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# Multistep Block Method for Solving Volterra Integro-Differential Equations

N.A.Mohamed *[1] and Z.A.Majid [2]

[1,2] *Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

*E-mail:atikahmohd89@gmail.com*
* *Corresponding author*

## ABSTRACT

In this paper, we use multistep block method for solving linear and non-linear Volterra integro-differential equations (VIDEs) of the second kind. The VIDEs will be solved by using the combination of multistep block method of order three and Newton-Cotes quadrature rule of suitable order. The proposed method will solve VIDEs for $K(x,s) = 1$ and $K(x,s) \neq 1$. The stability region of the method will be given. Numerical problems are included to represent the performance of the proposed method.

**Keywords:** Volterra integro-differential equations, multistep block method, quadrature rule.

# 1. Introduction

The Volterra integro-differential equations (VIDEs) that will be studied is in the subsequent form

$$y'(x) = F(x, y(x), z(x)), \quad y(0) = y_0, \ \ 0 \leq x \leq a, \tag{1}$$

$$z(x) = \int_0^x K(x, s, y(s))\,\mathrm{d}s. \tag{2}$$

**Theorem 1.** Linz (1969) *Assume that $F$ and $K$ are uniformly continuous and satisfy the subsequent condition:*

$$|F(x, y_1, z) - F(x, y_2, z)| \leq L_1|y_1 - y_2|,$$
$$|F(x, y, z_1) - F(x, y, z_2)| \leq L_2|z_1 - z_2|,$$
$$|K(x, s, y_1) - K(x, s, y_2)| \leq L_3|y_1 - y_2|.$$

*Under these conditions Equation (1) and Equation (2) possesses a unique solution in* $0 \leq x \leq a$.

VIDEs emerge in many applications such as fluid dynamics, biological models and ecology [Filiz (2014b)]. The development of numerical method for solving VIDEs began with the work by Day (1967). He solved integro-differential equations by using composite trapezoidal rule. The new combination of linear multistep method with quadrature formula for the numerical approximation of VIDEs has been introduced by Linz (1969). In 1974, the stability analysis of linear multistep method for the numerical approximation of VIDEs have been proposed by Brunner and Lambert (1974).

Chang (1982) has presented an extrapolation method for solving VIDEs by using two-step and three-step Adams-Moulton rule together with Euler-Maclaurin formula. In the same year, Makroglou (1982) has extended the theory of hybrid method in ODE for the numerical solution of VIDEs. He had also demonstrated the stability region of the proposed method. Rashed (2004) describes new method for the numerical solution of VIDEs by using Lagrange Interpolation. In Saadati et al. (2008) and Raftari (2010) VIDEs have been solved using finite difference method based upon Newton Cotes quadrature rule. They transform the VIDEs into a martix equations. Chen and Zhang (2011) proposed new quadrature rules generated by boundary value method for solving VIDEs.

Later Filiz (2013) has proposed the combination of the Runge-Kutta method of order four (RK4) with trapezoidal rule and Simpson's 1/3 rule for solving

VIDEs. He developed a new fourth order routine for the numerical approximation of VIDEs using Lagrange interpolating polynomial. In Filiz (2014b) and Filiz (2014a), he has solved VIDEs using Runge-Kutta Fehlberg Method and Cash-Karp method respectively.

In this paper, we define third order numerical method and suitable numerical integration method for solving Equation (1) and Equation (2). We show how to use this method to solve both linear and nonlinear VIDEs.

# 2. Multistep Block Method

We derive the method following the formation of the two point multistep block method from Majid and Suleiman (2011).

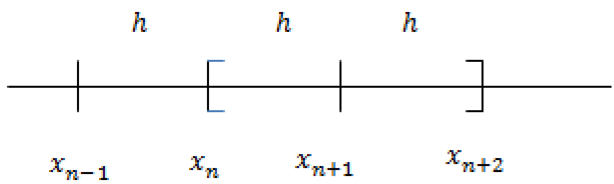


Figure 1: Two Point Multistep Block Method.

In Figure 1 the two formulas of $y_{n+1}$ and $y_{n+2}$ are approximated simultaneously in a block at $x_{n+1}$ and $x_{n+2}$ respectively. The first corrector formula will involve the set of points $\{x_n, x_{n+1}, x_{n+2}\}$ while the second corrector formula will involve the set of points $\{x_{n-1}, x_{n+1}, x_{n+2}\}$. Predictor formulas will include the set of points $\{x_{n-1}, x_n\}$. The predictor formulas and the corrector formulas are derived using Lagrange interpolation polynomial of order 2 and order 3 correspondingly. The two points of $y_{n+1}$ and $y_{n+2}$ will be attained after integrating $y' = f(x, y)$ at the interval of $[x_n, x_{n+1}]$ and $[x_n, x_{n+2}]$.

The generated multistep block method formulas are as follows:

$$y^p_{n+1} = y_n + \frac{h}{2}[3f_n - f_{n-1}], \tag{3}$$

$$y^p_{n+2} = y_n + h[4f_n - 2f_{n-1}], \tag{4}$$

$$y^c_{n+1} = y_n + \frac{h}{12}[-f_{n+2} + 8f_{n+1} + 5f_n], \tag{5}$$

$$y^c_{n+2} = y_n + \frac{h}{9}[2f_{n+2} + 15f_{n+1} + f_{n-1}]. \tag{6}$$

The order of this method is determined by applying the formula for the constants $C_q$. The formula is defined as follows:

$$\begin{aligned}
C_0 &= \sum_{j=0}^{k} \alpha_j, \\
C_1 &= \sum_{j=0}^{k} j\alpha_j - \beta_j, \\
C_2 &= \sum_{j=0}^{k} \frac{j^2\alpha_j}{2} - j\beta_j, \\
C_3 &= \sum_{j=0}^{k} \frac{j^3\alpha_j}{3!} - \frac{j^2\beta_j}{2}, \\
&\vdots \\
C_q &= \sum_{j=0}^{k} \frac{j^q\alpha_j}{q!} - \frac{j^{q-1}\beta_j}{(q-1)!}.
\end{aligned} \tag{7}$$

Thus, by using Equation (7) we can calculate for the order and error constant of the method.

For $q = 0$,

$$C_0 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $q = 1$,

$$\begin{aligned} C_1 &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ \frac{1}{9} \end{pmatrix} \\ &- \begin{pmatrix} \frac{5}{12} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{8}{12} \\ \frac{15}{9} \end{pmatrix} - \begin{pmatrix} -\frac{1}{12} \\ \frac{2}{9} \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

For $q = 2$,

$$\begin{aligned} C_2 &= \frac{1}{2}\begin{pmatrix} -1 \\ -1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{9}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{5}{12} \\ 0 \end{pmatrix} \\ &- 2\begin{pmatrix} \frac{8}{12} \\ \frac{15}{9} \end{pmatrix} - 3\begin{pmatrix} -\frac{1}{12} \\ \frac{2}{9} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

For $q = 3$,

$$\begin{aligned} C_3 &= \frac{1}{6}\begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{4}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{27}{6}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \frac{5}{12} \\ 0 \end{pmatrix} \\ &- 2\begin{pmatrix} \frac{8}{12} \\ \frac{15}{9} \end{pmatrix} - \frac{9}{2}\begin{pmatrix} -\frac{1}{12} \\ \frac{2}{9} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

For $q = 4$,

$$\begin{aligned} C_4 &= \frac{1}{24}\begin{pmatrix} -1 \\ -1 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{27}{8}\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} \frac{5}{12} \\ 0 \end{pmatrix} \\ &- \frac{4}{3}\begin{pmatrix} \frac{8}{12} \\ \frac{15}{9} \end{pmatrix} - \frac{9}{2}\begin{pmatrix} -\frac{1}{12} \\ \frac{2}{9} \end{pmatrix} = \begin{pmatrix} \frac{1}{24} \\ \frac{1}{9} \end{pmatrix}. \end{aligned}$$

The proposed multistep block method is said to be of order $q$ if $C_0 = C_1 = \cdots = C_q = 0$ and $C_{q+1} \neq 0$ is the error constant. So the corrector formula in Equation (5) and Equation (6) is of order 3 and the error constant is

$$C_{q+1} = C_4 = \begin{pmatrix} \frac{1}{24} \\ \frac{1}{9} \end{pmatrix}.$$

Considered the general linear multistep method as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j F_{n+j}, \tag{8}$$

where

$$F_n = F(x_n, y_n, z_n), \tag{9}$$

$$z_n = h\sum_{i=0}^{n} \omega_{ni} K(x_n, x_i, y_i). \tag{10}$$

**Definition 1.** Linz (1969) If $\rho(r) = \sum \alpha_j r^j$ and $\sigma(r) = \sum \beta_j r^j$ are characteristic polynomials, then a multistep method of the Equation (8) to Equation (10) is consistent if

1. $\rho(1) = 0$,
2. $\rho'(1) = \sigma(1)$ and
3. the weight $\omega_{ni}$ are bounded for all $n$ and $i \le n$ , $|\omega_{ni}| \le W$ , and are such that for any continuous function $f(x)$,

   $$\int_0^x f(t)\,\mathrm{d}t - h\sum_{i=0}^{n} \omega_{ni} f(x_i) = \theta(h),$$

   where $\theta(h) \to 0$ as $h \to 0$, $n \to \infty$, $nh = x$.

# 3. Implementation

In this part, we consider the use of multistep block method coupled with quadrature rules to obtain numerical methods for Equation (1) and Equation (2).

Integrating Equation (1) from $x_n$ to $x_{n+r}$,

$$y(x_{n+r}) = y(x_n) + \int_{x_n}^{x_{n+r}} F(x, y(x), z(x))\,\mathrm{d}x, \tag{11}$$

where

$$z(x) = \int_0^x K(x, s, y(s))\,\mathrm{d}s. \tag{12}$$

We can approximate Equation (11) by using multistep block method and obtain

$$\begin{aligned} y^p_{n+1} &= y_n + \frac{h}{2}[3F(x_n, y_n, z_n) \\ &- F(x_{n-1}, y_{n-1}, z_{n-1})], \qquad (13) \\ y^p_{n+2} &= y_n + h[4F(x_n, y_n, z_n) \\ &- 2F(x_{n-1}, y_{n-1}, z_{n-1})], \qquad (14) \\ y^c_{n+1} &= y_n + \frac{h}{12}[-F(x_{n+2}, y_{n+2}, z_{n+2}) \\ &+ 8F(x_{n+1}, y_{n+1}, z_{n+1}) + 5F(x_n, y_n, z_n)], \qquad (15) \\ y^c_{n+2} &= y_n + \frac{h}{9}[2F(x_{n+2}, y_{n+2}, z_{n+2}) \\ &+ 15F(x_{n+1}, y_{n+1}, z_{n+1}) \\ &+ F(x_{n-1}, y_{n-1}, z_{n-1})]. \qquad (16) \end{aligned}$$

To evaluate the integral in Equation (12) we consider two approaches:

## 3.1 Approach I

In the first approach we use Simpson's rule to evaluate the integral. If $k(x, s) = 1$, Simpson's 1/3 rule is applied:

$$z_{n+2} = z_n + \frac{h}{3}(y_{n+2} + 4y_{n+1} + y_n). \qquad (17)$$

## 3.2 Approach II

When $k(x, s) \neq 1$, it is more appropriate to apply composite Simpson's rule with interpolation schemes. Given for $n = 0, 2, 4, ...,$, we can write

$$\begin{aligned} z_{n+1} &= \frac{h}{3}\sum_{i=0}^{n} \omega^s_i K(x_{n+1}, x_i, y_i) \\ &+ \frac{h}{6}\{K(x_{n+1}, x_n, y_n) + 4K(x_{n+1}, x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}}) \\ &+ K(x_{n+1}, x_{n+1}, y_{n+1})\}, \qquad (18) \\ z_{n+2} &= \frac{h}{3}\sum_{i=0}^{n+2} \omega^s_i K(x_{n+2}, x_i, y_i), \qquad (19) \end{aligned}$$

where $y_{n+\frac{1}{2}}$ have unknown value that can be estimated by Lagrange interpolation at points $\{x_{n+1}, x_{n+2}, x_{n+3}\}$. Therefore we obtain

$$\begin{aligned} \ell_{n+1}(x_{\frac{1}{2}}) &= -\frac{1}{8}, \\ \ell_{n+2}(x_{\frac{1}{2}}) &= \frac{3}{4}, \\ \ell_{n+3}(x_{\frac{1}{2}}) &= \frac{3}{8}, \end{aligned} \tag{20}$$

and so we get

$$y_{n+\frac{1}{2}} = -\frac{1}{8}y_{n+1} + \frac{3}{4}y_{n+2} + \frac{3}{8}y_{n+3}. \tag{21}$$

## 4. Stability Region

Here is the discussion on the stability region of the multistep block method combined with Simpson's rule. The method is applied to the test equation

$$y'(x) = \xi y(x) + \eta \int_0^x y(s)\,\mathrm{d}s, \tag{22}$$

where $\xi = \lambda + \mu$ , $\eta = -\lambda\mu$ and obtained the following alternative form of Equation (22)

$$y'(x) = (\lambda + \mu)y(x) - \lambda\mu \int_0^x y(s)\,\mathrm{d}s. \tag{23}$$

From the proposed method for the numerical solution of Equation (1) and Equation (2), the correspond unique characteristic polynomials $\rho$, $\sigma$, $\tilde{\rho}$ and $\tilde{\sigma}$ are as follows:

1. First point of corrector formula

$$\rho(r) = r^2 - r \quad \sigma(r) = -\frac{1}{12}r^3 + \frac{8}{12}r^2 + \frac{5}{12}r \tag{24}$$

2. Second point of corrector formula

$$\rho(r) = r^3 - r \quad \sigma(r) = \frac{2}{9}r^3 + \frac{15}{9}r^2 + \frac{1}{9} \tag{25}$$

3. Simpson's 1/3 rule

$$\tilde{\rho}(r) = r^2 - 1 \quad \tilde{\sigma}(r) = \frac{1}{3}r^2 + \frac{4}{3}r + \frac{1}{3} \tag{26}$$

The stability polynomial of the method considered can be found after substituting Equation (24), Equation (25) and Equation (26) into this particular formula

$$\pi(r, h\xi, h^2\eta) := \tilde{\rho}(r)[\rho(r) - h\xi\sigma(r)] - h^2\eta\tilde{\sigma}(r)\sigma(r). \tag{27}$$

From the stability polynomial we can plot the region of absolute stability of the combinations method.

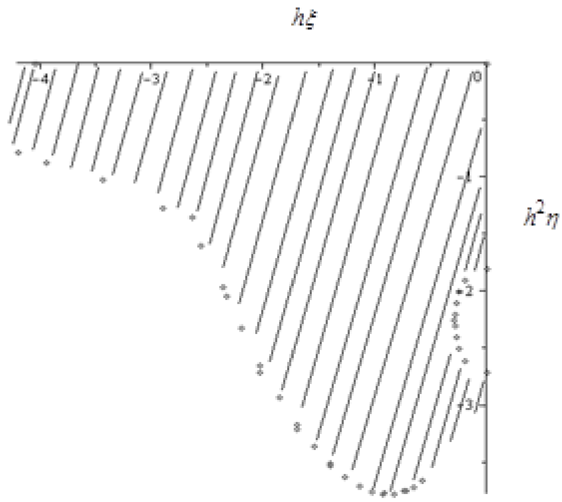


Figure 2: Stability region in the $h\xi$, $h^2\eta$ plane.

From Figure 2 it can be seen that the proposed method is stable within the shaded region.

## 5. Numerical Results

Some numerical problems are presented to study the performance of the proposed method.

Problem 1 ($k(x,s)=1$) Filiz (2013):

$$\begin{aligned} y'(x) &= -\int_0^x y(s)\,\mathrm{d}s, \qquad y(0)=1, \qquad 0\le x\le 1,\\ &\text{with the exact solution}\\ y(x) &= \cos x. \end{aligned}$$

Problem 2 ($k(x,s)=1$) Filiz (2013):

$$\begin{aligned} y'(x) &= 1-\int_0^x y(s)\,\mathrm{d}s, \qquad y(0)=0, \qquad 0\le x\le 1,\\ &\text{with the exact solution}\\ y(x) &= \sin x. \end{aligned}$$

Problem 3 ($k(x,s)\neq 1$) Chen and Zhang (2011):

$$\begin{aligned} y'(x) &= -\sin x-\cos x+\int_0^x 2\cos(x-s)y(s)\,\mathrm{d}s,\\ y(0) &= 1, \qquad 0\le x\le 5,\\ &\text{with the exact solution}\\ y(x) &= e^{-x}. \end{aligned}$$

Problem 4 ($k(x,s)\neq 1$) Chen and Zhang (2011):

$$\begin{aligned} y'(x) &= 1+y(x)-xe^{-x^2}-2\int_0^x xse^{-y^2(s)}\,\mathrm{d}s,\\ y(0) &= 0, \qquad 0\le x\le 5,\\ &\text{with the exact solution}\\ y(x) &= x. \end{aligned}$$

Notations used in the following tables are:

| | | |
|---|---|---|
| MAXE | : | Maximum error |
| $h$ | : | Step size |
| TS | : | Total steps |
| TF | : | Total functions |
| 2MVIDE | : | Multistep block method proposed in this paper |
| RK3 | : | Runge-Kutta method of order 3 with Simpson's 1/3 rule by Filiz (2013) |
| BVMs | : | Combination of BVMs and third order Generalized Adams Method by Chen and Zhang (2011) |

Table 1: Numerical results for Problem 1.

| $h$ | 0.025 | 0.0125 | 0.00625 |
|---|---|---|---|
| **MAXE** | | | |
| RK3 | 5.4120(-7) | 6.8067(-8) | 8.5342(-9) |
| 2MVIDE | 4.2996(-7) | 5.6627(-8) | 7.2698(-9) |
| **TS** | | | |
| RK3 | 40 | 80 | 160 |
| 2MVIDE | 21 | 41 | 81 |
| **TF** | | | |
| RK3 | 120 | 240 | 480 |
| 2MVIDE | 43 | 83 | 163 |

Table 2: Numerical results for Problem 2 .

| $h$ | 0.025 | 0.0125 | 0.00625 |
|---|---|---|---|
| **MAXE** | | | |
| RK3 | 2.7829(-7) | 3.4510(-8) | 1.3953(-9) |
| 2MVIDE | 4.2079(-7) | 4.9165(-8) | 5.9272(-9) |
| **TS** | | | |
| RK3 | 40 | 80 | 160 |
| 2MVIDE | 21 | 41 | 81 |
| **TF** | | | |
| RK3 | 120 | 240 | 480 |
| 2MVIDE | 43 | 83 | 163 |

Table 1 and Table 2 show the maximum errors between the method of RK3 and 2MVIDE for the case where $k(x,s) = 1$. The numerical problems are tested at three different step sizes. The results show that the maximum errors for both methods are comparable. However, 2MVIDE need less number of total steps and total functions call compared to RK3 for every different step size when solving the problems.

Table 3: Numerical results for Problem 3.

| $h$ | 0.25 | 0.125 | 0.0625 | 0.03125 |
|---|---|---|---|---|
| | | **MAXE** | | |
| BVMs | 2.1607(-1) | 2.8411(-2) | 3.6378(-3) | 4.6011(-4) |
| 2MVIDE | 5.5960(-2) | 1.1875(-3) | 3.7553(-4) | 8.4181(-5) |
| | | **TF** | | |
| 2MVIDE | 51 | 91 | 171 | 331 |

Table 4: Numerical results for Problem 4.

| $h$ | 0.25 | 0.125 | 0.0625 | 0.03125 |
|---|---|---|---|---|
| | | **MAXE** | | |
| BVMs | 1.0403(-1) | 3.0971(-2) | 5.0462(-3) | 7.0448(-4) |
| 2MVIDE | 5.1109(-1) | 4.2052(-2) | 2.9188(-3) | 1.9083(-4) |
| | | **TF** | | |
| 2MVIDE | 51 | 91 | 171 | 331 |

In Table 3 and Table 4 the numerical results for problem 3 and problem 4 are shown. These problems are solved using Approach II. In Table 3, 2MVIDE obtained smaller maximum error compared to BVMs and in Table 4 the maximum error for both methods are comparable. The accuracy of the 2MVIDE for solving the tested problems improved as the step sizes reduced. Thus 2MVIDE manage to solve and give acceptable results for all the tested problems.

# 6. Conclusion

In this paper, we have introduced and implemented the multistep block method combined with quadrature rules for solving linear and nonlinear VIDEs. From the results it can be concluded that the proposed multistep block method is appropriate for solving VIDEs.

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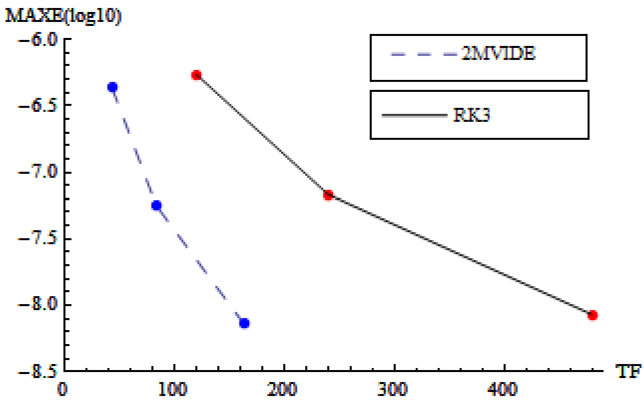


Figure 3: Graph of total functions versus maximum errors for RK3 and 2MVIDE when solving Problem 1.

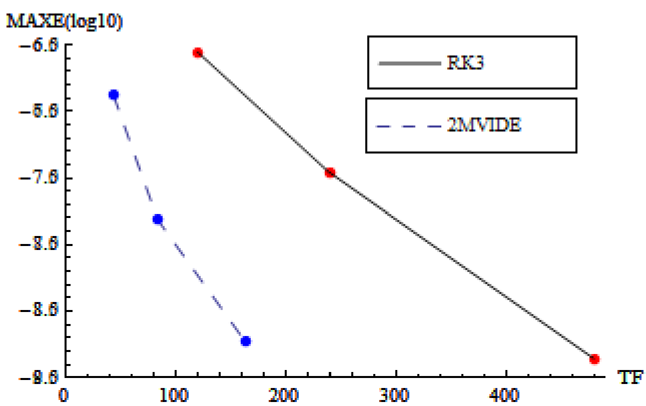


Figure 4: Graph of total functions versus maximum errors for RK3 and 2MVIDE when solving Problem 2.

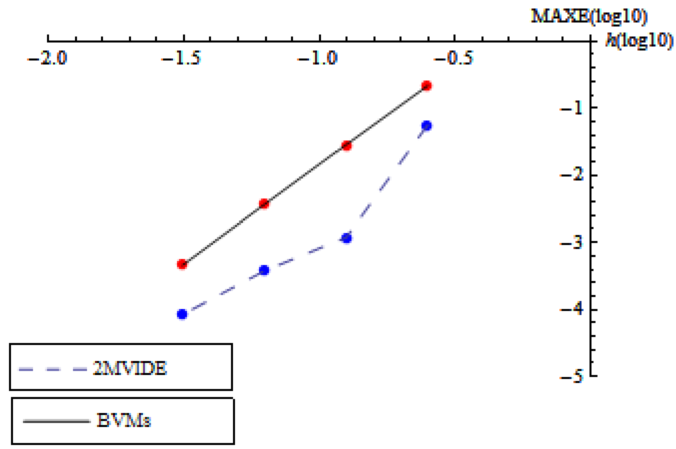


Figure 5: Graph of step sizes versus maximum errors for BVMs and 2MVIDE when solving Problem 3.

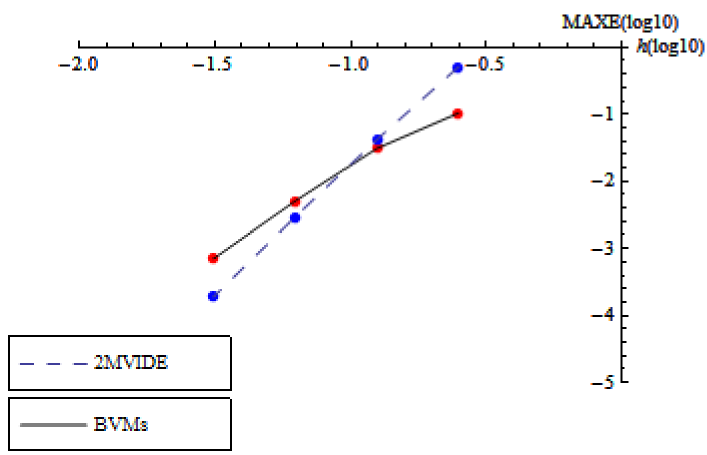


Figure 6: Graph of step sizes versus maximum errors for BVMs and 2MVIDE when solving Problem 4.

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