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The Point Procedure PRZSS1 for the Simultaneous Estimation of the Zeros of a Polynomial

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ABSTRACT

The order of convergence of the existing interval zoro symmetric single-step procedure is at least 4. The point version shares the same order of convergence. The point version of the interval zoro symmetric single-step procedure, aptly called point zoro symmetric single-step procedure, is modified by repeating the two forward and one backward steps r times. This modified procedure is named as point repeated zoro symmetric single-step procedure. It is shown that this procedure converges at the rate of at least $3r + 1$ where $r \geq 1$. This procedure and that of the point zoro symmetric single-step procedure are identical when $r = 1$. Numerical results show that the proposed point repeated zoro symmetric single-step procedure possesses higher rate of convergence than does the point zoro symmetric single-step procedure .

Keywords: Point procedure, R-order of convergence, simple zeros, simultaneous estimation.

1. Introduction

The ideas used in the point zoro symmetric single-step procedure (PZSS1) (Monsi et al. (2012)) and the the interval repeated symmetric single-step procedure (IRSS1) (Monsi and Wolfe (1988)) are compiled to establish a new procedure named as the point repeated zoro symmetric single-step procedure (PRZSS1). The term "zoro" or "zorro" is referred to a "z" or the "forward-backward-forward" pattern established in Monsi et al. (2012). The R-order of convergence of PZSS1 (Monsi et al. (2012)) is at least 4. The procedure PZSS1 may be upgraded by repeating the steps in PZSS1 (forward-backward-forward or FBF steps) so that the FBF steps are repeated r times. This established PRZSS1. We have analysed that the rate of convergence of PRZSS1 is at least $(3r+1)(r \geq 1)$. Other ideas for upgrading the performances of the methods in solving nonlinear equations can be seen in Bakar et al. (2012), Neta et al. (2012), Petkovic et al. (2010), Sharma et al. (2013), Thukral and Petkovic (2010), Wang and Liu (2010) and Wang et al. (2012). More recent advances are studied by Jamaludin et al. (2013a, 2014a,b, 2013b,c,d), Monsi et al. (2014), Sham et al. (2013a,b,c) and Rusli et al. (2014a,b).Most recent discussions on the rate of convergence can be found in Hanapiah et al. (2015a,b), Monsi and Hassan (2015), Monsi et al. (2015a,b), Rusli et al. (2015a,b,c), Sham et al. (2015) and Jamaludin et al. (2015a,b,c)

2. Estimating The Zeros of a Polynomial

The following definitions and theorem by Alefeld and Herzberger (1983) and Ortega and Rheinboldt (1970) are very useful for the evaluation of R-order of convergence.

Definition 2.1. *If there exists a $p \geq 1$ such that for any null sequence $w^{(k)}$ generated from $x^{(k)}$, then the R-factor of the sequence $w^{(k)}$ is defined to be*

$$R_p(w^{(k)}) = \begin{cases} \underset{k\to\infty}{lim\ sup} \left\| w^{(k)} \right\|^{\frac{1}{k}}, & p = 1, \\ \underset{k\to\infty}{lim\ sup} \left\| w^{(k)} \right\|^{\frac{1}{p^k}}, & p > 1, \end{cases}$$

where R_p is independent of the norm $\|.\|$.

Definition 2.2. *We next define the R-order of the procedure I in terms of the R-factor as*

$$O_R(I,x^*) = \begin{cases} +\infty \;\; if \; R_p(I,x^*) = 0, & for\; p \geq 1, \\ inf\{p | p \in [1,\infty), R_p(I,x^*) = 1\}, & otherwise. \end{cases}$$

Suppose that $R_p(w^{(k)}) < 1$, *then it follows from Ortega and Rheinboldt (1970) that the R-order of* I *satisfies the inequality* $O_R(I,x^*) \geq p$.

Theorem 2.1. *Let* I *be an iterative procedure and let* $\Omega(I,x^*)$ *be the set of all sequences* $x^{(k)}$ *generated by* I *which converges to the limit* x^*. *Suppose that there exists a* $p \geq 1$ *and a constant* γ *such that for any* $\{x^{(k)}\} \in \Omega(I,x^*)$,

$$\left\| x^{(k+1)} - x^* \right\| \leq \gamma \left\| x^{(k)} - x^* \right\|^p, \;\; k \geq k_0 = k_0(\{x^{(k)}\}).$$

Then, it follows that R-order of I *satisfies the inequality* $O_R(I,x^*) \geq p$.

We will use this result in order to calculate the R-order of convergence.

Let $p : C \rightarrow C$ *be a polynomial given by*

$$p(x) = \sum_{i=0}^{n} a_i x^i \quad (n > 1) \tag{1}$$

$a_i \in C \quad (i = 1, ..., n)$ *and* $a_n \neq 0$,*and let* $x^* \in (x_1^*, ..., x_n^*)^T$ *be the zeros vector of (1) in the form of*

$$p(x) = \prod_{i=1}^{n} (x - x_i^*) \tag{2}$$

with $a_n = 1$. *Taking the derivative of (2), we have*

$$p_i(x) = \prod_{j=1}^{i-1} (x - x_j^*) \prod_{i+1}^{n} (x - x_j^*) \tag{3}$$

Suppose that, for x_j *is an estimate of* x_j^* , *where* $x^* = (x_1^*, ..., x_n^*)^T$. *Then we can define* $q : C \rightarrow C$ *as*

$$q(x) = \prod_{j=1}^{n} (x - x_j) \tag{4}$$

and also its derivative

$$q^{'}(x) = \prod_{j \neq 1}^{n} (x_i - x_j) \;\; (i = 1, ..., n). \tag{5}$$

Then by (2), we have

$$x_i^* = x_i - \frac{p(x_i)}{\prod_{j \neq i} (x_i - x_j^*)} \; (i = 1, ..., n). \tag{6}$$

Now $x_j \approx x_j^* \quad (j = 1, ..., n)$ *so by (6),*

$$x_i^* \approx x_i - \frac{p(x_i)}{\prod_{j \neq i} (x_i - x_j)} \; (i = 1, ..., n). \tag{7}$$

An iteration procedure for (7) is given by

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j \neq i} (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, ..., n) \tag{8}$$

where the order of convergence of (8) is at least two. Equation (8) is modified by Alefeld and Herzberger (1974) to give

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1} (x_i^{(k)} - x_j^{(k+1)}) \prod_{j+i}^{n} (x_i^{(k)} - x_j^{(k)})} \quad (i = 1, ..., n) \tag{9}$$

where its rate of convergence is greater than two. The following procedure is in Monsi (2012) (see also Alefeld (1977), Alefeld and Herzberger (1974)), *for* $k \geq 0$,

$$x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k)})} \quad (i = 1, ..., n) \tag{10a}$$

$$x_i^{(k,2)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=i}^{i-1}(x_i^{(k)} - x_j^{(k,1)}) \prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,2)})} \quad (i = n, ..., 1) \tag{10b}$$

$$x_i^{(k+1)} = x_i^{(k+2)} \quad (i = 1, ..., n) \tag{10c}$$

where the R-order of convergence of (10) is at least three (Monsi (2012)). Further discussions on the rate of convergence can be found in Milovanovic and Petkovic (1983), Petkovic and Milosevic (2005) and Petkovic (1982).

3. The Procedure PRZSS1

Our proposed procedure PRZSS1 is defined as follows;

For $l = 1, ..., r,\ k \geq 0$,

$$x_i^{(k,3l-3)} = x_i^{(k)} \ (i = 1, ..., n) \tag{11a}$$

$$x_i^{(k,3l-2)} = x_i^{(k,3l-3)} - \frac{p(x_i^{(k,3l-3)})}{\prod_{j=i}^{i-1}(x_i^{(k,3l-3)} - x_j^{(k,3l-2)}) \prod_{j=i+1}^{n}(x_i^{(k,3l-3)} - x_j^{(k,3l-3)})} \quad (i = 1, ..., n) \tag{11b}$$

$$x_i^{(k,3l-1)} = x_i^{(k,3l-3)} - \frac{p(x_i^{(k,3l-3)})}{\prod_{j=i}^{i-1}(x_i^{(k,3l-3)} - x_j^{(k,3l-2)}) \prod_{j=i+1}^{n}(x_i^{(k,3l-3)} - x_j^{(k,3l-1)})} \quad (i = n, ..., 1) \tag{11c}$$

$$x_i^{(k,3l)} = x_i^{(k,3l-3)} - \frac{p(x_i^{(k,3l-3)})}{\prod_{j=i}^{i-1}(x_i^{(k,3l-3)} - x_j^{(k,3l)}) \prod_{j=i+1}^{n}(x_i^{(k,3l-3)} - x_j^{(k,3l-1)})} \quad (i = 1, ..., n) \tag{11d}$$

$$x_i^{(k+1)} = x_i^{(k,3r)} \quad (i = 1, ..., n) \tag{11e}$$

This procedure has the following attractive features;

- the terms $\prod_{j=i}^{i-1}(x_i^{(k,3l-3)} - x_j^{(k,3l-1)})$ $(i = 2, ..., n)$ $(l \geq 1)(k \geq 0)$ which are computed in (11b) can be reused in (11c).
- the terms $\prod_{i+1}^{n}(x_i^{(k,3l-3)} - x_j^{(k,3l-1)})$ $(i = n-1, ..., 1)$ $(l \geq 1)(k \geq 0)$ which are computed in (11c)can be reused in (11d).
- from (11b) and (11c), $x_n^{(k,3l-1)} = x_n^{(k,3l-2)}$ $(l \geq 1)(k \geq 0)$ need not be computed.
- from (11c) and (11d), $x_1^{(k,3l)} = x_1^{(k,3l-1)}$ $(l \geq 1)(k \geq 0)$ need not be computed.
- the procedure PRZSS1 is an extension of the procedure PZSS1 of Monsi et al.(2012).

Theorem 3.1. *If (i) $p : C \to C$ defined by (1) has n zeros x_i^* $(i = 1, ..., n)$; (ii) $\left|x_i^{(0)} - x_i^*\right| \leq \frac{\theta d}{2n-1}$ $(i = 1, ..., n)$ where $0 < \theta < 1$ and $d = min\left|x_i^* - x_j^*\right|$ $j \neq i$, the sequences $x_i^{(k)}$ are from (11), then $x_i^{(k)} \to x_i^*(k \to \infty)$ and $O_R(PRZSS1, x^*) \geq 3r + 1$ $(r \geq 1)$.*

Proof. If we are to consider $q_{(1,i)}, q_{(2,i)}$ and $q_{(3,i)}$ in Monsi et al. (2012), then we may define the following functions as follows. For $l = 1, ..., r$, let

$$q_{3l-2,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,3l-2)}) \prod_{m=i+1}^{n}(x - x_m^{(k,3l-3)}),$$

$$q_{3l-1,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,3l-2)}) \prod_{m=i+1}^{n}(x - x_m^{(k,3l-1)}),$$

and

$$q_{3l,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,3l)}) \prod_{m=i+1}^{n}(x - x_m^{(k,3l-1)}).$$

□

Then by considering functions $\phi_{(1,i)}, \phi_{(2,i)}$ and $\phi_{(3,i)}$ in Monsi et al. (2012), let for $i = 1, ..., n$; $l = 1, ..., r$,

$$\phi_{3l-2,i}(x) = q_{3l-2,i}(x)+\sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,3l-3)})q_{3l-2,i}(x)}{q'_{3l-2,i}(x_j^{(k,3l-2)})(x-x_j^{(k,3l-2)})}+\sum_{i+1}^{n} \frac{p_i(x_j^{(k,3l-3)})q_{3l-2,i}(x)}{q'_{3l-2,i}(x_j^{(k,3l-3)})(x-x_j^{(k,3l-3)})'}$$

$$\phi_{3l-1,i}(x) = q_{3l-1,i}(x)+\sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,3l-2)})q_{3l-1,i}(x)}{q'_{3l-1,i}(x_j^{(k,3l-2)})(x-x_j^{(k,3l-2)})}+\sum_{i+1}^{n} \frac{p_i(x_j^{(k,3l-1)})q_{3l-1,i}(x)}{q'_{3l-1,i}(x_j^{(k,3l-1)})(x-x_j^{(k,3l-1)})'}$$

and

$$\phi_{3l,i}(x) = q_{3l,i}(x)+\sum_{j=1}^{i-1} \frac{p_i(x_j^{(k,3l)})q_{3l,i}(x)}{q'_{3l,i}(x_j^{(k,3l)})(x-x_j^{(k,3l)})}+\sum_{i+1}^{n} \frac{p_i(x_j^{(k,3l)})q_{3l,i}(x)}{q'_{3l,i}(x_j^{(k,3l-1)})(x-x_j^{(k,3l-1)})'}$$

where $p_i(x)$ is defined by (3).The points $x_j^{(k,3l-3)}, x_j^{(k,3l-2)}, x_j^{(k,3l-1)}$ and $x_j^{(k,3l)}$, $(j = 1,...,n;\ j \neq i;\ l = 1,...,r;\ k \geq 0)$ satisfy Lemma 1 of Monsi et al.(2012).

By Lemma 1 and Lemma 2 of Monsi et al.(2012), with $q_i = q_{3l-2,i}, \check{x}_i = x_i^{(k,3l-3)}, \hat{x}_i = x_i^{(k,3l-3)}, \bar{x}_i = x_i^{(k,3l-2)}, \phi_i = \phi_{3l-2,i}$ $(i = 1,...,n)$,then we have

$$w_i^{(k,3l-2)} = w_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} a_{ij}^{(k,3l-2)} w_j^{(k,3l-2)} + \sum_{i+1}^{n} a_{ij}^{(k,3l-3)} w_j^{(k,3l-3)} \right\}, \quad (12)$$

where

$$w_i^{(k,s)} = x_i^{(k,s)} - x_i^* \ (s = 0,1,...,3r)$$

$$a_{i,j}^{(k,3l-2)} = \frac{\prod_{m\neq i,j}(x_j^{(k,3l-2)} - x_m^*)}{q'_{3l-2,i}(x_j^{(k,3l-2)})(x_j^{(k,3l-2)} - x_i^{(k,3l-3)})'} \quad (j = 1,...,i-1), \quad (13)$$

and

$$a_{i,j}^{(k,3l-3)} = \frac{\prod_{m\neq i,j}(x_j^{(k,3l-3)} - x_m^*)}{q'_{3l-2,i}(x_j^{(k,3l-3)})(x_j^{(k,3l-3)} - x_i^{(k,3l-3)})'} \quad (j = i+1,...,n). \quad (14)$$

Using Lemma 1 and Lemma 2 of Monsi et al. (2012), with $q_i = q_{3l-1,i}, \check{x}_i = x_i^{(k,3l-3)}, \hat{x}_i = x_i^{(k,3l-1)}, \bar{x}_i = x_i^{(k,3l-2)}, \phi_i = \phi_{3l-1,i}$, we have

$$w_i^{(k,3l-1)} = w_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} \beta_{ij}^{(k,3l-2)} w_j^{(k,3l-2)} + \sum_{i+1}^{n} \beta_{ij}^{(k,3l-1)} w_j^{(k,3l-1)} \right\}, \quad (15)$$

where

$$\beta_{i,j}^{(k,3l-2)} = \frac{\prod_{m \neq i,j} (x_j^{(k,3l-2)} - x_m^*)}{q'_{3l-1,i}(x_j^{(k,3l-2)})(x_j^{(k,3l-2)} - x_i^{(k,3l-3)})} \quad (j = 1, ..., i-1; i = 1, ..., n), \quad (16)$$

and

$$\beta_{i,j}^{(k,3l-1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,3l-1)} - x_m^*)}{q'_{3l-1,i}(x_j^{(k,3l-1)})(x_j^{(k,3l-1)} - x_i^{(k,3l-3)})} \quad (j = 1, ..., i-1; i = 1, ..., n), \quad (17)$$

Using Lemma 1 and Lemma 2 of Monsi et al.(2012), with $q_i = q_{3l,i}, \check{x}_i = x_i^{(k,3l-3)}, \hat{x}_i = x_i^{(k,3l-1)}, \bar{x}_i = x_i^{(k,3l)}, \phi_i = \phi_{3l,i}$, then

$$w_i^{(k,3l)} = w_i^{(k,3l-2)} \left\{ \sum_{j=1}^{i-1} \gamma_{ij}^{(k,3l)} w_j^{(k,3l)} + \sum_{i+1}^{n} \gamma_{ij}^{(k,3l-1)} w_j^{(k,3l-1)} \right\}, \quad (18)$$

where

$$\gamma_{i,j}^{(k,3l)} = \frac{\prod_{m \neq i,j} (x_j^{(k,3l)} - x_m^*)}{q'_{3l,i}(x_j^{(k,3l)})(x_j^{(k,3l-3)} - x_i^{(k,3l-3)})} \quad (j = 1, ..., i-1; i = 1, ..., n), \quad (19)$$

and

$$\gamma_{i,j}^{(k,3l-1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,3l-1)} - x_m^*)}{q'_{3l,i}(x_j^{(k,3l-1)})(x_j^{(k,3l-1)} - x_i^{(k,3l-3)})} \quad (j = i+1, ..., n; i = 1, ..., n). \quad (20)$$

By induction on l, from (12) - (14) and Lemma 3 of Monsi et al.(2012) we will have $|w_i^{(0,3l-2)}| \leq \theta^{3l-2} |w_i^{(0,0)}|$ $(i = 1, ..., n)$, and it follows from (15) - (17) and Lemma 3 that $|w_i^{(0,3l-1)}| \leq \theta^{3l-1} |w_i^{(0,0)}|$ $(i = 1, ..., n)$. Subsequently from (18) - (20) and Lemma 3 we can conclude that

$$|w_i^{(0,3l)}| \leq \theta^{3l} |w_i^{(0,0)}| \ (i = 1, ..., n),$$

where $|w_i^{(1,0)}| = |w_i^{(0,3r)}| \leq \theta^{3r} |w_i^{(0,0)}|$ $(i = 1, ..., n)$follows from (11e). By induction on k, finally we get the following result,

$$|w_i^{(k,0)}| \leq \theta^{(3r+1)^{k-1}} |w_i^{(0,0)}| \ (i = 1, ..., n),$$

where $x_i^{(k)} \to x_i^* \ (k \to \infty)$.Suppose that

$$h_i^{(k,m)} = \frac{(2n-1)}{d}|w_i^{(k,m)}| \ (i = 1, ..., n) \ (m = 0, 1, ..., 3r). \tag{21}$$

Then by (12)-(21),

$$h_i^{(k,3l-2)} \leq \frac{1}{(n-1)} h_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3l-2)} + \sum_{i+1}^{n} h_j^{(k,3l-3)} \right\} \ (i = 1, ..., n), \tag{22}$$

$$h_i^{(k,3l-1)} \leq \frac{1}{(n-1)} h_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3l-2)} + \sum_{i+1}^{n} h_j^{(k,3l-1)} \right\} \ (i = n, ..., 1), \tag{23}$$

and

$$h_i^{(k,3l)} \leq \frac{1}{(n-1)} h_i^{(k,3l-3)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,3l)} + \sum_{i+1}^{n} h_j^{(k,3l-1)} \right\} \ (i = 1, ..., n). \tag{24}$$

For $l = 1, ..., r$ let

$$u_i^{(1,3l-2)} = \begin{cases} 3l-1 & (i = 1, ..., n-1) \\ 3l & (i = n) \end{cases}, \tag{25}$$

$$u_i^{(1,3l-1)} = \begin{cases} 3l+1 & (i = 1) \\ 3l & (i = 2, ..., n) \end{cases}, \tag{26}$$

and

$$u_i^{(1,3l)} = \begin{cases} 3l+1 & (i = 1, ..., n-1) \\ 3l+2 & (i = n) \end{cases}, \tag{27}$$

For $m = 1, ..., 3r$, let

$$u_i^{(k+1,m)} = \begin{cases} (3r+1)u_i^{(k,m)} & (i = 1, ..., n-1) \\ (3r+1)u_i^{(k,m)} + 1 & (i = n) \end{cases}. \tag{28}$$

Then by (25) - (28),for $\forall k \geq 0$,

$$u_i^{(k,3l-2)} = \begin{cases} (3l-1)(3r+1)^{k-1} & (i = 1, ..., n-1) \\ (\frac{9lr+1}{3r})(3r+1)^{k-1} - \frac{1}{3r} & (i = n) \end{cases}, \forall k \geq 0, \tag{29}$$

$$u_i^{(k,3l-1)} = \begin{cases} (3l+1)(3r+1)^{k-1} & (i = 1) \\ 3l(3r+1)^{k-1} & (i = 2, ..., n-1) \\ (\frac{9lr+1}{3r})(3r+1)^{k-1} - \frac{1}{3r} & (i = n) \end{cases} \tag{30}$$

and

$$u_i^{(k,3l)} = \begin{cases} (3l+1)(3r+1)^{k-1} & (i=1,...,n-1) \\ (\frac{9lr+6r+1}{3r})(3r+1)^{k-1} - \frac{1}{3r} & (i=n) \end{cases}. \quad (31)$$

Suppose that

$$h_i^{(0,0)} \leq h < 1 \ \ (i=1,...,n). \quad (32)$$

Therefore by induction on i, k and l, and from (21) - (32) we will have

$$h_i^{(k,3l-2)} \leq h^{u_i^{(k+1,3l-2)}},\ h_i^{(k,3l-1)} \leq h^{u_i^{(k+1,3l-1)}},\ h_i^{(k,3l)} \leq h^{u_i^{(k+1,3l)}} \ \ (i=1,...,n,\ k \geq 0). \quad (33)$$

where by (31)-(33)and (11e), $(\forall k \geq 0)$

$$h_i^{(k)} \leq h^{(3r+1)^k} (i=1,...,n). \quad (34)$$

By (21) for $m = 3r$,

$$|w_i^{(k,3r)}| = \frac{d}{(2n-1)} h_i^{(k,3r)} \ (i=1,...,n),$$

then by (11e),

$$|w_i^{(k+1)}| = |w_i^{(k+1,0)}| = |w_i^{(k,3r)}|;\ h_i^{(k,3r)} = h_i^{(k+1,0)} = h_i^{(k+1)} \ (i=1,...,n).$$

So, for $(i=1,...,n)$ $(k \geq 0)$,

$$|w_i^{(k)}| = \frac{d}{(2n-1)} h_i^{(k)}. \quad (35)$$

For $k \geq 0$, let

$$w^{(k)} = \max_{1 \leq i \leq n} \{|w_i^{(k)}|\}, \quad (36)$$

and

$$h^{(k)} = \max_{1 \leq i \leq n} \{h_i^{(k)}\}. \quad (37)$$

Then, by (21)-(37)

$$w^{(k)} \leq \frac{d}{(2n-1)} h^{(3r+1)^k} \ (\forall k \geq 0).$$

$$\begin{aligned} R_{3r+1}(w^{(k)}) &= \lim_{k\to\infty} sup\{(w^{(k)})^{1/(3r+1)^k}\} \\ &\leq \lim_{k\to\infty} sup\{\frac{d}{(2n-1)}^{1/(3r+1)^k} h\} \\ & h < 1. \end{aligned}$$

Therefore, by Milovanovic and Petkovic (1983), Ortega and Rheinboldt (1970) and Alefeld and Herzberger (1983), we thus have

$$O_R(PRZSS1, x_i^*) \geq (3r+1)\ (r \geq 1).\blacksquare$$

4. Numerical Result

Table 1 shows the CPU times to run the three algorithms PSS1, PZSS1 and PRZSS1 using ten test polynomials of Rusli et al. (2011), with initial points $x_i^{(0)}$ being the midpoints of the respective intervals $X_i^{(0)}$ in Alefeld (1977). The convergence criterion used is $|x_i^{(k)} - x_i^*| < 10^{-12} \quad (i = 1, ..., n)$. The results show that the procedure PRZSS1 is most efficient compared to PSS1(Monsi 2012) and PZSS1 (Monsi et al.2012). These can be seen clearly in Figures 1 and 2.

Table 1: CPU Times and Number of Iterations k and r

Polynomial	n	**PSS1**(k)	**PZSS1**(k)	**PRZSS1**(k,r)
1	4	0.20280 (3)	0.19500 (2)	0.14352 (2,2)
2	4	0.20280 (3)	0.19968 (2)	0.16848 (2,2)
3	5	0.21216 (3)	0.24024 (2)	0.22152 (2,2)
4	6	0.32136 (3)	0.31824 (2)	0.30576 (2,2)
5	9	0.59280 (3)	0.59280 (3)	0.58968 (2,2)
6	5	0.24648 (3)	0.24336 (2)	0.22776 (2,2)
7	6	0.42120 (4)	0.40560 (4)	0.42900 (3,2)
8	3	0.17160 (2)	0.16120 (2)	0.12480 (2,2)
9	6	0.31200 (3)	0.29640 (2)	0.26520 (2,2)
10	10	0.57720 (3)	0.53040 (2)	0.64740 (2,2)

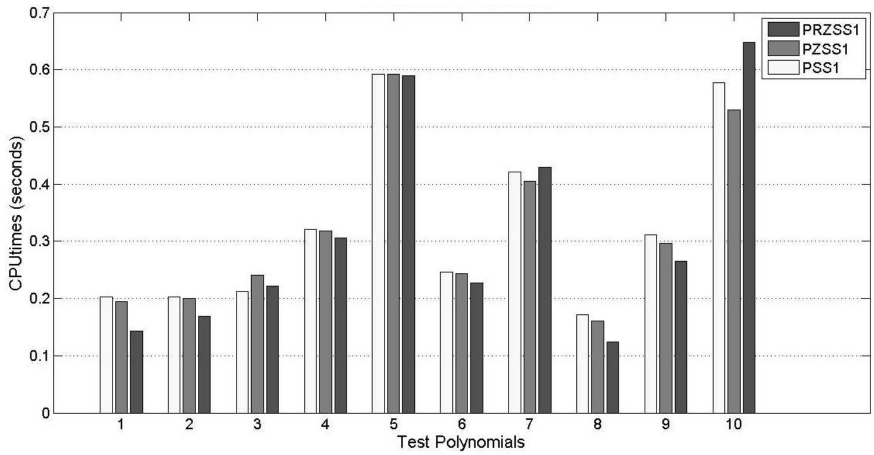

Figure 1: Comparison of CPU times

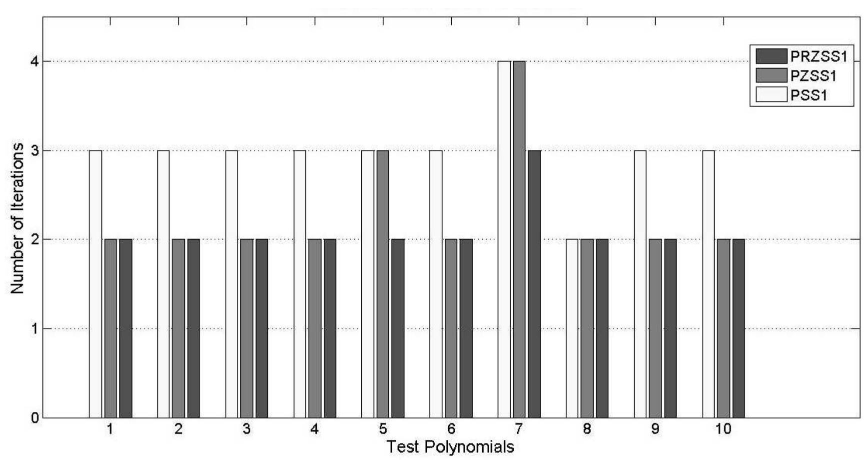

Figure 2: Comparison of Number of Iterations

5. Conclusion

We have shown that convergence rate of PRZSS1 is at least$(3r+1)$ $(r \geq 1)$ that is higher than that of PZSS1 (Monsi et al.2012). The results of CPU times show that the efficiency of PRZSS1 occurs when $r = 2$. For other values of r

, we found that the CPU times of PRZSS1 did not yield better results. The procedure PRZSS1 produced number of iterations significantly much less than that of PSS1 for almost all test polynomials. Between PRZSS1 and PZSS1, the numbers of iterations for both are almost unchanged except for test polynomials 5 and 7 where PRZSS1 takes less number of iterations compared to that of PZSS1. Furthermore, PRZSS1 is more efficient because it performs an inner iteration twice ($r = 2$) without further function evaluations. This reduces the number of CPU times for PRZSS1 compared to that of PZSS1 for all test polynomials, except for test polynomials 7 and 10. Also note that the procedure PRZSS1 reduces to PZSS1 when $r = 1$ (see equation (11) for $l = 1$ and algorithm PZSS1).

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