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A Method for Determining p - Adic Orders of Factorials

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ABSTRACT

In this paper, with a prime p the $p-$adic size of $n!$ where n is a positive integer is determined for $ord_p n = 0$ and $ord_p n > 0.$ The discussion begins with the determination of the $p-$adic sizes of factorial functions $p^{\alpha}!, q^{\alpha}!$ and $\left(p^{\alpha}q^{\beta}\right)!$ with $\alpha, \beta > 0$ and q a prime different from $p.$ It is found that $ord_p p^{\alpha}! = \frac{p^{\alpha}-1}{p-1}$ with $\alpha > 0.$ Results are then used to obtain the explicit form of $p-$adic sizes of n from works of earlier authors. It is also found that the $p-$adic orders of ${}^{n}C_r = \frac{n!}{(n-r)!r!}$ is given by

$$ord_p {}^{p^{\alpha}}C_{p^{\theta}} = \frac{p^{\alpha}-p^{\theta}}{p-1} - \sum_{t=0}^{\left[\frac{\ln\left(p^{\alpha}-p^{\theta}\right)}{\ln p}\right]} t\left(\left[p^{\alpha-t}-p^{\theta-t}\right]-\left[p^{\alpha-t-1}-p^{\theta-t-1}\right]\right)$$

where $n = p^{\alpha}$ and $r = p^{\theta}$ with $\alpha > \theta > 0.$

Keywords: Factorial functions, $p-$adic sizes.

1. INTRODUCTION

In this paper, we present a method for determining $p-$adic orders of $n!$, for any poitive integer n. We use the notation $ord_p x$, where p is a prime and x is any rational number to denote the highest power of p dividing x. We refer to $ord_p x$ as the $p-$adic order or the $p-$adic size of x. It follows that, for two rational numbers of x and y, $ord_p(xy) = ord_p x + ord_p y$, $ord_p\left(\frac{x}{y}\right) = ord_p x - ord_p y$ and $ord_p(x \pm y) \geq \min\{ord_p x, ord_p y\}$. By convention $ord_p x = \infty$ if $x = 0$. The notation $[x]$ will as usual denote the greatest integer function. With $n > r$, ${}^nC_r$ will denote the quotient $\frac{n!}{r!(n-r)!}$ where in general $x! = x(x-1)(x-2)...3.2.1.$

Lengyel, 2003 discussed the order of lacunary sums of binomial coefficient of the form $G_{n,l(k)} = \sum_{t=0}^{\left[\frac{k}{n}\right]} \binom{k}{l+nt}$ where $\binom{n}{r}$ indicates the factorial function $\frac{n!}{r!(n-r)!}$. His study involves integers n of the form $n = p^\alpha$ where p is an odd prime and $\alpha > 0$. Adelberg, 1996 examined the $p-$adic orders of $n!$ and $\binom{n}{r}$ where $n > r$ and established some new congruence relations associated with $p-$adic integer order Bernoulli numbers. He applied the relations to prove the irreducibility property of certain Bernoulli polynomials with orders that are divisible by p. Wagstaff, 1996 discussed the Aurifeullian factorizations and the period of the Bell numbers modulo a prime. He showed that $\frac{p^p - 1}{p-1}$ is the minimum period modulo p of the Bell exponential integers.

Koblitz, 1977 determined the $p-$adic sizes of $n!$ where n can be expressed as a number in base prime number p in the form $n = a_0 + a_1 p + a_2 p^2 + ... + a_s p^s$ that is $ord_p\, n! = \frac{n - S_n}{p-1}$ with $S_n = \sum a_i$ the summation of the coefficient in n for $0 \le a_i \le p-1$. Berend, 1997 stated that there exist infinitely many integer positive n such that $ord_{p_1} n! \equiv ord_{p_1} n! \equiv \ldots \equiv ord_{p_k} n! \equiv 0 \pmod 2$ where $p_1, p_2, ..., p_k$ are prime factorsin $n!$ in ascending order. Later, Yong, 2003 improved the result obtained by Sander, 2001 and showed that there exist initial values of n for any prime factorization of $n!$. Suppose k is any integer and $\varepsilon_i \in \{0,1\}$ for $i = 1, 2, ..., k$. It is shown that there exist infinitely many positive integer n with $ord_{p_1} n! \equiv \varepsilon_1 \pmod 2$, $ord_{p_2} n! \equiv \varepsilon_2 \pmod 2, \ldots, ord_{p_k} n! \equiv \varepsilon_k \pmod 2$. In 2007, Yong and Wei proved that if q is prime and b, l any positive integers then $ord_q \left(lq^b\right)! = l\ ord_q\, q^b ! + ord_q\, l!$.

Based on the works of Koblitz, 1997 and Mohd Atan and Loxton, 1986, Sapar and Mohd Atan, 2002 examined the coefficients of linear partial derivative polynomials f_x and f_y associated with the quadratic polynomial $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + m$ and gave the estimate of the $p-$adic sizes of their common zeros in terms of the $p-$adic orders of the coefficients. Later in 2007, they examined the coefficients of a 6^{th} degree polynomial to arrive at the $p-$adic estimate of common zeros of f_x and f_y in terms of the $p-$adic orders of the coefficients in the dominant terms of $f(x, y)$. For a polynomial in the binomial form, the coefficients are expressible in terms of the factorials. That is, $f(x, y) = (ax + by)^n = \sum_{i=0}^{n} C_i^n (ax)^{n-i} (by)^i$ where ${}^nC_i = \frac{n!}{(n-i)!i!}$. Such cases necessitate a method to determine the $p-$adic orders of the factorials in ${}^nC_i$. We begin our discussion with Section 2.0 for determining the $p-$adic orders of $p^\alpha!$ and $q^\alpha!$where p, q primes with $p \ne q$ and $\alpha > 0$. We also derive a formula for $p-$adic orders of $\left(p^\alpha q^\beta\right)!$ with $\alpha, \beta > 0$ in this section. In the subsequent Section 3.0, we present a method for determining the $p-$adic sizes of $n!$ where $n=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_k^{\alpha_k}$ with $\alpha_k>0$ for $ord_p\,n=0$ and $ord_p\,n>0$. At the end of this section, we discuss the findings of previous researchers regarding the determination of $p-$adic sizes for any factorials. In the following Section 4.0, we present a method for determining the $p-$adic sizes of the factorial function ${}^nC_r$ where $n=p^\alpha$ and $r=p^\theta$ with $\alpha>\theta>0$. This is with a view to determine the $p-$adic sizes of coefficients of the terms in the expansion of the polynomial $f(x,y)=(ax+by)^n$ where n is positive.

2. *p* - ADIC ORDERS OF FACTORIAL FUNCTIONS

Let n be a positive integer, p and q primes. We will first consider the cases $n=p^\alpha$ and $n=q^\beta$ where $p\neq q$ and $\alpha,\beta>0$. In this section, we will determine the $p-$adic sizes of $p^\alpha!$ and $q^\alpha!$ followed by the $p-$adic sizes of $\left(p^\alpha q^\beta\right)!$ with $\alpha,\beta>0$.

2.1 *p* - Adic Orders of $p^\alpha!$ and $q^\beta!$

We begin our discussion by introducing a lemma for determining the number of factors $\left(p^\alpha-k\right)$ in $p^\alpha!$ where $0\le k\le p^\alpha-1$ such that $ord_p\left(p^\alpha-k\right)=t$ where t a non-negative integer as follows.

Lemma 2.1.1. Suppose p is a prime, $\alpha>0$, $1\le k\le p^\alpha-1$ and $0\le t\le\alpha-1$. Then, there exist $(p-1)p^{\alpha-(t+1)}$ factors $\left(p^\alpha-k\right)$ in $p^\alpha!$ such that $ord_p\left(p^\alpha-k\right)=t$.

Proof. Let $p^\alpha!=\prod_{k=0}^{p^\alpha-1}\left(p^\alpha-k\right)$. Then $ord_p\,p^\alpha!=\sum_{k=0}^{p^\alpha-1}ord_p\left(p^\alpha-k\right)$.

Let $\left(p^\alpha-k\right)$ be a factor in $p^\alpha!$ such that $ord_p\left(p^\alpha-k\right)=t$ where $0\le t\le\alpha-1$.

The number of such factors is the same as the number of k such that $ord_p k = t$ since clearly $\alpha > ord_p k$.

Now, $ord_p k = t$ when $k = p^t \ell$ with $ord_p \ell = 0$ for some integer ℓ.

Hence, $(p^\alpha, k) = (p^\alpha, p^t \ell) = p^t$. It follows that $(p^{\alpha-t}, \ell) = 1$.

Now, $1 \le k < p^\alpha$ implies that $1 \le \frac{k}{p^t} < p^{\alpha-t}$. That is $1 \le \ell < p^{\alpha-t}$.

The number of ℓ such that $1 \le \ell < p^{\alpha-t}$ and $(p^{\alpha-t}, \ell) = 1$, is given by the Euler totient function, $\varphi(p^{\alpha-t}) = p^{\alpha-t-1}(p-1) = (p-1)p^{\alpha-(t+1)}$.

Hence, there exist $(p-1)p^{\alpha-(t+1)}$ values of k, $1 \le k \le p^\alpha - 1$ such that $ord_p k = t$.

Our assertion follows. □

Corollary 2.1.1. If p is a prime, $\alpha > 0$, $1 \le k \le p^\alpha - 1$ and $0 \le t \le \alpha - 1$, then $ord_p p^\alpha! = \alpha + p^\alpha \sum_{t=0}^{\alpha-1} \frac{(p-1)t}{p^{(t+1)}}$.

Proof. Let $p^\alpha ! = p^\alpha \prod_{k=1}^{p^\alpha - 1} (p^\alpha - k)$.

Then $ord_p p^\alpha ! = ord_p p^\alpha + \sum_{k=1}^{p^\alpha-1} ord_p (p^\alpha - k) = \alpha + \sum_{k=1}^{p^\alpha-1} ord_p (p^\alpha - k)$.

Let t be a non-negative integer. By Lemma 2.1.1, for each t there exist $(p-1)p^{\alpha-(t+1)}$ factors $(p^\alpha - k)$ where $1 \le k \le p^\alpha - 1$ in $p^\alpha!$ such that $ord_p (p^\alpha - k) = t$.

Therefore, $ord_p p^\alpha ! = \alpha + \sum_{t=0}^{\alpha-1} t\left((p-1)p^{\alpha-(t+1)}\right) = \alpha + \sum_{t=0}^{\alpha-1} \frac{(p-1)p^\alpha t}{p^{t+1}}$.

That is $ord_p p^{\alpha}! = \alpha + p^{\alpha} \sum_{t=0}^{\alpha-1} \frac{(p-1)t}{p^{t+1}}$. □

This corollary will be applied for the proof of the following theorem:

Theorem 2.1.1. Let p be a prime and $\alpha > 0$. Then $ord_p p^{\alpha}! = \frac{p^{\alpha}-1}{p-1}$.

Proof. By Corollary 2.1.1,

$$
\begin{aligned}
ord_p p^{\alpha}! &= \alpha + p^{\alpha} \sum_{t=0}^{\alpha-1} \frac{(p-1)t}{p^{t+1}} \\
&= \alpha + p^{\alpha}\left(\frac{(p-1)0}{p} + \frac{(p-1)1}{p^2} + \frac{(p-1)2}{p^3} + \frac{(p-1)3}{p^4} + \ldots + \frac{(p-1)(\alpha-1)}{p^{\alpha}} \right) \\
&= \alpha + (p-1)p^{\alpha}\left(\frac{1}{p^2} + \frac{2}{p^3} + \frac{3}{p^4} + \ldots + \frac{(\alpha-1)}{p^{\alpha}} \right) \\
&= \alpha + (p-1)p^{\alpha}\left(\begin{array}{l} \left(\frac{1}{p^2} + \frac{1}{p^3} + \frac{1}{p^4} + \ldots + \frac{1}{p^{\alpha}} \right) + \left(\frac{1}{p^3} + \frac{1}{p^4} + \ldots + \frac{1}{p^{\alpha}} \right) + \ldots \\ + \left(\frac{1}{p^{k+1}} + \frac{1}{p^{k+2}} + \ldots + \frac{1}{p^{\alpha}} \right) + \ldots + \left(\frac{1}{p^{\alpha}} \right) \end{array} \right) \\
&= \alpha + (p-1)p^{\alpha}\left(\frac{p^{\alpha-1}-1}{(p-1)p^{\alpha}} + \frac{p^{\alpha-2}-1}{(p-1)p^{\alpha}} + \ldots + \frac{p^{\alpha-k}-1}{(p-1)p^{\alpha}} + \ldots + \frac{1}{p^{\alpha}} \right) \\
&= \alpha + \frac{(p-1)p^{\alpha}}{(p-1)p^{\alpha}}\Big((\alpha-2)(-1) + \left(p^{\alpha-1} + p^{\alpha-2} + \ldots + p^{\alpha-k} + \ldots + p^{\alpha-(\alpha-2)} \right) \Big) \\
&\quad + (p-1)p^{\alpha}\left(\frac{1}{p^{\alpha}} \right) \\
&= \alpha + (\alpha-2)(-1) + \left(\frac{p^{\alpha}-p^2}{p-1} \right) + (p-1).
\end{aligned}
$$

On simplifying we obtain $ord_p p^{\alpha}! = \frac{p^{\alpha}-1}{p-1}$. □

Next, we investigate the $p-$adic sizes of $q^{\alpha}!$ where q is a prime and $p \neq q$ and $\alpha > 0$. At first, we introduce the following lemma that will be used for the next theorem. For a positive integer n we determine the number of integers m where $0 < m \leq n$, whose $p-$adic orders are non-vanishing.

Lemma 2.1.2 Let p be a prime, $n > 0$ and m is an integer with $0 < m \leq n$. Then there exist $\left[\frac{n}{p}\right]$ integers m such that $ord_p\, m \neq 0$.

Proof. Since m is an integer and $ord_p\, m \neq 0$, m is of the form $m = pk$ where $ord_p k \geq 0$.

Now, let $S = \{m | m = pk, k = 1, 2, 3, ...\}$ be the set of integers m such that $ord_p k \geq 0$. Then, k gives the number of integers m such that $ord_p\, m \neq 0$.

Now, $0 < m \leq n$ implies that $0 < pk \leq n$. That is, $0 < k \leq \frac{n}{p}$.

Since k is an integer, $0 < k \leq \left[\frac{n}{p}\right]$.

Therefore, $S = \left\{ m \middle| m = pk, k = 1, 2, 3, ..., \left[\frac{n}{p}\right] \right\}$.

Clearly, there exist $\left[\frac{n}{p}\right]$ elements in S. It follows that, there exist $\left[\frac{n}{p}\right]$ integers m such that $ord_p\, m \neq 0$. □

We can now apply this lemma for determining the number of factors $(q^{\alpha} - k)$ in $q^{\alpha}!$ whose $p-$adic orders are non-negative, as follows:

Theorem 2.1.2. Let p, q be any prime and $p < q$, $\alpha > 0$ and $0 \leq k \leq q^{\alpha} - 1$. Then, there exist $\left[\frac{q^{\alpha}}{p^{t}}\right] - \left[\frac{q^{\alpha}}{p^{t+1}}\right]$ factors $(q^{\alpha} - k)$ in $q^{\alpha}!$ such that $ord_p (q^{\alpha} - k) = t$ where $0 \leq t \leq \left[\frac{\ln q^{\alpha}}{\ln p}\right]$.

Proof. Let $q^{\alpha}! = \prod_{k=0}^{q^{\alpha}-1}\left(q^{\alpha}-k\right)$. Then $ord_p q^{\alpha}! = \sum_{k=0}^{q^{\alpha}-1} ord_p\left(q^{\alpha}-k\right)$.

For each factor $\left(q^{\alpha}-k\right)$ in $q^{\alpha}!$, let $ord_p\left(q^{\alpha}-k\right)=t$ where $t \geq 0$. Now, $ord_p\left(q^{\alpha}-k\right)=t$ when $q^{\alpha}-k = p^t m$ where $ord_p m = 0$.

Thus, the number of such factors of $q^{\alpha}!$ is given by the number of integers m such that $ord_p m = 0$ for every t.

We will determine this number as follows:

Clearly, $k = q^{\alpha} - p^t m$. Since $0 \leq k < q^{\alpha}$, we have $0 \leq q^{\alpha} - p^t m < q^{\alpha}$ from which $0 < m \leq \frac{q^{\alpha}}{p^t}$. Since m is an integer, $0 < m \leq \left[\frac{q^{\alpha}}{p^t}\right]$. By Lemma 2.1.2, there exist $\left[\frac{q^{\alpha}}{p^{t+1}}\right]$ integers m such that $ord_p m \neq 0$. Therefore, there exist $\left[\frac{q^{\alpha}}{p^t}\right] - \left[\frac{q^{\alpha}}{p^{t+1}}\right]$ integers m such that $ord_p m = 0$. Hence, the number of factors $\left(q^{\alpha}-k\right)$ in $q^{\alpha}!$ such that $ord_p\left(q^{\alpha}-k\right)=t$ is given by $\left[\frac{q^{\alpha}}{p^t}\right] - \left[\frac{q^{\alpha}}{p^{t+1}}\right]$. Since m is an integer with $1 \leq m \leq \left[\frac{q^{\alpha}}{p^t}\right]$, we have $p^t < q^{\alpha}$.

Therefore, $0 \leq t \leq \left[\frac{\ln q^{\alpha}}{\ln p}\right]$. □

We will recover the result of Lemma 2.1.1 by letting $p = q$ in the first part of the above theorem. The following theorem gives the $p-$adic order of $q^{\alpha}!$ where $p < q$ and $\alpha > 0$.

Theorem 2.1.3. Let p, q be any prime, $p < q$ and $\alpha > 0$. Then

$$ord_p q^{\alpha}! = \sum_{t=0}^{\left[\frac{\ln q^{\alpha}}{\ln p}\right]} t\left(\left[\frac{q^{\alpha}}{p^t}\right] - \left[\frac{q^{\alpha}}{p^{t+1}}\right]\right).$$

Proof. Let $q^{\alpha}! = \prod_{k=0}^{q^{\alpha}-1}\left(q^{\alpha}-k\right)$. Then $ord_p q^{\alpha}! = \sum_{k=0}^{q^{\alpha}-1} ord_p\left(q^{\alpha}-k\right)$.

Let t be a non-negative integer. By Theorem 2.1.2, for each t there exist $\left[\frac{q^{\alpha}}{p^{t}}\right]-\left[\frac{q^{\alpha}}{p^{t+1}}\right]$ factors $\left(q^{\alpha}-k\right)$ in $q^{\alpha}!$ such that $ord_p\left(p^{\alpha}-k\right)=t$, where $0 \le t \le \left[\frac{\ln q^{\alpha}}{\ln p}\right]$. Thus, $ord_p q^{\alpha}! = \sum_{t=0}^{\left[\frac{\ln q^{\alpha}}{\ln p}\right]} t\left(\left[\frac{q^{\alpha}}{p^{t}}\right]-\left[\frac{q^{\alpha}}{p^{t+1}}\right]\right)$. □

2.2 $p-$ Adic Orders of $\left(p^{\alpha}q^{\beta}\right)!$

Theorem 2.2.1 gives the $p-$adic orders of $\left(p^{\alpha}q^{\beta}\right)!$ with $\alpha,\beta>0$ using the result from Theorem 2.1.3.

Theorem 2.2.1. Suppose p and q prime with $p<q$, $\alpha,\beta>0$ and $t>0$. Then $ord_p\left(p^{\alpha}q^{\beta}\right)! = q^{\beta}\left(\frac{p^{\alpha}-1}{p-1}\right)+\sum_{t=0}^{\left[\frac{\ln q^{\beta}}{\ln p}\right]} t\left(\left[\frac{q^{\beta}}{p^{t}}\right]-\left[\frac{q^{\beta}}{p^{t+1}}\right]\right)$.

Proof. From the definition, $\left(p^{\alpha}q^{\beta}\right)! = \prod_{k=0}^{p^{\alpha}q^{\beta}-1}\left(p^{\alpha}q^{\beta}-k\right)$.

This equation can be rewritten as

$\left(p^{\alpha}q^{\beta}\right)! = \prod_{j=0}^{p^{\alpha-1}q^{\beta}-1}\left(p^{\alpha}q^{\beta}-pj\right)\prod_{i=1,\, p\nmid i}^{p^{\alpha}q^{\beta}-1}\left(p^{\alpha}q^{\beta}-i\right)$ with $p\nmid i$ indicating $ord_p i = 0$.

Now, $\prod_{j=0}^{p^{\alpha-1}q^{\beta}-1}\left(p^{\alpha}q^{\beta}-pj\right) = \prod_{j=0}^{p^{\alpha-1}q^{\beta}-1} p\left(p^{\alpha-1}q^{\beta}-j\right) = p^{p^{\alpha-1}q^{\beta}}\left(p^{\alpha-1}q^{\beta}\right)!$.

Therefore, $\left(p^{\alpha}q^{\beta}\right)! = p^{p^{\alpha-1}q^{\beta}}\left(p^{\alpha-1}q^{\beta}\right)!\prod_{i=1,\, p\nmid i}^{p^{\alpha}q^{\beta}-1}\left(p^{\alpha}q^{\beta}-i\right)$ with $p\nmid i$.

Then

$$ord_p\left(p^{\alpha}q^{\beta}\right)! = ord_p\, p^{p^{\alpha-1}q^{\beta}}\left(p^{\alpha-1}q^{\beta}\right)! + ord_p\prod_{i=1,\, p\nmid i}^{p^{\alpha}q^{\beta}-1}\left(p^{\alpha}q^{\beta}-i\right)$$

$$= p^{\alpha-1}q^{\beta} + ord_p\left(p^{\alpha-1}q^{\beta}\right)! + \sum_{i=1,\, p\nmid i}^{p^{\alpha}q^{\beta}-1} ord_p\left(p^{\alpha}q^{\beta}-i\right).$$

Since $p\nmid i$, $ord_p\left(p^{\alpha}q^{\beta}-i\right)=0$ for $1\le i\le p^{\alpha}q^{\beta}-1$. It follows that $\sum_{i=1,\, p\nmid i}^{p^{\alpha}q^{\beta}-1} ord_p\left(p^{\alpha}q^{\beta}-i\right)=0$. Thus, $ord_p\left(p^{\alpha}q^{\beta}\right)! = p^{\alpha-1}q^{\beta} + ord_p\left(p^{\alpha-1}q^{\beta}\right)!$.

From this equation, by replacing $\alpha-i$ we have $ord_p\left(p^{\alpha-i}q^{\beta}\right)! = p^{\alpha-(i+1)}q^{\beta} + ord_p\left(p^{\alpha-(i+1)}q^{\beta}\right)!$ where $i<\alpha$.

Therefore,

$$ord_p\left(p^{\alpha}q^{\beta}\right)! = p^{\alpha-1}q^{\beta} + p^{\alpha-2}q^{\beta} + p^{\alpha-3}q^{\beta} + \ldots + p^{\alpha-(\alpha-1)}q^{\beta} + q^{\beta} + ord_p\, q^{\beta}!$$

$$= p^{\alpha}q^{\beta}\left(\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\ldots+\frac{1}{p^{i+1}}+\ldots+\frac{1}{p^{\alpha-1}}+\frac{1}{p^{\alpha}}\right) + ord_p\, q^{\beta}!\,.$$

Summing the geometric progression, we obtain $ord_p\left(p^{\alpha}q^{\beta}\right)! = p^{\alpha}q^{\beta}\left(\frac{p^{\alpha}-1}{p^{\alpha}\left(p-1\right)}\right) + ord_p\, q^{\beta}!$.

That is, $ord_p\left(p^{\alpha}q^{\beta}\right)! = q^{\beta}\left(\frac{p^{\alpha}-1}{p-1}\right) + ord_p\, q^{\beta}!\,.$

From Theorem 2.1.3, with $p < q$, we have $ord_p\, q^{\beta}! = \sum_{t=0}^{\left[\frac{\ln q^{\beta}}{\ln p}\right]} t\left(\left[\frac{q^{\beta}}{p^t}\right]-\left[\frac{q^{\beta}}{p^{t+1}}\right]\right)$. It follows that

$$ord_p\left(p^{\alpha}q^{\beta}\right)! = q^{\beta}\left(\frac{p^{\alpha}-1}{p-1}\right) + \sum_{t=0}^{\left[\frac{\ln q^{\beta}}{\ln p}\right]} t\left(\left[\frac{q^{\beta}}{p^t}\right]-\left[\frac{q^{\beta}}{p^{t+1}}\right]\right). \qquad \square$$

3. p -ADIC ORDERS OF $n!$

Let p be any prime and n a positive integer. In this section, we present our main results on determining the $p-$adic sizes of $n!$ where n is expressed in its prime power decomposition of the form $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_k^{\alpha_k}$ with $\alpha_i > 0$ for $i = 1, 2, ..., k.$ In order to determine the $p-$adic sizes of $n!$, we need to consider the value of n which can be divided into two cases. They are integers n such that $ord_p\, n = 0$ and $ord_p\, n > 0.$

3.1 $p-$ Adic Orders of $n!$ with $ord_p\, n = 0$

In this section, we discuss the $p-$adic orders of $n!$ with $ord_p\, n = 0$ as in the following theorem.

From the definition, $n! = \prod_{k=0}^{n-1}(n-k)$. Therefore, $ord_p\, n! = \sum_{k=0}^{n-1} ord_p\,(n-k)$.

Suppose $(n-k)$ is a factor in $n!$ with $0 \le k < n$. The following lemma and theorem show that the $p-$adic sizes of $n!$ depends on the number of these factors. The result of Lemma 2.1.2 is used in the proof of Lemma 3.1.1 as follows:

Lemma 3.1.1. Suppose p is a prime and n be a positive integer with $ord_p\, n = 0$ and $0 \le k < n$. Let t be a non-negative integer. Then, there exist

$\left[\frac{n}{p^t}\right]-\left[\frac{n}{p^{t+1}}\right]$ factors $(n-k)$ in $n!$ such that $ord_p(n-k)=t$ with $0 \le t \le \left[\frac{\ln n}{\ln p}\right]$.

Proof.

From the definition, $n! = \prod_{k=0}^{n-1}(n-k)$, $ord_p\, n! = \sum_{k=0}^{n-1} ord_p(n-k)$.

Suppose $(n-k)$ is a factor in $n!$ with $k = 0,1,2,...,n-1$ and t a non-negative integer.

Now, $ord_p(n-k)=t$ when $n-k = p^t m$ with $ord_p m = 0$.

Thus, the number of such factors of $(n-k)$ such that $ord_p(n-k)=t$ in $n!$ is given by the number of integers m $ord_p m = 0$ for each t.

Now, consider the factors $(n-k)$ such that $ord_p(n-k) \ge t$. Thus $(n-k) = p^t m$ where $ord_p m \ge 0$. Since $k \ge 0$ and $k = n - p^t m$, we have $0 < m \le \frac{n}{p^t}$. Since m is an integer, $0 < m \le \left[\frac{n}{p^t}\right]$, there exist $\left[\frac{n}{p^t}\right]$ integers m for every k such that $ord_p m \ge 0$.

From Lemma 2.1.2, there exist $\left[\frac{n}{p^{t+1}}\right]$ integers m such that $ord_p m \ne 0$. It follows that there exist $\left[\frac{n}{p^t}\right]-\left[\frac{n}{p^{t+1}}\right]$ integers m such that $ord_p m = 0$ for every t.

Thus, the number of such factors $(n-k)$ with $k = 0,1,2,...,n-1$ in $n!$ such that $ord_p(n-k)=t$ is given by $\left[\frac{n}{p^t}\right]-\left[\frac{n}{p^{t+1}}\right]$ for every t.

Since m is an integer with $1 \le m \le \left[\frac{n}{p^t}\right]$, we have $p^t < n$. Hence, $0 \le t \le \left[\frac{\ln n}{\ln p}\right]$. □

The following Theorem 3.1.1 gives the $p-$adic sizes of $n!$ for $ord_p\, n = 0$ using the result from Lemma 3.1.1.

Theorem 3.1.1. Suppose p is a prime, n a positive integer with $ord_p\, n = 0$ and t a non-negative integer. Then, $ord_p n! = \sum_{t=0}^{\left[\frac{\ln n}{\ln p}\right]} t\left(\left[\frac{n}{p^t}\right] - \left[\frac{n}{p^{t+1}}\right]\right)$.

Proof. Since $n! = \prod_{k=0}^{n-1}(n-k)$, we have $ord_p\, n! = \sum_{k=0}^{n-1} ord_p\,(n-k)$.

From Lemma 3.1.1, there exist $\left[\frac{n}{p^t}\right] - \left[\frac{n}{p^{t+1}}\right]$ factors $(n-k)$ with $k = 0,1,2,...,n-1$ in $n!$ such that $ord_p\,(n-k) = t$ with $0 \le t \le \left[\frac{\ln n}{\ln p}\right]$. It follows that $ord_p n! = \sum_{t=0}^{\left[\frac{\ln n}{\ln p}\right]} t\left(\left[\frac{n}{p^t}\right] - \left[\frac{n}{p^{t+1}}\right]\right)$. □

3.2 $p-$ Adic Orders of $n!$ with $ord_p\, n > 0$

In this section, we present the case in which the $p-$adic orders of $n!$ is positive. The following theorem gives a result on the $p-$adic sizes of such $n!$ by using result from Theorem 3.1.1.

Theorem 3.2.1. Let p be any prime and n a positive integer such that $ord_p\, n = \alpha$ where $\alpha > 0$. Then,

$$ord_p\, n! = n\left(\frac{p^{\alpha}-1}{p^{\alpha}(p-1)}\right) + \sum_{t=0}^{\left[\frac{\ln np^{-\alpha}}{\ln p}\right]} t\left(\left[\frac{n}{p^{\alpha+t}}\right] - \left[\frac{n}{p^{\alpha+t+1}}\right]\right).$$

Proof. Given $ord_p\, n = \alpha$ with $\alpha > 0$. Then, $n = p^{\alpha} n_1$ with $ord_p\, n_1 = 0$. Clearly, $n_1 = \frac{n}{p^{\alpha}}$.

From the definition, $n! = \left(p^{\alpha} n_1\right)! = \prod_{k=0}^{p^{\alpha}n_1-1}\left(p^{\alpha} n_1 - k\right)$.

The product on the right hand side of the equation can be rewritten as a product of two factors according to the $p-$adic sizes of k, which are $ord_p\, k > 0$ and $ord_p\, k = 0$.

Thus, $n! = \left(p^{\alpha} n_1\right)! = \prod_{j=0}^{p^{\alpha-1}n_1-1}\left(p^{\alpha} n_1 - pj\right)\prod_{i=1,\, p\nmid i}^{p^{\alpha}n_1-1}\left(p^{\alpha} n_1 - i\right)$ where $p \nmid i$ indicates that $ord_p\, i = 0$.

Now, $\prod_{j=0}^{p^{\alpha-1}n_1-1}\left(p^{\alpha} n_1 - pj\right) = \prod_{j=0}^{p^{\alpha-1}n_1-1} p\left(p^{\alpha-1} n_1 - j\right) = p^{p^{\alpha-1}n_1}\left(p^{\alpha-1} n_1\right)!$.

Therefore, $\left(p^{\alpha} n_1\right)! = p^{p^{\alpha-1}n_1}\left(p^{\alpha-1} n_1\right)!\prod_{i=1,\, p\nmid i}^{p^{\alpha}n_1-1}\left(p^{\alpha} n_1 - i\right)$.

Then $ord_p\left(p^{\alpha} n_1\right)! = ord_p\, p^{p^{\alpha-1}n_1} + ord_p\left(p^{\alpha-1} n_1\right)! + \sum_{i=1,\, p\nmid i}^{p^{\alpha}n_1-1} ord_p\left(p^{\alpha} n_1 - i\right)$.

That is, $ord_p\left(p^{\alpha} n_1\right)! = p^{\alpha-1} n_1 + ord_p\left(p^{\alpha-1} n_1\right)! + \sum_{i=1,\, p\nmid i}^{p^{\alpha}n_1-1} ord_p\left(p^{\alpha} n_1 - i\right)$.

Since $p \nmid i, ord_p\left(p^{\alpha} n_1 - i\right) = 0$ for $1 \le i \le \left(p^{\alpha} n_1 - 1\right)$. It follows that $\sum_{i=1,\, p\nmid i}^{p^{\alpha}n_1-1} ord_p\left(p^{\alpha} n_1 - i\right) = 0$.

Thus,

$$ord_p\left(p^{\alpha}n_1\right)! = p^{\alpha-1}n_1 + ord_p\left(p^{\alpha-1}n_1\right)!. \qquad (1)$$

Let i be an integer in the range $1 \le i \le \alpha - 1$.

Then by replacing α by $\alpha - i$ in Equation (1) we would have

$$ord_p\left(p^{\alpha-i}n_1\right)! = p^{\alpha-(i+1)}n_1 + ord_p\left(p^{\alpha-(i+1)}n_1\right)!. \qquad (2)$$

Therefore by (1) and (2),

$$\begin{aligned} ord_p\, n! &= ord_p\left(p^{\alpha}n_1\right)! \\ &= \left(p^{\alpha-1}n_1\right) + \left(p^{\alpha-2}n_1\right) + \left(p^{\alpha-3}n_1\right) + \ldots + \left(p^{\alpha-(i+1)}n_1\right) + \ldots + \left(p^{\alpha-(\alpha-1)}n_1\right) \\ &\quad + n_1 + ord_p\, n_1! \\ &= n_1\left(p^{\alpha-1} + p^{\alpha-2} + p^{\alpha-3} + \ldots + p^{\alpha-(i+1)} + \ldots + p^{\alpha-(\alpha-1)} + 1\right) + ord_p\, n_1!. \end{aligned}$$

Hence,

$$ord_p\, n! = n_1 p^{\alpha}\left(\frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \ldots + \frac{1}{p^{(i+1)}} + \ldots + \frac{1}{p^{\alpha-1}} + \frac{1}{p^{\alpha}}\right) + ord_p\, n_1!.$$

On simplifying, we obtain $ord_p\, n! = n_1\left(\dfrac{p^{\alpha}-1}{p-1}\right) + ord_p\, n_1!$.

Since $ord_p\, n_1 = 0$, from Theorem 3.1.1, we have

$$ord_p\, n_1! = \sum_{t=0}^{\left[\frac{\ln n_1}{\ln p}\right]} t\left(\left[\frac{n_1}{p^t}\right] - \left[\frac{n_1}{p^{t+1}}\right]\right).$$

Thus, $ord_p\, n! = n_1\left(\frac{p^{\alpha}-1}{p-1}\right) + \sum_{t=0}^{\left[\frac{\ln n_1}{\ln p}\right]} t\left(\left[\frac{n_1}{p^t}\right]-\left[\frac{n_1}{p^{t+1}}\right]\right)$. Letting $n_1 = \frac{n}{p^{\alpha}}$, it follows that $ord_p\, n! = n\left(\frac{p^{\alpha}-1}{p^{\alpha}\left(p-1\right)}\right) + \sum_{t=0}^{\left[\frac{\ln np^{-\alpha}}{\ln p}\right]} t\left(\left[\frac{n}{p^{\alpha+t}}\right]-\left[\frac{n}{p^{\alpha+t+1}}\right]\right)$. □

The following corollary of Theorem 3.2.1 shows the $p-$adic orders of $n!$ where $ord_p\, n = \alpha$ with $\alpha > 0$.

Corollary 3.2.1. Suppose n is any positive integer with prime power decomposition $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$.

Let $n_i = \frac{n}{p_i^{\alpha_i}}$. Then, $ord_{p_i}\, n! = n_i\left(\frac{p^{\alpha_i}-1}{p-1}\right) + \sum_{t=0}^{\left[\frac{\ln n_i}{\ln p_i}\right]} t\left(\left[\frac{n_i}{p_i^t}\right]-\left[\frac{n_i}{p_i^{t+1}}\right]\right)$ for $i = 1, 2, \ldots, k$.

Proof. Given $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$ with $\alpha_i > 0$ for $i = 1, 2, \ldots, k$. Clearly, $ord_{p_i}\, n = \alpha_i > 0$.

As in the proof of Theorem 3.2.1, we obtain $ord_{p_i}\, n! = n_i\left(\frac{p_i^{\alpha_i}-1}{p_i-1}\right) + ord_{p_i}\, n_i!$ with $ord_{p_i}\, n_i = 0$ for $i = 1, 2, ..., k$.

Since $ord_{p_i}\, n_i = 0$, by Theorem 3.1.1 we have $ord_{p_i}\, n_i! = \sum_{t=0}^{\left[\frac{\ln n_i}{\ln p_i}\right]} t\left(\left[\frac{n_i}{p_i^t}\right]-\left[\frac{n_i}{p_i^{t+1}}\right]\right)$.

Hence, $ord_{p_i}\, n! = n_i\left(\frac{p^{\alpha_i}-1}{p-1}\right) + \sum_{t=0}^{\left[\frac{\ln n_i}{\ln p_i}\right]} t\left(\left[\frac{n_i}{p_i^t}\right]-\left[\frac{n_i}{p_i^{t+1}}\right]\right)$ with $n_i = \frac{n}{p_i^{\alpha_i}}$. □

Koblitz , 1977 showed that if p is any prime and n positive integer where n is expressed as a number in base prime p of the form $n = a_0 + a_1 p + a_2 p^2 + ... + a_s p^s$ and $0 \le a_i \le p-1$, then $ord_p\, n! = \frac{n - S_n}{p-1}$, where $S_n = \sum a_i$ the summation of the coefficients of p in $n, i \ge 0$. By comparison, Theorem 3.1.1 and Theorem 3.2.1 give

$$ord_p n! = \sum_{t=0}^{\left[\frac{\ln n}{\ln p}\right]} t\left(\left[\frac{n}{p^t}\right] - \left[\frac{n}{p^{t+1}}\right]\right) \text{ for } ord_p n = 0 \text{ and}$$

$$ord_p\, n! = n\left(\frac{p^\alpha - 1}{p^\alpha (p-1)}\right) + \sum_{t=0}^{\left[\frac{\ln np^{-\alpha}}{\ln p}\right]} t\left(\left[\frac{n}{p^{\alpha+t}}\right] - \left[\frac{n}{p^{\alpha+t+1}}\right]\right) \text{ for } ord_p n > 0$$

respectively. Both formulae give alternative ways of determining $ord_p\, n!$ based on certain conditions with the latter without having to express the value of n in base prime number p.

In the previous findings of $p-$adic sizes of particular factorials by Yong and Wei, 2007, it is shown that for all primes q and any positive integers b and l, $e_q\left(lq^b\right)! = l\, e_q q^b ! + e_q\, l!$. The notation $e_q\, x$ represents the $q-$adic order of an integer x. Based on Theorem 2.2.1, by interchanging primes p, q and $\beta = 1$, we obtain $ord_q\left(q^\alpha p\right)! = p\left(\frac{q^\alpha - 1}{q-1}\right) + \sum_{t=0}^{\left[\frac{\ln p}{\ln q}\right]} t\left(\left[\frac{p}{q^t}\right] - \left[\frac{p}{q^{t+1}}\right]\right)$ where $p < q < p^\alpha$ and $\alpha > 0$. This gives the $q-$adic sizes of $\left(q^\alpha p\right)!$ for any prime p and q. Now, we apply our method to obtain explicit result for $ord_q\left(lq^b\right)!$ for any positive integer l. We need only to determine the value of $ord_q\, l!$ since that of $ord_q\, q^b!$ is readily available from Theorem 2.1.1. In order to evaluate $ord_q\, l!$, there are two cases to consider; they are the case when $ord_q\, l = 0$ and $ord_q\, l > 0$.

The following theorem gives the $q-$adic sizes of $\left(lq^b\right)!$ with $q<l$ and $ord_q\, l=0$ using the result from Theorem 3.1.1.

Theorem 3.2.2 . Suppose q is any prime and b,l are positive integers with $q<l$ and $ord_q\, l=0$. Suppose t be a non-negative integer. Then

$$ord_q\left(lq^b\right)!=l\left(\frac{q^b-1}{q-1}\right)+\sum_{t=0}^{\left[\frac{\ln l}{\ln q}\right]} t\left(\left[\frac{l}{q^t}\right]-\left[\frac{l}{q^{t+1}}\right]\right).$$

Proof. Yong and Wei, 2007 showed that $ord_q\left(lq^b\right)!=l\, ord_q\, q^b!+ord_q\, l!$ with $b>0$. From Theorem 2.1.1, with q a prime and $b>0$, we have

$$ord_q\, q^b!=\left(\frac{q^b-1}{q-1}\right).$$

As well as from Theorem 3.1.1 with $ord_q\, l=0$, we have

$$ord_q\, l!=\sum_{t=0}^{\left[\frac{\ln l}{\ln q}\right]} t\left(\left[\frac{l}{q^t}\right]-\left[\frac{l}{q^{t+1}}\right]\right).$$

Thus, $ord_q\left(lq^b\right)!=l\left(\frac{q^b-1}{q-1}\right)+\sum_{t=0}^{\left[\frac{\ln l}{\ln q}\right]} t\left(\left[\frac{l}{q^t}\right]-\left[\frac{l}{q^{t+1}}\right]\right).$ □

Now, the next theorem gives the $q-$adic sizes of $\left(lq^b\right)!$ with $q<l$ and $ord_q\, l=\alpha$, $\alpha>0$ using the result from Theorem 3.2.1.

Theorem 3.2.3. Suppose q is any prime and b and l are positive integers with $q<l$.

Suppose $ord_q\, l=\alpha$ with $\alpha>0$ and t is a non-negative integer. Then,

$$ord_q\left(lq^b\right)!=l\left(\frac{q^{\alpha+b}-1}{q^{\alpha}\left(q-1\right)}\right)+\sum_{t=0}^{\left[\frac{\ln lq^{-\alpha}}{\ln q}\right]} t\left(\left[\frac{l}{q^{\alpha+t}}\right]-\left[\frac{l}{q^{\alpha+t+1}}\right]\right).$$

Proof. From the works of Yong and Wei, 2007, it is shown that $ord_q\left(lq^b\right)!=l\, ord_q\, q^b!+ord_q\, l!$ with $b>0$. From Theorem 2.1.1, with q prime and $b>0$, we obtain $ord_q\, q^b!=\left(\frac{q^b-1}{q-1}\right)$.

As well as from Theorem 3.2.1, with $ord_q\, l=\alpha$, $\alpha>0$, we have

$$ord_q\, l!=l\left(\frac{q^{\alpha}-1}{q^{\alpha}\left(q-1\right)}\right)+\sum_{t=0}^{\left[\frac{\ln lq^{-\alpha}}{\ln q}\right]} t\left(\left[\frac{l}{q^{\alpha+t}}\right]-\left[\frac{l}{q^{\alpha+t+1}}\right]\right).$$

Then, $ord_q\left(lq^b\right)!=l\left(\frac{q^b-1}{q-1}\right)+l\left(\frac{q^{\alpha}-1}{q^{\alpha}\left(q-1\right)}\right)+\sum_{t=0}^{\left[\frac{\ln lq^{-\alpha}}{\ln q}\right]} t\left(\left[\frac{l}{q^{\alpha+t}}\right]-\left[\frac{l}{q^{\alpha+t+1}}\right]\right).$

That is, $ord_q\left(lq^b\right)!=l\left(\frac{q^{\alpha+b}-1}{q^{\alpha}\left(q-1\right)}\right)+\sum_{t=0}^{\left[\frac{\ln lq^{-\alpha}}{\ln q}\right]} t\left(\left[\frac{l}{q^{\alpha+t}}\right]-\left[\frac{l}{q^{\alpha+t+1}}\right]\right).$ □

4. p - ADIC ORDERS OF ^nC_r

Let n, r be integers with $n>r$. In this section, we will discuss the $p-$adic sizes of $^nC_r=\frac{n!}{r!\left(n-r\right)!}$, for the case $n=p^{\alpha}$ and $r=p^{\theta}$. Clearly $\alpha>\theta$ since $n>r$.

Now, $^{p^{\alpha}}C_{p^{\theta}}=\frac{p^{\alpha}!}{p^{\theta}!\left(p^{\alpha}-p^{\theta}\right)!}$.

Therefore, $ord_p\, {}^{p^{\alpha}}C_{p^{\theta}}=ord_p\, p^{\alpha}!-ord_p\, p^{\theta}!-ord_p\left(p^{\alpha}-p^{\theta}\right)!$.

By Theorem 2.1.1, we have

$$ord_p \; {}^{p^\alpha}C_{p^\theta} = \frac{p^\alpha - 1}{p-1} - \frac{p^\theta - 1}{p-1} - ord_p\left(p^\alpha - p^\theta\right)!$$

$$= \frac{p^\alpha - p^\theta}{p-1} - ord_p\left(p^\alpha - p^\theta\right)! \qquad (3)$$

The value of $ord_p\left(p^\alpha - p^\theta\right)!$ is determined by the following Theorem 4.1.1.

First, we have the following assertion:

Lemma 4.1.1. Suppose p is a prime, $\alpha > \theta > 0$ and $0 \le k \le p^\alpha - p^\theta - 1$. Then, there exist $\left[p^{\alpha-t} - p^{\theta-t}\right] - \left[p^{\alpha-t-1} - p^{\theta-t-1}\right]$ factors $\left(p^\alpha - p^\theta - k\right)$ in $\left(p^\alpha - p^\theta\right)!$ such that $ord_p\left(p^\alpha - p^\theta - k\right) = t$ where $0 \le t \le \left[\frac{\ln\left(p^\alpha - p^\theta\right)}{\ln p}\right]$.

Proof. Let $\left(p^\alpha - p^\theta\right)! = \prod_{k=0}^{p^\alpha - p^\theta - 1}\left(p^\alpha - p^\theta - k\right)$.

Then, $ord_p\left(p^\alpha - p^\theta\right)! = \sum_{k=0}^{p^\alpha - p^\theta - 1} ord_p\left(p^\alpha - p^\theta - k\right)$.

Let $\left(p^\alpha - p^\theta - k\right)$ be a factor in $\left(p^\alpha - p^\theta\right)!$ such that $ord_p\left(p^\alpha - p^\theta - k\right) = t$ where $t \ge 0$.

Now, $ord_p\left(p^\alpha - p^\theta - k\right) = t$ when $p^\alpha - p^\theta - k = p^t m$ where $ord_p m = 0$.

Thus, the number of such factors in $\left(p^\alpha - p^\theta\right)!$ is given by the number of integers m such that $ord_p m = 0$ for every t.

Now, $k = p^\alpha - p^\theta - p^t m$. That is, $p^t m = p^\alpha - p^\theta - k$ with $k = 0, 1, 2, ..., p^\alpha - p^\theta - 1$.

Thus, $0 < p^t m \le p^\alpha - p^\theta$.

Since m is integer then $0 < m \le \left[p^{\alpha-t} - p^{\theta-t} \right]$.

By Lemma 2.1.2**,** there exist $\left[p^{\alpha-t-1} - p^{\theta-t-1} \right]$ integers m such that $ord_p\, m \ne 0$.

Therefore, there exist $\left[p^{\alpha-t} - p^{\theta-t} \right] - \left[p^{\alpha-t-1} - p^{\theta-t-1} \right]$ integers m such that $ord_p\, m = 0$ **.**

Hence, the number of factors of $\left(p^\alpha - p^\theta - k \right)$ in $\left(p^\alpha - p^\theta \right)!$ such that $ord_p \left(p^\alpha - p^\theta - k \right) = t$ is given by $\left[p^{\alpha-t} - p^{\theta-t} \right] - \left[p^{\alpha-t-1} - p^{\theta-t-1} \right]$.

Since m is an integer and $1 \le m \le p^{\alpha-t} - p^{\theta-t}$, we have $p^t < p^\alpha - p^\theta$.

Therefore, $0 \le t \le \left[\dfrac{\ln\left(p^\alpha - p^\theta \right)}{\ln p} \right]$. □

Theorem 4.1.1. Let p be any prime, $\alpha > \theta > 0$, then

$$ord_p \left(p^\alpha - p^\theta \right)! = \sum_{t=0}^{\left[\frac{\ln\left(p^\alpha - p^\theta \right)}{\ln p} \right]} t\left(\left[p^{\alpha-t} - p^{\theta-t} \right] - \left[p^{\alpha-t-1} - p^{\theta-t-1} \right] \right)$$

Proof.

$$ord_p \left(p^\alpha - p^\theta \right)! = ord_p \prod_{k=0}^{p^\alpha - p^\theta - 1} \left(p^\alpha - p^\theta - k \right) = \sum_{k=0}^{p^\alpha - p^\theta - 1} ord_p \left(p^\alpha - p^\theta - k \right).$$

From Lemma 4.1.1, there exist $\left[p^{\alpha-t}-p^{\theta-t}\right]-\left[p^{\alpha-t-1}-p^{\theta-t-1}\right]$ factors $\left(p^{\alpha}-p^{\theta}-k\right)$ in $\left(p^{\alpha}-p^{\theta}\right)!$ such that $ord_p\left(p^{\alpha}-p^{\theta}-k\right)=t$, where $0\le t\le\left[\dfrac{\ln\left(p^{\alpha}-p^{\theta}\right)}{\ln p}\right]$.

Thus,

$$ord_p\left(p^{\alpha}-p^{\theta}\right)!=\sum_{t=0}^{\left[\frac{\ln\left(p^{\alpha}-p^{\theta}\right)}{\ln p}\right]} t\left(\left[p^{\alpha-t}-p^{\theta-t}\right]-\left[p^{\alpha-t-1}-p^{\theta-t-1}\right]\right). \qquad \square$$

We next determine the value of $ord_p\,{}^{p^{\alpha}}C_{p^{\theta}}$ as follows:

Theorem 4.1.2. Let p be a prime, $\alpha>\theta>0$, then

$$ord_p\,{}^{p^{\alpha}}C_{p^{\theta}}=\frac{p^{\alpha}-p^{\theta}}{p-1}-\sum_{t=0}^{\left[\frac{\ln\left(p^{\alpha}-p^{\theta}\right)}{\ln p}\right]} t\left(\left[p^{\alpha-t}-p^{\theta-t}\right]-\left[p^{\alpha-t-1}-p^{\theta-t-1}\right]\right).$$

Proof. The proof follows from equation (3) and Theorem 4.1.1. □

5. CONCLUSION

In this paper, we have presented a method for determining the $p-$adic sizes of $n!$ where n is a positive integer and p is a prime. The results obtained are in explicit forms and the method of obtaining them offers an alternative way to finding $ord_p\,n!$. As presented in this paper, the method does not require the integer n to be expressed in base p as it is usually done. It also enables one to obtain more explicit results of $p-$adic sizes of $\left(lq^{b}\right)!$ where l is an integer, q a prime and $b>0$. To illustrate application of results obtained in this paper, $p-$adic sizes of ${}^{n}C_{r}$ where $n=p^{\alpha}$ and $r=p^{\theta}$ with $\alpha>\theta>0$ are determined. This method is extendable to determining ${}^{n}C_{r}$ for any positive n and r. The $p-$adic sizes of other expressions containing factorial factors may also be found by applying the results in this paper.

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