UNIVERSITI PUTRA MALAYSIA

RUNGE-KUTTA METHODS FOR SOLVING ORDINARY AND DELAY DIFFERENTIAL EQUATIONS

RAE'D ALI AHMED ALKHASAWNEH

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RUNGE-KUTTA METHODS FOR SOLVING ORDINARY AND DELAY DIFFERENTIAL EQUATIONS

By

RAE'D ALI AHMED ALKHASAWNEH

Thesis Submitted to the School of Graduate Studies, Universiti Putra Malaysia, in Fulfillment of the Requirements for the Degree of Doctor of Philosophy

November 2006

DEDICATION

To my Great Father and Mother,

To my beloved Wife,

To my Brothers and Sisters.

Raed

Abstract of thesis presented to the Senate of Universiti Putra Malaysia in fulfilment of the requirement for the degree of Doctor of Philosophy

RUNGE-KUTTA METHODS FOR SOLVING ORDINARY AND DELAY DIFFERENTIAL EQUATIONS

By

RAE'D ALI AHMED ALKHASAWNEH

November 2006

Chairman: Associate Professor Fudziah bt Ismail, PhD

Faculty : Science

An introduction to Runge-Kutta methods for the solution of ordinary differential equations (ODEs) is introduced. The technique of using Singly Diagonally Implicit Runge-Kutta (SDIRK) method for the integration of stiff and non-stiff ODEs has been widely accepted, this is because SDIRK method is computationally efficient and stiffly stable. Consequently embedded SDIRK method of fourth-order six stage in fifth-order seven stage which has the property that the first row of the coefficient matrix is equal to zero and the last row of the coefficient matrix is equal to the vector output value is constructed. The stability region of the method when applied to linear ODE is given. Numerical results when stiff and non-stiff first order ODEs are solved using the method are tabulated and compared with the method in current use.

Introduction to delay differential equations (DDEs) and the areas where they arise are given. A brief discussion on Runge-Kutta method when adapted to delay differential equation is introduced**.** SDIRK method which has been derived previously is used to solve delay differential equations; the delay term is approximated using divided difference interpolation. Numerical results are tabulated and compared with the existing methods. The stability aspects of SDIRK method when applied to DDEs using Lagrange interpolation are investigated and the region of stability is presented.

Runge-Kutta-Nystróm (RKN) method for the solution of special second-order ordinary differential equations of the form $y'' = f(x, y)$ is discussed. Consequently, Singly Diagonally Implicit Runge-Kutta Nystróm (SDIRKN) method of third-order three stage embedded in fourth-order four stage with small error coefficients is constructed. The stability region of the new method is presented. The method is then used to solve both stiff and non-stiff special second order ODEs and the numerical results suggest that the new method is more efficient compared to the current methods in use.

Finally, introduction to general Runge-Kutta-Nystrom (RKNG) method for the solution of second-order ordinary differential equations of the form $y'' = f(x, y, y')$ is given. A new embedded Singly Diagonally Implicit Runge-Kutta-Nystróm General (SDIRKNG) method of third-order four stage embedded in fourth-order five stage is derived. Analysis on the stability aspects of the new method is given and numerical results when the method is used to solve both stiff and non-stiff second order ODEs are presented. The results indicate the superiority of the new method compared to the existing method.

Abstrak tesis dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

KAEDAH RUNGE-KUTTA UNTUK MENYELESAIKAN PERSAMAAN PEMBEZAAN BIASA DAN LENGAH

Oleh

RAE'D ALI AHMED ALHASAWNEH

November 2006

Pengerusi: Profesor Madya Fudziah bt Ismail, PhD

Fakulti : Sains

Pengenalan kepada kaedah Runge-Kutta untuk menyelesaikan Persamaan Pembezaan Biasa (PPB) diperkenalkan. Teknik yang menggunakan kaedah Runge-Kutta Pepenjuru Tunggal Tersirat (RKPTT) untuk kamiran PPB kaku dan tak kaku telah diterima pakai kerana kaedah RKPTT ini cekap dan sangat stabil. Seterusnya, kaedah terbenam RKPTT bagi peringkat empat tahap enam dalam peringkat lima tahap tujuh dengan ciri-ciri baris pertama bagi matriks pekalinya bersamaan dengan sifar dan baris terakhir bagi matriks pekalinya bersamaan dengan nilai vektor outputnya dibina. Rantau kestabilan bagi kaedah ini apabila digunakan ke atas PPB linear diberikan. Keputusan berangka apabila PPB kaku dan tak kaku diselesaikan menggunakan kaedah itu dibentangkan dan dibandingkan dengan kaedah yang sedang digunakan sekarang.

Pengenalan kepada Persamaan Pembezaan Lengah (PBL) dan bidang di mana ianya timbul diberikan. Perbincangan ringkas mengenai kaedah Runge-Kutta apabila diadaptasikan kepada persamaan pembezaan lengah diperkenalkan. Kaedah RKPTT yang telah diperolehi sebelum ini digunakan untuk menyelesaikan PBL; sebutan

lengahnya dianggarkan menggunakan interpolasi beza bahagi. Keputusuan berangka dibentangkan dan dibandingkan dengan kaedah yang sedia ada. Aspek kestabilan kaedah RKPTT apabila digunakan ke atas PPL menggunakan interpolasi Lagrange diselidik dan rantau kestabilannya dipersembahkan.

Kaedah Runge-Kutta-Nystróm (RKN) bagi penyelesaian PBB khas peringkat kedua dalam bentuk $y'' = f(x, y)$ dibincangkan. Seterusnya, kaedah peringkat tiga tahap tiga terbenam dalam kaedah peringkat empat tahap empat Runge-Kutta Nystrom Pepenjuru Tunggal Tersirat (RKNPTT) dengan pekali ralat yang kecil dibina. Rantau kestabilan bagi kaedah yang baru ini dipersembahkan. Kaedah ini kemudiannya digunakan untuk menyelesaikan kedua-dua PPB kaku dan tak kaku peringkat kedua dan keputusan berangkanya menunjukkan bahawa kaedah ini lebih cekap berbanding kaedah yang sedang digunakan sekarang.

Akhir sekali, pengenalan kepada kaedah umum Runge-Kutta-Nystrom (RKNU) untuk menyelesaikan persamaan pembezaan biasa peringkat kedua dalam bentuk $y'' = f(x, y, y')$ diberikan. Kaedah umum Runge-Kutta Nystrom Pepenjuru Tunggal Tersirat (RKNUPTT) yang baru, peringkat tiga tahap empat terbenam dalam kaedah peringkat empat tahap lima diterbitkan. Analisis bagi aspek kestabilan kaedah baru ini diberikan dan keputusan berangka apabila kaedah ini digunakan untuk menyelesaikan kedua-dua PBB umum kaku dan tak kaku peringkat kedua dipersembahkan. Keputusannya menunjukkan kaedah baru tersebut adalah lebih baik berbanding dengan kaedah yang sedia ada.

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Raed Al-Khasawneh,

September 2006.

I certify that an Examination Committe has met on 24 November 2006 to conduct the final examination of Rae'd Ali Ahmed Alkhasawneh on his Doctor of Philosophy thesis entitled "Runge-Kutta Methods for Solving Ordinary and Delay Differential Equations" in accordance with Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The Committee recommends that the candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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This thesis submitted to the Senate of Universiti Putra Malaysia and has been accepted as fulfilment of the requirement for the degree of Doctor of Philosophy. The members of the Supervisory Committee are as follows

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

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RAE'D ALI AHMED ALKHASAWNEH

Date:

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LIST OF ABBREVIATIONS

IVP	Initial Value Problem
ODE	Ordinary Differential Equation
ODEs	Ordinary Differential Equations
IMEXRK	Implicit Explicit Runge-Kutta
DDE	Delay Differential Equation
DDEs	Delay Differential Equations
RDE	Retarded Delay Differential Equation
NDE	Neutral Delay Differential Equation
RKN	Runge-Kutta-Nystróm
RKNG	Runge-Kutta-Nystróm General
SDIRK	Singly Diagonally Implicit Runge-Kutta
DIRKN	Diagonally Implicit Runge-Kutta-Nystróm
SDIRKN	Singly Diagonally Implicit Runge-Kutta-Nystróm
SDIRKNG	Singly Diagonally Implicit Runge-Kutta-Nystróm General
SODEs	Stiff Ordinary Differential Equations
CRK	Continuous Runge-Kutta
RKF	Runge-Kutta Felhberg
FSAL	First Stage As Last

ABSTRACT

NUMERICAL SOLUTION OF ORDINARY AND DELAY DIFFERENTIAL EQUATIONS BY RUNGE-KUTTA METHODS

By

RAED ALI AHMED ALHASAWNEH

Chairman: Associated Professor Fudziah bt Ismail, Ph.D.

Faculty: Science.

An introduction to Runge-Kutta methods for the solution of ordinary differential equations (ODEs) is introduced. It is widely accepted that using Singly Diagonally Implicit Runge-Kutta (SDIRK) method become as an efficient technique for the integration of many stiff and non-stiff problems and at the same time, it can overcome the difficulties for using the fully implicit Runge-Kutta method and the limitations for explicit one. The derivation for embedded SDIRK method of fourth-order six stages in fifth-order seven stages is illustrated. The stability region is presented and the numerical results are compared with the other existing methods.

Introduction to delay differential equations (DDEs) and the areas where they arise are given. A brief discussion on Runge-Kutta method when adapted to delay differential equation is introduced**.** SDIRK method which derived in Chapter III is used to solve delay differential equations. The delay term is approximated using divided difference interpolation. Numerical results are tabulated and compared with the other existing methods. The stability properties of SDIRK method when applied to DDEs using Lagrange interpolation are investigated and their regions of stability are presented.

Runge-Kutta-Nystróm (RKN) method for the solution of special second-order ordinary differential equations of the form $y'' = f(x, y)$ is described. Consequently, singly diagonally implicit Runge-Kutta Nystróm (SDIRKN) of third-order three stages embedded in fourth-order four stages is constructed. The stability region of the new method is presented and numerical results are compared with the same method of lower order.

Finally, introduction to general Runge-Kutta-Nystrom (RKNG) method for the solution of second-order ordinary differential equations of the form $y'' = f(x, y, y')$ is given. A new singly diagonally implicit Runge-Kutta-Nystróm general (SDIRKNG) method of third-order embedded in fourth-order is derived. Analysis the stability region of the new method is discussed and numerical results are presented.

ABSTRAK

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