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On the Convergence of the Point Repeated Symmetric Single-Step Procedure for Simultaneous Estimation of Polynomial Zeros

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ABSTRACT

The point symmetric single-step procedure established by Monsi (2012) has R-order of convergence at least 3. This procedure is modified by repeating the steps in the procedure r times without involving function evaluations. This modified procedure is called the point repeated symmetric single-step PRSS1. The R-order of convergence of PRSS1 is at least $(2r + 1)$ $(r \geq 1)$. Computational experiences in the implementation of the interval version of PRSS1 (see Monsi and Wolfe, 1988) showed that the repeated symmetric single-step procedure is more efficient than the total step (Kerner, 1966) and the single-step (Alefeld and Herzberger, 1974) methods.

Keywords: Point procedure, R-order of convergence, simple zeros, simultaneous estimation.

1. INTRODUCTION

The iterative procedures for estimating simultaneously the polynomial zeros of a polynomial of degree n were discussed in Ehrlich, 1967, Aberth, 1973, Alefeld and Herzberger, 1974, Farmer and Loizou, 1975, Milovanovic and Petkovic, 1983 and Petkovic and Stefanovic, 1986. In this paper, we adopt the methods established by Kerner, 1966, Alefeld and Herzberger, 1983, Monsi, 2012, Monsi et al., 2012 and Bakar et al., 2012 in order to increase the rate of convergence of the point repeated symmetric single-step method PRSS1. The convergence analysis of this procedure is given in Section 3. This procedure needs some pre-conditions for initial points $x_i^{(0)}\ (i = 1, \ldots n)$ to converge to the zeros $x_i^*\ (i = 1, \ldots, n)$ respectively, as shown subsequently in this paper. We also list the attractive features of PRSS1 in Section 3.

2. METHODS OF ESTIMATING THE POLYNOMIAL ZEROS

Consider a polynomial of degree $n > 1$

$$p(x) = \sum_{i=0}^{n} a_i x^i = \prod_{j=1}^{n} (x - x_j^*) \tag{1}$$

$a_i \in C\ \ (i = 1, \ldots, n)$ and $a_n \neq 0$ with distinct zeros $x_i^*\ \ (i = 1, \ldots, n)$.

Let $x_1, \ldots, x_n$ be distinct approximations for these zeros and let $q: C \to C$ be defined by

$$q(x) = \prod_{j=1}^{n} (x - x_j). \tag{2}$$

Then

$$q'(x) = \prod_{j \neq i}^{n} (x_i - x_j) \qquad (i = 1, \ldots, n). \tag{3}$$

By (1), if for $i = 1, \ldots, n,\ \ x_i \neq x_j\ \ (j = 1, \ldots, n; j \neq i)$, then

$$x_i^* = x_i - \frac{p(x_i)}{\prod_{j \neq i}(x_i - x_j^*)}\,. \tag{4}$$

Now $x_j \approx x_j^*\ \ (j = 1, \ldots, n)$ so by (3) and (4),

$$x_i^* \approx x_i - \frac{p(x_i)}{\prod_{j \neq i}(x_i - x_j)} \quad (i = 1, \ldots, n). \tag{5}$$

The iterative procedures of (5) defined by

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j \neq i}\left(x_i^{(k)} - x_j^{(k)}\right)} \quad (i = 1, \dots, n)(k \geq 0), \tag{6}$$

is the total step procedure of Kerner, 1966 and

$$x_i^{(k+1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k+1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k)}\right)} \tag{7}$$

$$(i = 1, \dots, n)\ (k \geq 0)$$

is the single-step procedure of Alefeld and Herzberger, 1974.

Furthermore, the point symmetric single-step procedure PSS1 (Monsi, 2012) is defined by

$$x_i^{(k,0)} = x_i^{(k)}\ (i = 1, \dots n), \tag{8a}$$

$$x_i^{(k,1)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,1)})\prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,0)})} \tag{8b}$$

$$(i = 1, \dots, n),$$

$$x_i^{(k,2)} = x_i^{(k)} - \frac{p(x_i^{(k)})}{\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,1)})\prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,2)})} \tag{8c}$$

$$(i = n, \dots, 1),$$

$$x_i^{(k+1)} = x_i^{(k,2)}(i = 1, \dots n). \tag{8d}$$

The procedure (6) has R-order of convergence at least 2 or $O_R(PT1, x^*) \geq 2$, while the R-order of convergence of (7) is greater than 2 or $O_R(PS1, x^*) > 2$. However, the R-order of convergence of PSS1 is at least 3 or $O_R(PSS1, x^*) \geq 3$.

Ortega and Rheinboldt, 1970 have introduced the following theorem and definition of the convergence order.

Theorem 1. Let I be an iterative procedure and let $\Omega(I, x^*)$ be the set of all sequences $\{x^{(k)}\}$ generated by I which converge to the limit x^*. Suppose that there exists a $p \geq 1$ and a constant γ such that for any $\{x^{(k)}\} \in \Omega(I, x^*)$,

$$\left\|x^{(k+1)} - x^*\right\| \leq \gamma \left\|x^{(k)} - x^*\right\|^p, k \geq k_0 = k_0(\{x^{(k)}\}).$$

Then it follows that R-order of I satisfies the inequality $O_R(I, x^*) \geq p$, where the R-order of convergence of the procedure I is at least p. ■

Definition 1. If there exists a $p \geq 1$ such that for any null sequence $\{w^{(k)}\}$ generated from $\{x^{(k)}\}$, then the R-factor of such sequence is defined to be

$$R_p\left(w^{(k)}\right) = \begin{cases} \lim\limits_{k\to\infty} \sup \left\|w^{(k)}\right\|^{1/k}, & p = 1 \\ \\ \lim\limits_{k\to\infty} \sup \left\|w^{(k)}\right\|^{1/p^k}, & p > 1, \end{cases}$$

where R_p is independent of the norm $\|\cdot\|$. ■

Suppose that $R_p(w^{(k)}) < 1$ then it follows from Ortega and Rheinboldt, 1970 that the R-order of I satisfies the inequality $O_R(I, x^*) \geq p$. We shall use this definition for analysis of the convergence of the iterative procedure PRSS1 in the subsequent section.

3. THE POINT REPEATED SYMMETRIC SINGLE –STEP PROCEDURE PRSS1

The value of $x_i^{(k,2)}$ which is computed from (8c) requires $(n-i)$ multiplications, one division, and $(n-i+1)$ subtractions, increasing the lower bound on the *R*-order by unity compared with the *R*-order of PS1. This observation gives rise to the idea that it might be advantageous to repeat the steps (8b) and (8c) r times in each iteration where the integer $r \geq 1$. This leads to the repeated point symmetric single-step procedure PRSS1 which consists of generating the sequences $\left\{x_i^{(k)}\right\}$ $(i=1,\dots,n)(k \geq 0)$ from (9a) to (9d) below. For $l = 1,\dots,r$;

$$x_i^{(k,0)} = x_i^{(k)} \quad (i = 1,\dots,n) \tag{9a}$$

$$x_i^{(k,2l-1)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,2l-1)})\prod_{j=i+1}^{n}(x_i^{(k)} - x_j^{(k,2l-2)})} \quad (i = 1,\dots,n) \tag{9b}$$

$$x_i^{(k,2l)} = x_i^{(k)} - \frac{p\left(x_i^{(k)}\right)}{\prod_{j=1}^{i-1}\left(x_i^{(k)} - x_j^{(k,2l-1)}\right)\prod_{j=i+1}^{n}\left(x_i^{(k)} - x_j^{(k,2l)}\right)} \quad (i = n,\dots,1) \tag{9c}$$

$$x_i^{(k+1)} = x_i^{(k,2l)} \; (i = 1,\dots,n) \quad (k \geq 0). \tag{9d}$$

If $r = 1$ $(\forall k \geq 0)$ then PRSS1 is reduced to PSS1 of Monsi, 2012. The procedure PRSS1 has the following attractive features:

(i) From $(9b)$ and $(9c)$ it follows that for

$$l \geq 1, k \geq 0, \ \ x_n^{(k,2l)} = \ x_n^{(k,2l-1)}$$

and

$$x_1^{(k,2l+1)} = x_1^{(k,2l)}$$

so that $x_n^{(k,2l)}$ and $x_1^{(k,2l+1)}$ need not be computed.

(ii) The product

$$\prod_{j=1}^{i-1}(x_i^{(k)} - x_j^{(k,2l-1)}) \ \ (i = 2, \dots, n)$$

which are computed for use in (9b) are re-used in (9c).

The following lemmas of Monsi, 2012 are required in the proof of Theorem 2.

Lemma 1.

If (i) $p: C \to C$ is defined by (1);

(ii) $p_i: C \to C$ is defined by

$$p_i(x) = \prod_{m=1}^{i-1}(x - x_m^*)\,(x - x_m^*) \tag{10}$$

$$(i = 1, \dots, n),$$

(iii) $q_i: C \to C$ is defined by

$$q_i(x) = \prod_{m=1}^{i-1}(x - \bar{x}_m) \prod_{m=i+1}^{n} (x - \hat{x}_m) \tag{11}$$

$$(i = 1, \dots, n),$$

where $\bar{x}_j \neq \bar{x}_m$ and $\hat{x}_j \neq \hat{x}_m$ $(j, m = 1, \dots, n; j \neq m)$;

(iv) $\emptyset_i : C \to C$ is defined by

$$\emptyset_i(x) = q_i(x) + \sum_{j=1}^{i-1} \frac{p_i(\bar{x}_j) q_i(x)}{q_i'(\bar{x}_j)(x - \bar{x}_j)} + \sum_{j=i+1}^{n} \frac{p_i(\hat{x}_j) q_i(x)}{q_i'(\hat{x}_j)(x - \hat{x}_j)} \quad (i = 1, \dots, n), \tag{12}$$

then

$$\emptyset_i(x) = p_i(x) \quad (\forall x \in C)(i = 1, \dots, n). \blacksquare$$

Lemma 2.

If hypotheses (i) – (iv) of Lemma 1 are valid; (v) $\check{x}_i$ $(i = 1, \dots, n)$ are such that

$$p(\check{x}_i) \neq 0 \ (i = 1, \dots, n),$$
$$\check{x}_i \neq \bar{x}_m \ (m = 1, \dots, i-1),$$
$$\check{x}_i \neq \hat{x}_m \ (m = i+1, \dots, n),$$

and

$$\bar{x}_i = \check{x}_i - \frac{p(\check{x}_i)}{\prod_{m=1}^{i-1}(\check{x}_i - \bar{x}_m) \prod_{m=i+1}^{n}(\check{x}_i - \hat{x}_m)} \quad (i = 1, \dots, n);$$

(vi) $\breve{w}_i = \check{x}_i - x_i^*$, $\widehat{w}_i = \hat{x}_i - x_i^*$, and $\overline{w}_i = \bar{x}_i - x_i^*$ $(i = 1, \dots, n)$, then

$$\overline{w}_i = \breve{w}_i \left\{ \sum_{j=1}^{i-1} \bar{\gamma}_{ij} \overline{w}_j + \sum_{j=i+1}^{n} \hat{\gamma}_{ij} \widehat{w}_j \right\} \quad (i = 1, \dots, n),$$

where

$$\bar{\gamma}_{ij} = \frac{\prod_{m \neq i,j}(\bar{x}_j - x_m^*)}{q_i'(\bar{x}_j)(\bar{x}_j - \check{x}_i)} \quad (j = 1, \dots, i-1),$$

and

$$\hat{\gamma}_{ij} = \frac{\prod_{m \neq i,j}(\hat{x}_j - x_m^*)}{q_i'(\hat{x}_j)(\hat{x}_j - \check{x}_i)} \quad (j = i+1, \dots, n). \blacksquare$$

Lemma 3.

If hypotheses (i) – (vi) of Lemma 2 are valid; (vii) $|\breve{x}_i - x_i^*| \leq \theta d/(2n-1)$ and $|\hat{x}_i - x_i^*| \leq \frac{\theta d}{2n-1}(i = 1, \dots, n)$ where $d = \min\{|x_i^* - x_j^*||i, j = 1, \dots n; j \neq i\}$ and $0 < \theta < 1$, then $|\overline{w}_i| \leq \theta|\breve{w}_i|$ $(i = 1, \dots, n)$. ■

Theorem 2.

If (i) $p: C \to C$ defined by (1) has n distinct zeros x_i^* $(i = 1, \dots, n)$;

(ii) $\left|x_i^{(0)} - x_i^*\right| \leq \frac{\theta d}{2n-1}(i = 1, \dots, n)$ where $0 < \theta < 1$ and $d = \min\{|x_i^* - x_j^*||i, j = 1, \dots, n: j \neq i\}$ such that the sequences $\left\{x_i^{(k)}\right\}(i = 1, \dots, n)(k \geq 0)$ are generated from PRSS1 (i.e. from (9)),

then

$$x_i^{(k)} \to x_i^* \ (k \to \infty) \ (i = 1, \dots, n) \text{ and } O_R(\text{PRSS1}, x^*) \geq 2r + 1 \text{ for } r \geq 1.$$

Proof.

For $l = 1, \dots, r,\ i = 1, \dots, n,$ let

$$q_{2l-1,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,2l-1)}) \prod_{m=i+1}^{n}\left(x - x_m^{(k,2l-2)}\right),$$

$$q_{2l,i}(x) = \prod_{m=1}^{i-1}(x - x_m^{(k,2l-1)}) \prod_{m=i+1}^{n}\left(x - x_m^{(k,2l)}\right),$$

$$\emptyset_{2l-1,i}(x) = q_{2l-1,i}(x) + \sum_{j=1}^{i-1} \frac{p_i\left(x_j^{(k,2l-1)}\right) q_{2l-1,i}(x)}{q'_{2l-1,i}\left(x_j^{(k,2l-1)}\right)(x - x_j^{(k,2l-1)})} +$$

$$\sum_{j=i+1}^{n} \frac{p_i\left(x_j^{(k,2l-2)}\right) q_{2l-1,i}(x)}{q'_{2l-1,i}\left(x_j^{(k,2l-2)}\right)(x - x_j^{(k,2l-2)})},$$

and

$$\emptyset_{2l,i}(x) = q_{2l,i}(x) + \sum_{j=1}^{i-1} \frac{p_i\left(x_j^{(k,2l-1)}\right) q_{2l,i}(x)}{q'_{2l,i}\left(x_j^{(k,2l-1}\right)(x - x_j^{(k,2l-1)})} +$$

$$\sum_{j=i+1}^{n} \frac{p_i\left(x_j^{(k,2l)}\right) q_{2l,i}(x)}{q'_{2l,i}\left(x_j^{(k,2l)}\right)(x - x_j^{(k,2l)})},$$

where $p_i(x)$ is defined by (10).

By Lemma 1 and Lemma 2 with $q_i = q_{2l-1,i}, \check{x}_i = x_i^{(k)}, \hat{x}_i = x_i^{(k,2l-2)}, \bar{x}_i = x_i^{(k,2l-1)}, \emptyset_i = \emptyset_{2l-1,i}$ $(i = 1, \dots, n)(l = 1, \dots, r)$, it follows that for $i = 1, \dots, n,\ l = 1, \dots, r,$ and $k \geq 0$,

$$w_i^{(k,2l-1)} = w_i^{(k)} \tag{13}$$

$$\left\{\sum_{j=1}^{i-1} \alpha_{ij}^{(k,2l-1)} w_j^{(k,2l-1)} + \sum_{j=i+1}^{n} \alpha_{ij}^{(k,2l-2)} w_j^{(k,2l-2)}\right\},$$

where

$$w_i^{(k,s)} = x_i^{(k,s)} - x_i^* \quad (s = 0, \dots, r), \tag{14}$$

$$\alpha_{ij}^{(k,2l-1)} = \frac{\prod_{m \neq i,j}(x_j^{(k,2l-1)} - x_m^*)}{q'_{2l-1,i}\left(x_j^{(k,2l-1)}\right)(x_j^{(k,2l-1)} - x_i^{(k)})},$$

$$(j = 1, \dots, i-1),$$

and

$$\alpha_{ij}^{(k,2l-2)} = \frac{\prod_{m \neq i,j}(x_j^{(k,2l-2)} - x_m^*)}{q'_{2l-1,i}\left(x_j^{(k,2l-2)}\right)(x_j^{(k,2l-2)} - x_i^{(k)})}, \tag{15}$$

$$(j = i+1, \dots, n).$$

Similarly, by Lemma 1 and Lemma 2, with $q_i = q_{2l,i}, \check{x}_i = x_i^{(k)}, \hat{x}_i = x_i^{(k,2l)}, \bar{x}_i = x_i^{(k,2l-1)}, \emptyset_i = \emptyset_{2l,i}$ $(i = 1, \dots, n)(l = 1, \dots, r)$, it follows that for $i = 1, \dots, n, l = 1, \dots, r$ and $k \geq 0$,

$$w_i^{(k,2l)} = w_i^{(k)} \left\{ \sum_{j=1}^{i-1} \beta_{ij}^{(k,2l-1)} w_j^{(k,2l-1)} + \sum_{j=i+1}^{n} \beta_{ij}^{(k,2l)} w_j^{(k,2l)} \right\}, \tag{16}$$

where

$$\beta_{ij}^{(k,2l-1)} = \frac{\prod_{m \neq i,j} (x_j^{(k,2l-1)} - x_m^*)}{q'_{2l,i}\left(x_j^{(k,2l-1)}\right)(x_j^{(k,2l-1)} - x_i^{(k)})} \tag{17}$$

$$(j = 1, \dots, i-1),$$

and

$$\beta_{ij}^{(k,2l)} = \frac{\prod_{m \neq i,j} (x_j^{(k,2l)} - x_m^*)}{q'_{2l,i}\left(x_j^{(k,2l)}\right)(x_j^{(k,2l)} - x_i^{(k)})} \quad (j = i+1, \dots, n). \tag{18}$$

It follows from (13) – (15) and Lemma 3 that $\left|w_i^{(0,1)}\right| \leq \theta \left|w_i^{(0,0)}\right|$ $(i = 1, \dots, n)$, and it follows from (16) – (18) and Lemma 3 that $\left|w_i^{(0,2)}\right| \leq \theta^2 \left|w_i^{(0,0)}\right|$ $(i = 1, \dots, n)$. Then it follows from (13) – (18) by finite induction on l that

$$\left|w_i^{(0,2l)}\right| \leq \theta^{2l} \left|w_i^{(0.0)}\right| \;\; (i = 1, \dots, n) \; (l \geq 1).$$

Then, for $l = r$ and from (8d)

$$\left|w_i^{(0,2r)}\right| \leq \theta^{2r} \left|w_i^{(0,0)}\right| \;\; (i = 1, \dots, n),$$

whence $\left|w_i^{(1,0)}\right| \leq \theta^{2r} \left|w_i^{(0,0)}\right|$ $(i = 1, \dots, n)$. It then follows by induction on k that $\forall k \geq 0$

$$\left|w_i^{(k,0)}\right| \leq \theta^{(2r+1)^k - 1} \left|w_i^{(0,0)}\right| \;\; (i = 1, \dots, n),$$

whence $x_i^{(k)} \to x_i^* \ (k \to \infty), (i = 1, \dots, n)$. Let

$$h_i^{(k,m)} = \frac{(2n-1)}{d} \left| w_i^{(k,m)} \right| \quad (i = 1, \dots, n)(m = 0, \dots, 2r). \tag{19}$$

Then by (13), (15) and (19), for $i = 1, \dots, n$,

$$h_i^{(k,2l-1)} \leq \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,2l-1)} + \sum_{j=i+1}^{n} h_j^{(k,2l-2)} \right\}, \tag{20}$$

and for $i = n, \dots, 1$,

$$h_i^{(k,2l)} \leq \frac{1}{(n-1)} h_i^{(k,0)} \left\{ \sum_{j=1}^{i-1} h_j^{(k,2l-1)} + \sum_{j=i+1}^{n} h_j^{(k,2l)} \right\}. \tag{21}$$

For $l = 1, \dots, r$ let

$$u_i^{(1,2l-1)} = \begin{cases} 2l & (i = 1, \dots, n-1) \\ 2l+1 & (i = n) \end{cases}, \tag{22}$$

and

$$u_i^{(1,2l-1)} = \begin{cases} 2l & (i = 1, \dots, n-1) \\ 2l+1 & (i = n) \end{cases}, \tag{23}$$

and for $p = 1, \dots, 2r$ let

$$u_i^{(k+1,p)} = \begin{cases} (2r+1)u_i^{(k,p)} + 1 & (i = 1) \\ (2r+1)u_i^{(k,p)} & (i = 2, \dots, n) \end{cases} \tag{24}$$

Then for $l = 1, \dots, r$ $(\forall k \geq 0)$, by (22) – (24) ,

$$u_i^{(k,2l-1)} = \begin{cases} \left(\frac{4rl+1}{2r}\right)(2r+1)^{k-1} - \frac{1}{2r} & (i = 1) \\ 2l(2r+1)^{k-1} & (i = 2, \dots, n-1), \\ (2l+1)(2r+1)^{k-1} & (i = n) \end{cases} \tag{25}$$

and

$$u_i^{(k,2l)} = \begin{cases} \left(\frac{4rl+4r+1}{2r}\right)(2r+1)^{k-1} - \frac{1}{2r} & (i = 1) \\ (2l+1)(2r+1)^{k-1} & (i = 2, \dots, n) \end{cases}. \tag{26}$$

Suppose, without loss of generality, that

$$h_i^{(0,0)} \leq h < 1 \quad (i = 1, \dots, n). \tag{27}$$

Then by a lengthy inductive argument, it follows from (13) – (27) that for

$$l = 1, \dots, r, \quad i = 1, \dots, n, \qquad \text{and} \quad k \geq 0,$$

$$h_i^{(k,2l-1)} \leq h^{u_i^{(k+1,2l-1)}},$$

and

$$h_i^{(k,2l)} \leq h^{u_i^{(k+1,2l)}},$$

whence, by (26) with $l = r$ and (9d), $(\forall k \geq 0)$

$$h_i^{(k)} \leq h^{(2r+1)^k} \quad (i = 1, \dots, n). \tag{28}$$

So $(\forall k \geq 0)$, by (19) for $m = 2r$ and (9d),

$$\left|w_i^{(k,2r)}\right| = \frac{d}{(2n-1)} h_i^{(k,2r)}$$

or

$$\left|w_i^{(k)}\right| = \frac{d}{(2n-1)} h_i^{(k)}.$$

Let

$$w^{(k)} = \max_{1\le i\le n}\{\left|w_i^{(k)}\right|\}, \tag{29}$$

and

$$h^{(k)} = \max_{1\le i\le n}\{h_i^{(k)}\}\,. \tag{30}$$

Then, by (28)-(30)

$$w^{(k)} \le \frac{d}{(2n-1)} h^{(2r+1)^k} \quad (\forall k \ge 0). \tag{31}$$

So by Definition 1 and equations (27) and (31), we have

$$\begin{aligned} R_{2r+1}\left(w^{(k)}\right) &= \lim_{k\to\infty} \sup\left\{(w^{(k)})^{1/(2r+1)^k}\right\} \\ &\le \lim_{k\to\infty} \sup\left\{\left(\frac{d}{2n-1}\right)^{1/(2r+1)^k} h\right\} \\ &= h < 1. \end{aligned}$$

Therefore from Alefeld and Herzberger, 1983 and Ortega and Rheinboldt, 1970, it can be concluded that

$$O_R(PRSS1, x_i^*) \ge (2r+1)\ (r \ge 1)\ (i = 1, \dots, n). \blacksquare$$

4. CONCLUSION

We have shown that the procedure PRSS1 has a higher rate of convergence compared to the previous methods. The attractive features of this procedure will give less computational time. Our experiences in the implementation of the interval version of PRSS1 of Monsi and Wolfe, 1988 showed that this procedure is more efficient for bounding the zeros in terms of CPU times (in seconds) and number of iterations k (see Table 1 below).

Note that for PSS1, no inner iteration can occur since $r = 1$. The polynomials used in Table 1 are the same as those in Monsi *et al*. (2014).

TABLE 1: Number of Iterations (*k*,*r*) and CPU Times

Polynomial	**n**	**PSS1**		**PRSS1**	
		No. of iterations	**CPU Times (seconds)**	**No. of iterations**	**CPU Times (seconds)**
1	4	(3, 1)	0.2215214	(2,2)	0.1466409
2	6	(3, 1)	0.3213620	(2,2)	0.2745616
3	9	(3, 1)	0.5460034	(2,2)	0.5428834
4	5	(3, 1)	0.2714416	(2,2)	0.2090413
5	6	(3, 1)	0.3120020	(2,2)	0.2714417

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