ 0, k \neq i. \end{cases}$$
ε_{fij}	- Levi-Civita alternating symbol, a completely antisymmetric form of weight -1 and rank 3 whose values in any coordinate system is +1 or -1 if *fij* is an even or odd permutation of 234, respectively, and zero otherwise. Note that $\varepsilon_{234} = 1$.
$g_{\mu\nu}$	- Symmetric metric coefficients. If $g_{\mu\nu} = g(\zeta_\mu, \zeta_\nu)$ is Remannian, it must follow the following axioms [1] at each point x in a manifold M: i) $g(\zeta_\mu, \zeta_\nu) = g(\zeta_\nu, \zeta_\mu)$, ii) $g(\zeta_\mu, \zeta_\mu) \geq 0$ where the equality holds only when $\zeta_\mu = 0$. Thus g is a symmetric positive-definite bilinear form.

CHAPTER 1

INTRODUCTION

1.1 Homogeneous Manifolds with Extra Dimensions

Since the unification of electromagnetic and weak forces, theoretical physicists struggled with the idea of unifying of all forces including gravity. With gravity unification seems impossible without introducing new ideas such as supersymmetry, strings and extra dimensions. The history of extra dimensions can be traced back to the Kaluza-Klein theory in 1920s for which the dimensions are compactified in this scheme. Unlike ten spacetime dimensions [1] and thus six extra dimensions in string theory, Kaluza's unified theory of gravity and electromagnetism has only one extra dimension [2]. Another scheme known as brane worlds appeared in 1999 for which our universe is localized in a higher dimensional space.

Extra spatial dimensions might be one of the universe's secrets. In this research, the simplest four-dimensional spatial sections of spacetimes have been derived. They are three-manifolds with one extra dimension which admit group of motions whose subgroups are Bianchi type N and an additional translational group. This classification relies on the work of W. Killing in 1892, namely the Killing equations. The Killing equations were later used by L. Bianchi in his classification of homogeneous three-manifolds [3].

We assume that our four-manifolds are homogeneous since our universe is homogeneous on scales larger than 10^8 light-years, which are large enough to include so many clusters of galaxies. Observations show that the universe is currently pervaded by a very low energy and very uniform radiation, called cosmic microwave background (CMB) as a thermal residue of the hot big bang. From CMB measurements, the fluctuations are extremely small, representing deviations from homogeneity of only about 10^{-6} of the average temperature of the observed microwave background.

1.2 Objectives

The research is meant for;

1. Classifying four-manifolds admitting both the three dimensional group of motions which act simply transitively on three dimensional orbits, and translational symmetries.

2. Providing the spacelike models of the spacetime for future understanding about the interactions involved (i.e. by finding the stress-energy tensors involved in each model).

3. Extending a few of three-dimensional group of motions of homogeneous cosmologies to those of four dimensions by means of the Killing vectors Lie algebra.

1.3 The Research Approach

Usually, researchers attempt to solve the Einstein equations by searching the explicit values for the metric coefficient functions of a spacetime. This means that one chooses first what stress-energy tensor (thus, the interactions) involved in each case. But in this work, an attempt was made to find the solutions of the spacelike four-manifold metrics without prior setting the type of interactions involved. The Killing equations need to be solved for this aim. In doing so, an additional translational symmetry was chosen for each four-manifold. There are four types of four-manifolds which have been derived in this work. Basically, any solution of the spatial section of four dimensional cosmology was locally embedded in a higher dimensional manifold, as suggested by string theory.

The original idea was to seek all nine four-manifolds with an extra dimension. However, due to time constraint, almost half of them were only successfully found. They are Type N four-manifolds (N=I, II, V, IX). Solving a set of equations with Lie derivatives, it was extremely difficult to find all the explicit values for each four-manifold metric coefficient, since there are a large number of unknown parameters to be determined simultaneously. Assuming that the four-manifolds admit as a subgroup of motions the translational symmetry over extra variable simplifies the calculations.

A major goal of this research direction is to find a systematic classification of cosmological models with extra dimensions.

CHAPTER 2

LITERATURE REVIEW

2.1 Symmetry

A systematic discussion of affine, conformal and curvature symmetries in spacetime was given by Hall [4]. The presentation of affine symmetry included that of isometric and homothetic. He divided the global vector fields X on a four dimensional, Hausdorff, connected smooth manifold M into three kinds. Each kind of X generates different symmetries. One of them called affine vector field is subdivided into proper affine, homothetic and Killing vector fields. They can be distinguished from each other by using the Lie derivative

$$\mathcal{L}_X g = 2h,$$

where g is a metric, h is a (global) covariantly constant, second order, symmetric tensor on M. If h is a constant multiple of g and a zero, X is called homothetic and Killing vector fields, respectively. X becomes a proper affine vector field when h is neither a constant multiple of g nor a zero.

It is interesting to note that both the homothetic and Killing vector fields can also be classified under conformal vector field, as described by Apostolopoulos-Tsamparlis (AT) [5] (Their original symmetry related work will be discussed later). In this case, their Lie derivatives are given by

$$\mathcal{L}_X g = 2\phi g,$$

where ϕ is a real valued function on M. Obviously, ϕ=constant$\neq 0$ and ϕ=0 are the conditions for homothetic and Killing vector fields, respectively. We will have the special conformal Killing vector field if $\phi_{;ab} = 0$.

The other kind of X is called curvature collineation, defined by

$$\mathcal{L}_X R^a{}_{bcd} = 0,$$

where $R^a{}_{bcd}$ is Riemannian tensor of the second kind.

Symmetry plays important roles in the classification as well as in the study of the dynamics of homogeneous cosmologies. In 1898, Bianchi [3] classified three-dimensional manifolds according to the group of motions they admit. There are nine types of those groups, called Bianchi types, and three of them admit three-manifolds of constant curvatures. Cosmological models whose spacelike sections admit the Bianchi types are known as the Bianchi models.

Cotsakis et al. [6] geometrically reformulated five-dimensional homogeneous cosmologies for vacuum, scalar field with zero potential, constant potential and arbitrary potential cases. Having found the Killing and Noether symmetries for Bianchi models, they raised the question on how these symmetries behave as the models expand.

Analysis on the roles of each particular kind of symmetries is still active. For instance, Vilasi-Vitale [7] analyzed the $SO(2,1)$ symmetry in General Relativity. Compared to the $SO(3)$, $SO(2,1)$ is inflexible - there is only one possible invariant cosmology admitting $SO(2,1)$ as a subgroup of six-dimensional group of motions, which is the Lorentz invariant, constant negative curvature Friedmann-Robertson-Walker model.

AT [5], using purely geometrical methods determined the metrics that admit the proper and not proper (as special case) conformal Killing vectors (CKV_s). The results hold for any type of matter, and were applied to achieve two purposes - to classify the CKV_s of a static spherically symmetric spacetimes[1], and to find the non-tilted locally rotationally symmetric perfect fluid spacetime admitting proper CKV_s and its basic properties.

One of the more advanced studies on symmetry has been done by Cotăescu [8], at the level of the relativistic quantum mechanics in the sense of general relativity, by formulating the external symmetry as a combination of isometries and suitable tetrad gauge transformations. The external symmetry, whose transformations leave the equations of the fields with spin invariant, is divided into central and maximal. Cotăescu showed that the Lie algebra of the isometry group is isomorphic to that of the external symmetry of the spacetime, having the same structure constants. It is known that the fields with spin transform according to the representations of the

[1] Spherical symmetric spacetimes obey the cosmological principle.

external symmetry are induced by linear representations of $SL(2,C)$. Thus the calculation for the generators of the external symmetry transformations is possible. These generators, with specific spin terms, represent new physical observables. In this formulation, de Sitter and anti-de Sitter spacetimes in (4+1)-dimensional or (3+2)-dimensional cosmologies were considered as examples.

2.2 String Theory

The emergence of string and Kaluza-Klein theories gave rise to the need of extra dimensions. Earlier understanding uses compact extra dimensions but later allows macroscopic ones particularly the brane world scenario [9]. In recent years, Townsend-Wohlfarth (TW) [10], Chen *et al*. [11] and Djordjević-Něsí (DN) [12] are among the theoretical cosmologists and string theorists who circumvent these problems and systematically study the accelerating cosmological models.

2.2.1 The Accelerating Vacuum Universe

In the search of a new set of vacuum accelerating cosmologies, Chen *et al.* [11] considered the cases where the spacetimes are products of flat and hyperbolic spaces. They showed that the flat external dimensions do not lead to accelerating cosmologies.

For the case where the usual four-dimensional spacetime is hyperbolic and the internal space is flat as well as the case where both of them are hyperbolic, the perturbative expansions about the solutions indicate that the eternal acceleration is possible if the internal dimension is equal to seven and above. The behaviour is qualitatively the same with that of four-dimensional Friedmann-Lemaitré-Robertson-Walker (FLRW) universe as a single hyperbolic space.

2.2.2 The Accelerating Universe with Matter

In order to obtain the accelerating models with matter from compactification, TW [10] considered n-dimensional ($n \geq 2$) compact Einstein manifold with metric

$$ds_n{}^2 = g_{mn} dg^m dg^n .$$

For negative t in a certain interval, the rate of change of the scale factor S with respect to the proper time η and its second derivative give

$$\frac{ds}{d\eta} > 0, \frac{d^2 s}{d\eta^2} > 0.$$

These results show that the universe inflates and experiences accelerating expansion. The conditions for the accelerating cosmologies are as follows;

- The internal metric should be time-dependent.

- The internal space should be hyperbolic and compact in which massless Kaluza-Klein vector fields are absent. The absence allows matter to live on space filling branes and to be supersymmetric.

Note that in this case, $n = 7,$ has relevance to M-theory compactifications. In other words, the universe has eleven dimensions, including time. Relativists hypothesize the existence of energy with strong negative pressure permeating all of space. At large scales, this so-called dark energy acts in opposition to gravity such that the universe appears to be inflating at an accelerating rate.

If t approaches $-\infty$ and t approaches 0 (from $t < 0$), the universe decelerates. Hence, the epoch of accelerated expansion is in between the two epochs of deceleration due to scalar field of four-dimensions arising from the mode of Kaluza-Klein scaling the compact internal space volume.

At the end (i.e. at $t = 0$), TW [10] considered the following case;

- If $n = 4,$ one has Einstein de-Sitter Universe.
- If $n = 6, 7,$ the universe will have negative pressure matter. It is not impossible to assume that this is the beginning of the universe contraction.