UNIVERSITI PUTRA MALAYSIA

INTERPOLATION AND APPROXIMATION OF NON DIFFERENTIABLE FUNCTION USING POLYNOMIAL

KOO LEE FENG

INTERPOLATION AND APPROXIMATION OF NON DIFFERENTIABLE FUNCTION USING POLYNOMIAL

KOO LEE FENG

MASTER OF SCIENCE
UNIVERSITI PUTRA MALAYSIA

2007

INTERPOLATION AND APPROXIMATION OF NON DIFFERENTIABLE FUNCTION USING POLYNOMIAL

By

KOO LEE FENG

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfilment of the Requirement for the Degree of Master of Science

February 2007

DEDICATION

To my God Jehovah

for his grace, kindness and love.

All glories, praises and thanksgivings

to Him in the highest.

"For the LORD gives wisdom;

From His mouth come knowledge and understanding."

Proverbs 2: 6

Scripture

Psalm 40: 1-2

"*I waited patiently for the Lord; And He inclined to me, And heard my cry. He also brought me up out of a horrible pit, Out of the miry clay, And set my feet upon a rock, And established my steps.*"

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Master of Science

INTERPOLATION AND APPROXIMATION OF NON DIFFERENTIABLE FUNCTION USING POLYNOMIAL

By

KOO LEE FENG

February 2007

Chairman: Associate Professor Adem Kilicman, PhD

Faculty: Science

In any approximation problem, we concerned with a measure of closeness of polynomial $p(x)$ to a function $f(x)$. Approximations are often obtained by power series expansions in which the higher order terms are dropped.

One of the fundamental idea in the differential calculus is that a function can be "locally" approximated by its tangent line. If f be a function defined on an open interval I and let $c \in I$ and $n \in N$. Suppose that the function has n-th derivative at all $x \in I$. Then, the polynomial $T_n(f,c)(c)$ is called as Taylor polynomial of order n of function at the point c. If the function is infinitely differentiable on I, so the series is called Taylor Series of f at point c. Taylor Series is one of the fundamental idea in differential calculus. However, Taylor Series only can apply if and only if the function f be differentiated on its interval stated.

A differentiable function is a continuous function. But, this is not always true that a continuous function is a differentiable function. Weierstass Approximation Theorem states that every continuous function defined on an interval $[a,b]$ can be uniformly approximated as closely as the decided function. Thus, this theorem ensure that every function can be approximate by a polynomial.

Consequently, in the research, we develop a new approximation method to approximate the non differentiable function which has singularity at one point, two points and three points by using Fourier series, Lagrange Interpolation and convolution method. We will also discover the asymptotic of Lagrange Interpolation for function $f_\lambda(t) = |t|^\lambda$ where $\lambda > 0$ with equidistant nodes and discover the convergence of $f_\lambda(t) = |t|^\lambda, \lambda > 0$ at point t =0. Lastly, we compare the effective of Fourier series and Lagrange method in approximate the non-differentiable function.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Master Sains

INTERPOLASI DAN PENGHAMPIRAN FUNGSI TAK TERBEZAKAN DENGAN MENGGUNAKAN POLYNOMIAL

Oleh

KOO LEE FENG

Februari 2007

Pengerusi: **Profesor Madya Adem Kilicman, PhD**

Falkuti: **Sains**

Dalam sebarang masalah penyelesaian hampiran, merujuk kepada pengukuran yang terdekat polinomial $p(x)$ terhadap fungsi asal $f(x)$. Biasanya, suatu penghampiran $f(x)$ diperoleh apabila perluasan kuasa siri kuasa yang lebih tinggi dihapuskan.

Salah satu idea asa dalam kalkulus pembezaan ialah suatu fungsi dapat dihampiri secara setempat dengan menggunakan garisan tangen fungsi tersebut. Sekiranya, sesuatu fungsi, f ditakrifkan dalam selang terbuka I sedemikian hingga $c \in I$ dan $n \in N$. Jika fungsi tersebut mempunyai terbitan ke-n pada semua nlai $x \in I$, maka polynomial $T_n(f,c)(c)$ dikenali sebagai fungsi polinomial hampiran Taylor berdarjah n pada titik c. Jikalau fungsi tesebut dapat dibezakan infiniti pada selang I, maka siri tersebut dikenali sebagai siri Taylor fungsi f pada titik c. walaupun,siri Taylor merupakan idea asas dalam terbitan kalkulus,namun begitu, penghampiran dengan kaedah siri Taylor hanya dapat diaplikasikan jika dengan hanya jika fungsi f tersebut dapat diterbitkan dalam selang yang dinyatakan.

Dikatakan bahawa sesuatu fungsi terbitan adalah fungsi selanjar.Namun begitu, kenyataan “jika fungsi selanjar mengimplikasikan bahawa fungsi tersebut juga mempunyai fungsi terbitan” adalah tidak benar. Theorem Penghampiran Weierstrass menyatakan bahawa setiap fungsi selanjar yang tertakrif dalam selang tertutup $[a,b]$ dapat dihampirkan secara seragam, sehampir mungkin dengan fungsi yang diputuskan. Dengan itu, theorem ini memastikan bahawa setiap fungsi dapat dihampiri dengan menggunakan polinomial.

Justeru itu, dalam kajian ini telah membangunkan satu cara penghampiran yang baru untuk menghampiri fungsi yang tidak dapat dibezakan pada satu, dua dan tiga titik dengan menggunakan siri Fourier, interpolasi Lagrange dan konvolasi.Seterusnya,kita juga akan membincangkan asimptot Interpolasi Lagrange pada fungsi $f_\lambda(t) = |t|^\lambda$ di mana $\lambda > 0$ berdasarkan titik sama jarak dan seterusnya membincangkan penumpuan fungsi $f_\lambda(t) = |t|^\lambda$ di mana $\lambda > 0$ pada titik t=0. Akhirnya, perbandingan keberkesanan tentang cara penggunaan siri Fourier dan interpolasi Lagrange dalam proses penghampiran fungsi tanpa terbitan dibuat.

ACKNOWLEDGEMENTS

It’s my pleasure to acknowledge the cooperation of those who involved directly or indirectly in the journey of completing thesis. My deepest appreciation and gratitude goes to all mighty God, Jesus Christ for His wonderful guidance and blessing.

I am extremely grateful to my supervisor, Assoc Prof Dr Adem Kilicman for his excellent supervision, invaluable guidance, patience, helpful discussions and continuous encouragement. My thanks also goes to the members of my supervisor committee Assoc Prof Dr Fudziah Ismail and Dr Zainidin K. Eshkuvatov for their invaluable discussion, comment and help. They had given their precious time to share their expertise in preparing, researching and writing up this thesis.

Also, I would like to extend my thanks to all in Mathematics Department, University Putra Malaysia for their kind assistance during my studies. Also, my appreciation goes to my best buddies who endured with patience and understanding in the ups and downs throughout the preparation of this thesis.

Last but not least, I would like to thank my beloved family especially my parents for their unconditional love, support and understanding throughout the course of my studies.

DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

KOO LEE FENG

Date : 26APRIL2007

This thesis submitted to the Senate of Universiti Putra Malaysia and has been accepted ad fulfillment of the requirement for the degree of Master of Science. The members of the Supervisory Committee as follows :

Adem Kilicman, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Fudziah Ismail, PhD
Associate Professor
Faculty of Science
Universiti Putra Malaysia
(Member)

Zainidin K. Eshkuvatov, PhD
Faculty of Science
Universiti Putra Malaysia
(Member)

AINI IDERIS, PhD
Professor / Dean
School of Graduate Studies
University of Putra Malaysia

Date : 10MAY2007

I certify that an Examination Committee has met on 2nd February2007 to conduct the final examination of Koo Lee Feng on her master thesis entitled “Interpolation and Approximation of Non Differentiable Function using Polynomial” in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981.The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

Nik Asri Nik Long, PhD
Faculty of Science
Universiti Putra Malaysia
(Chairman)

Azmi Jaafar, PhD
Associate Professor
Faculty of Computer Science and
Information Technology
Universiti Putra Malaysia
(Internal Examiner)

Leong Wah June, PhD
Faculty of Science
Universiti Putra Malaysia
(Internal Examiner)

Maslina Darus, PhD
Professor
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
(External Examiner)

HASANAH MOHD. GHAZALI, PhD
Professor/Deputy Dean
School of Graduate Studies
Universiti Putra Malaysia

Date :

LIST OF TABLES

Table		Page
2.1	Lagrange Interpolation	8
5.1	Lagrange Interpolation With Function $f(t)=\|t\|$	57
5.4	Lagrange Interpolation With Function $f(t)=\|t\|$ on [-1,1] with Equidistant Nodes	59
5.13	Lagrange Interpolation With Function $f(t)=\|t^2-4\|$	65
5.16	Lagrange Interpolation With Function $f(t)=\|t^2-4\|$	66
5.19	Lagrange Interpolation With Function $f(t)=\|(t^2-4)\times(t-3)\|$	68

LIST OF FIGURES

Figure		Page
3.1	Approximation By Fourier Polynomial $f(t)=\|t\|$	23
3.2	Error Estimation of $f(t)=\|t\|$ With Fourier Polynomial	25
3.3	Approximation By Fourier Polynomial $f(t)=\|t-2\|$	26
3.4	Error Estimation of $f(t)=\|t-2\|$ With Fourier Polynomial	27
3.5	Approximation By Fourier Polynomial $f(t)=\|4-t^2\|$	31
3.6	Error Estimation of $f(t)=\|4-t^2\|$ With Fourier Polynomial	31
3.7	Approximation By Fourier Polynomial $f(t)=\|(t-3)(4-t^2)\|$	34
3.8	Error Estimation of $f(t)=\|(t-3)(4-t^2)\|$ With Fourier Polynomial	35
4.1	Convolution of $(f*g)(t)$ where $f(x)=\|x\|, g(x)=\|x\|^3$	41
4.2	Convolution of $(f*g)(t)$ where $f(x)=\|x\|, g(x)=\|x\|^2$	42
5.2	Lagrange Interpolation Polynomial at Several Degree of Function $f(t)=\|t\|$	58
5.3	Error Estimate of $f(t)=\|t\|$ With Lagrange Interpolation Polynomial	58
5.5	Lagrange Interpolation Polynomial at Several Degree of Function $f(t)=\|t\|$	60
5.6	Error Estimation of $f(t)=\|t\|$ With Lagrange Interpolation Polynomial	61
5.7	Interpolation of $f(t)=\|t\|$ by using Lagrange and Fourier at Degree 3	62
5.8	Error Estimate of $f(t)=\|t\|$ by using Lagrange and Fourier at Degree 3	62
5.9	Interpolation of $f(t)=\|t\|$ by using Lagrange and Fourier at Degree 4	63
5.10	Error Estimate of $f(t)=\|t\|$ by using Lagrange and Fourier at Degree 4	63
5.11	Interpolation of $f(t)=\|t\|$ by using Lagrange and Fourier at Degree 10	64

5.12	Error Estimate of $f(t)=\|t\|$ by using Lagrange and Fourier at Degree 10	64
5.14	Lagrange Interpolation Polynomial at Several Degree of Function $f(t)=\|t^2-4\|$	65
5.15	Error Estimate of $f(t)=\|t^2-4\|$ With Lagrange Interpolation Polynomial	66
5.17	Lagrange Interpolation Polynomial at Several Degree of Function $f(t)=\|t^2-4\|$	67
5.18	Error Estimate of $f(t)=\|t^2-4\|$ With Lagrange Interpolation Polynomial	67
5.20	Lagrange Interpolation Polynomial at Several Degree of Function $f(t)=\|(t^2-4)\times(t-3)\|$	68
5.21	Error Estimate of $f(t)=\|(t^2-4)\times(t-3)\|$ With Lagrange Interpolation Polynomial	69
5.22	$D_{\pm\varepsilon}=\{z=x\pm i\varepsilon : a\le x\le\sqrt{M^2-\varepsilon^2}\}$	75
5.23	Graph *y=ax+b* and *y=cos x*	139

TABLE OF CONTENTS

		Page
DEDICATION		ii
ABSTRACT		iii
ABSTRAK		v
ACKNOWLEDGEMENTS		vii
DECLARATION		viii
APPROVAL		ix
LIST OF TABLES		xi
LIST OF FIGURES		xii

CHAPTER

1	**INTRODUCTION**	1
2	**MATHEMATICS BACKGROUND AND LITERATURE REVIEW**	5
	2.1 Taylor Approximation	5
	2.2 Lagrange Interpolation	7
	2.3 Fourier Series	9
	2.4 Convolution	10
	2.5 Weierstrass Approximation Theorem	12
3	**APPROXIMATION TO THE NON DIFFERENTIABLE FUNCTION**	15
	3.1 Orthogonal Polynomial	16
	3.2 Approximation Of The Function Non Differentiable At One Point By Fourier Polynomial	17
	3.3 Approximation Of The Function Non Differentiable At Two Points By Fourier Polynomial	28
	3.4 Approximation Of The Function Non Differentiable At Three Points By Fourier Polynomial	32
	3.5 Conclusion	37
4	**APPROXIMATION AND CONVOLUTION METHOD**	38
	4.1 Convolution Of Non-Differentiable Function	38
	4.2 Convolution By Using Approximate Identity	45
	4.3 Continuous Property Of Function	48
	4.4 Limit Of Non-Differentiable Function	53
5	**APPROXIMATION BY LAGRANGE METHOD**	56
	5.1 The Function Non-Differentiable At One Point	57
	5.2 The Function Non-Differentiable At Two Points	65
	5.3 The Function Non-Differentiable At Three Points	68

	5.4 Strong Asymptotics In Lagrange Interpolation With Equidistant Nodes	69
	5.5 Conclusion	141
6	**CONCLUSION AND FURTHER RESEARCH**	142

REFERENCES 144
APPENDICES 146
BIODATA OF THE AUTHOR 156

CHAPTER ONE

INTRODUCTION

Numerical method is a procedure which is often useful, either to approximate a mathematical problem with a numerical problem or to solve a numerical problem or at least to reduce the complex numerical problem to a simple problem [9].

Approximation and interpolation are two important studies in Numerical analysis.

Approximation is a method to say anything significant about how well $\hat{f}$ approximates f requires qualitative information about f under appropriate conditions such as its continuity and differentiability. In approximation, we prefer numerical measure of closeness. Different approximation schemes correspond to different notions of convergence [9, 18].

Besides, it sometimes may be a very hard task for us to solve a problem by using exact formula, thus we often solve it by approximate it. Hence, approximation is an alternative technique for this problem.

In engineering and science, one often has a number of data points, obtained by sampling or some experiment, and tried to construct a function which closely fits those data points. We called this process as curve fitting. Interpolation is specific case of curve fitting.

Thus, we can say that interpolation is a method of constructing new data points from discrete set of known data points.

Interpolation consists fundamentally in finding a simple function which known as an interpolating function which takes the given function values at the given values of x and represents $y = f(x)$ adequately in a range including the non-tabular value of x. The value of the interpolating function at the latter value of x is taken to the value of $y = f(x)$.

The difference between function approximation and interpolation is the interpolating function f is used to replace or simply the original function g with certain smooth property preserved by the discrete interpolation nodes and their neighborhood.

The behavior of Lagrange interpolation polynomial of function $f(t) = |t|^{\lambda}, \lambda > 0$ where $t \in (-1,1)$ have attracted much attention of several generations of mathematicians [8]. It began with Bernstein in 1916 proved that the sequence of Lagrange interpolation polynomial to $|t|$ based on the equidistant nodes diverges at "any of the interval" of $[-1,1]$. The investigations of this problem continue by a few mathematicians and concluded that Lagrange interpolation polynomials based on equidistant nodes may have very poor approximation properties [10, 11, 12,20].

The function of $|t|$ has two tangent lines. Thus, it is a type of non-differentiable function. Hence, in this thesis, our main objective is to develop a new approximation method in approximating the possible of non-differentiable for the function $|f(t)|^{\lambda}$ where $\lambda = 1$.

Based on the problem, we will review the mathematical background which will assists us to have a clear picture in our research. This includes the review of a few types of approximation and interpolation and the Weierstrass Approximation Theorem, which provide the basic idea approaches our target that each continuous function is always can be approximated by the polynomial.

We obtain our main result through a few examples in order to show that the non-differentiable function can be approximated by using Fourier polynomial. First of all, we generate the orthogonal polynomial, and then we approximate the non-differentiable function by using the orthogonality and the approximation of Fourier polynomial. For each of the example given, we will start with the function has singularity at one point, then two points and following by three points. For each example given, we provide two graphs. The first graph is the approximation by using Fourier polynomial of the non-differentiable function at different orders and following by the error estimation graph of the function with Fourier polynomial at several orders.

Next, we continue to investigate a few ideas in the process of approximate the non-differentiable function by using convolution and discuss the continuous property of non-differentiable function.

We approximated the non-differentiable function by using the Lagrange interpolation polynomial method. We plot the graph of Lagrange interpolation function at different order and following by the graph of error estimation of the function with Lagrange interpolation at several orders for each example that we used. After that, we will discuss the result regarding the error estimation of Lagrange interpolation for function $f_\lambda(t) = |t|^\lambda, \lambda > 0$ with Equidistant Nodes, describe the rate of divergence of the Lagrange interpolation $L_N(f_\lambda, t)$ for $0 < |t| < 1$, and discuss their convergence at $t = 0$. Lastly, we compare the effective of using Lagrange Interpolation method and Fourier Polynomial method in approximation of the non-differentiable function.

CHAPTER TWO

MATHEMATICS BACKGROUND AND LITERATURE REVIEW

Approximation to non-differentiable function have a very close relationship to the differentiable function, Fourier Series[5,8], Convolution[2,14,5], Weierstrass Approximation Theorem[6,8,17,18,19], Taylor Approximation Polynomial[6,7,16] and Lagrange Interpolation[1,7]. Thus, in this chapter, we will review topics which we will use in our investigation.

2.1 Taylor Approximation

Taylor approximation is the most basic approximation technique compared to other methods. We use Taylor approximation method to approximate a function with another function. If a function $f(x)$ is given, we try to approximate function f by function g by finding the derivative of function f in the order n that we wish to find.

Definition 2.1 [6]

If function f has n derivatives at point $x_0 \in [a,b]$, the Taylor polynomial of order n for f at x_0 is written as

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots\ldots \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$
$$= \sum_{k=0}^{n} \frac{f^{(k)}(x - x_0)^k}{k!}$$

Theorem 2.2 Taylor Theorem With Cauchy Remainder [6]

Let function $f(x)$ has $n+1$ continuous derivatives on interval $[a,b]$ and let $x_0, x_1 \in [a,b]$.

Then $f(x) = P_n(x) + R_n(x)$ where $P_n(x) = \sum_{k=0}^{n} \frac{(x - x_0)^k}{k!} f^{(k)}(x_0)$ is Taylor expansion at n degree while $R_n(x)$ is the remainder with the formula

$$R_n(x) = \frac{1}{n!} \int_{x_0}^{x} (x - t)^n f^{(n+1)}(t)\,dt \,. \qquad (2.1)$$

This integral form of remainder does require the additional hypothesis that the $(n+1)$st derivative is Riemann integrabled. This theorem ensures that the series converges to the function.

Since Taylor approximation requires many derivatives, and it only uses information at a single point, so it will not be an ideal method for uniform approximation over an interval. In addition, Taylor polynomial is very accurate near the middle of the interval and much less accurate near the ends. Further, Taylor approximation can only be applied when the function is a differentiable function.

2.2 Lagrange Interpolation

Lagrange approximation is one approach for interpolating with polynomials.

If we want to approximate a function, $f \in C[a,b]$ with a polynomial

$p(x)=\sum_{i=0}^{n} c_i x_i$, $a \le x \le b$, the most straightforward method is to calculate the value of f at the n+1 distinct points $\{x_i ; i = 0,1,2,3,...n\}$ of $[a,b]$ and satisfied the equations $p(x_i) = f(x_i), i = 0,1,2...n$ .We might have problems with specifying the coefficients $\{c_i : i = 0,1,2,...n\}$. Thus, Lagrange interpolating polynomial is the best solution for it.

Definition 2.3 [2]

Lagrange interpolation polynomial is in the form $P(x)=\sum_{j=0}^{n} f(x_j)L_j(x_j)$ where $L_j(x)=\prod_{i\neq j}\frac{x-x_j}{x_k-x_j}, j=0:n$. $P(x)$ can be written as

$$P(x)=\frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)...(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)...(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)...(x_1-x_n)}y_1+\frac{(x-x_0)(x-x_1)...(x-x_n)}{(x_2-x_0)(x_2-x_1)...(x_2-x_n)}y_2+...\frac{(x-x_0)(x-x_1)(x-x_2)...(x-x_n)}{(x_n-x_1)(x_n-x_2)...(x_n-x_{n-1})}y_n$$