



UNIVERSITI PUTRA MALAYSIA

***SYMPLECTIC TECHNIQUES IN GEOMETRIC QUANTUM
MECHANICS AND NONLINEAR QUANTUM MECHANICS***

SAEID MOLLADAVOUDI

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**SYMPLECTIC TECHNIQUES IN GEOMETRIC
QUANTUM MECHANICS AND NONLINEAR
QUANTUM MECHANICS**

By

SAEID MOLLADAVOUDI

Thesis Submitted to the School of Graduate Studies, Universiti Putra
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DEDICATIONS

To Somayeh



Abstract of thesis presented to the Senate of Universiti Putra Malaysia in
fulfilment of the requirement for the degree of Doctor of Philosophy

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June 2013

Chair: Associate Professor Hishamuddin B Zainuddin, PhD

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In this thesis we study the roles played by symplectic geometry in quantum mechanics, in particular quantum dynamics and quantum information theory treated as two separate parts. The common ground for both parts is the geometrical formulation of quantum mechanics. In Chapter 2, we review the associated complex projective Hilbert space of quantum pure states, with symplectic and Riemannian structures and their roles in quantum dynamics and kinematics.

In Chapter 3, we motivate the idea of information-theoretic constraint on the differentiable manifold of probability distributions through the maximum uncertainty principle and the linear Schrödinger equation by introduction of the wavefunction.

In Chapter 4, we review both regular and singular symplectic reduction of a symplectic manifold, which is acted upon properly and symplectically by a compact Lie group.

Chapter 5 contains the author's original contributions to the first part of the thesis. In this chapter, by using the same information-theoretic discussion of the Chapter

3 we propose a non-relativistic, spin-less, non-linear quantum dynamical equation, with the Fisher information metric replaced by the Jensen-Shannon distance information. Furthermore, we show that the non-linear Schrödinger equation is in fact a Hamiltonian dynamics, namely it preserves the symplectic structure of the complex Hilbert space. The projected dynamics on the corresponding projective Hilbert space is derived and its properties are highlighted in further details.

Chapter 6 contains the author's primary contributions to the second part of the thesis. In particular, by using the singular symplectic reduction of the Chapter 4 we explicitly construct the space of entanglement types of three-qubit pure states with a specific (shifted) spectra of single-particle reduced density matrices. Moreover, we obtain the image of the symplectic quotient under the induced Hilbert map, by using local unitary invariant polynomials. Then the symplectic structure on the principal stratum of the symplectic quotient is derived. Finally, it is discussed that other lower dimensional strata are relative equilibria on the original manifold and their stability properties are investigated under compact subgroups of the local unitary transformations.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk ijazah Doktor Falsafah

**TEKNIK SIMPLEKTIK DALAM MEKANIK KUANTUM
BERGEOMETRI DAN MEKANIK KUANTUM TAK LINEAR**

Oleh

SAEID MOLLADAVOUDI

Jun 2013

Pengerusi: Profesor Madya Hishamuddin B Zainuddin, PhD

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Dalam tesis ini, kita kaji peranan yang dimainkan oleh geometri simplektik dalam mekanik kuantum, khususnya dalam dinamik kuantum dan teori maklumat kuantum yang dikaji sebagai dua bahagian berasingan. Yang menjadi asas sama-sama bagi kedua-dua bahagian adalah formulasi bergeometri mekanik kuantum. Dalam Bab 2, kita tinjau kembali ruang Hilbert unjuran kompleks bagi keadaan tulen kuantum, bersama dengan struktur-struktur simplektik dan Riemannian dan peranan mereka dalam dinamik dan kinematik kuantum.

Dalam Bab 3, kita bangunkan idea kekangan berteori maklumat ke atas manifold terbezakan bagi taburan kebarangkalian melalui prinsip ketakpastian maksimum dan persamaan linear Schrodinger dengan memperkenalkan fungsi gelombang. Dalam Bab 4, kita tinjau semula penurunan simplektik nalar dan singular bagi manifold simplektik yang ditindak secara wajar dan simplektik oleh satu kumpulan Lie yang padat.

Bab 5 mengandungi sumbangan asli penulis bagi bahagian pertama tesis. Dalam bab ini, menggunakan perbincangan berteori maklumat Bab 3, kita usulkan satu per-

samaan dinamik kuantum yang bukan kereatifan, tanpa spin dan tak linear, dengan menggantikan metrik maklumat Fisher dengan maklumat jarak Jensen-Shannon. Tambahan itu, kita tunjukkan bahawa persamaan Schrodinger tak linear adalah sebenarnya dinamik Hamiltonan, iaitu ia mengekalkan struktur simplektik ruang Hilbert kompleks. Dinamik yang terunjur ke atas ruang Hilbert terunjur yang berpadanan diterbitkan dan sifatnya disorot dengan terperinci.

Bab 6 mengandungi sumbangan utama penulis ke bahagian kedua tesis. Khususnya, dengan menggunakan penurunan simplektik singular dari Bab 4, kita bangunkan secara eksplisit ruang jenis keterbelitan bagi keadaan tulen tiga-qubit dengan spektrum (teranjak) khas bagi matriks ketumpatan terturun zarah tunggal. Tambahan lagi, kita memperoleh imej hasil bahagi simplektik di bawah pemetaan Hilbert teraruh dengan menggunakan polinomial tak varian unitary setempat. Kemudian itu, struktur simplektik di atas stratum utama hasil bahagi simplektik diterbitkan. Akhir sekali, dibincangkan strata berdimensi lebih rendah adalah equilibria relatif di atas manifold asal dan ciri kestabilan mereka dikaji di bawah subkumpulan padat bagi transformasi unitary setempat.

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I certify that a Thesis Examination Committee has met on 10 June 2013 to conduct the final examination of Saeid Molladavoudi on his thesis entitled “Symplectic Techniques in Geometric Quantum Mechanics and Nonlinear Quantum Mechanics” in accordance with the Universities and University Colleges Act 1971 and the Constitution of the Universiti Putra Malaysia [P.U.(A) 106] 15 March 1998. The Committee recommends that the student be awarded the Doctor of Philosophy.

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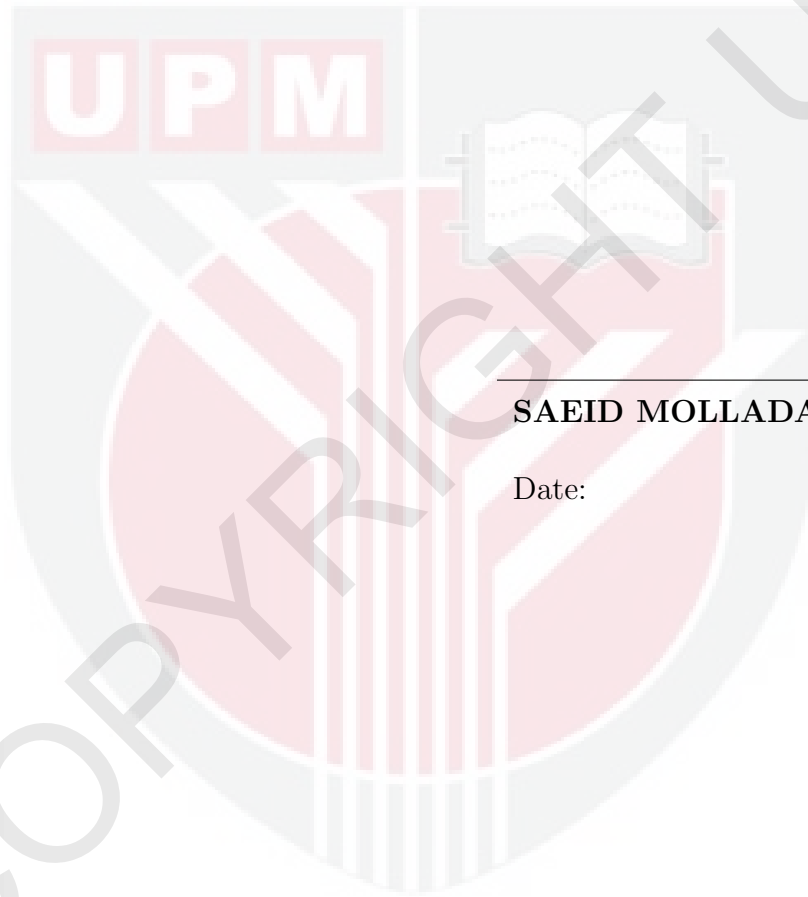
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DECLARATION

I declare that the thesis is my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously, and is not concurrently, submitted for any other degree at Universiti Putra Malaysia or at any other institution.



SAEID MOLLADAVOUDI

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