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An Implicit 2-point Block Extended Backward Differentiation Formula for Integration of Stiff Initial Value Problems

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ABSTRACT

A class of implicit 2-point block extended backward differentiation formula (BEBDF) of order 4 is presented. The stability region of the method is constructed and shown to be A – stable. Results obtained are compared with an existing block backward differentiation formula (BBDF). The comparison shows that using constant step size and the same number of integration steps, our method achieves greater accuracy than the 2-point BBDF and is suitable for solving stiff initial value problems.

Keywords: 2–point block method, Stability region, Block extended backward differentiation formula, Stiff.

1. INTRODUCTION

Stiff differential equations arise in many areas of science and technology. Their solutions are known to be numerically unstable with many numerical methods, unless the step size taken is extremely small (Brugnano *et al.* (2011)). Thus, to overcome this stability restriction on the step size, numerical methods that possess unbounded region of absolute stability (A–stable or stiffly stable) have been recommended for the solution of stiff initial value problems (IVPs). One of the most popular methods for solving stiff ordinary differential equations (ODEs) of the form

$$y' = (f(x,y) \qquad y(a) = y_0, \qquad x \in [a,b] \tag{1}$$

is the backward differentiation formula (BDF) by Curtiss and Hirschfelder (1952). The BDF is a linear multistep method (LMM) of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} \tag{2}$$

Relevant research on numerical solution of stiff IVPs can also be found in Alt (1978), Alvarez and Rojo (2002), Aminikhah and Hemmatnezhad (2011), Kushnir and Rokhlin (2012), Suleiman et al (2013) and Musa et al (2013). A famous result due to Dahlquist (1963) has shown that no A–stable linear multistep method (LMM) can have order greater than 2. However, strategies for improving accuracy, order and efficiency of multistep methods have been suggested by Hairer and Wanner (2004) which include addition of a future point, off-step point and using higher derivatives. Cash (1980) attempted to circumvent the Dahlquist barrier by developing an extended backward differentiation formula (EBDF); characterized by the use of a super 'future point'. The formula has the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_k f_{n+k} + h\beta_{k+1} f_{n+k+1} \tag{3}$$

The method is L – stable up to order 4 and L (α) – stable up to order 9.

The procedures for implementing the formula (3) involve predicting the required solution using the conventional BDF and correcting the solution using EBDF method of higher order. The procedures as outlined in Cash (1980) and Hairer and Wanner (2004) are as follows:

(1) Computing $\overset{-n}{y}_{n+k}$ as the solution of the conventional k-step BDF

$$y_{n+k} - h\overline{\beta}_k f_{n+k} = -\sum_{j=0}^{k-1} \overline{\alpha}_j y_{n+j} \tag{4}$$

(2) Computing $y_{n+k+1}^{(n)}$ as the solution of

$$\overline{y}_{n+k+1} - h\overline{\beta}_k f_{n+k+1} = -\overline{\alpha}_{k-1} \overline{y}_{n+k}^{(n)} - \sum_{j=0}^{k-1} \overline{\alpha}_j y_{n+j} \tag{5}$$

(3) Computing $\overline{f}_{n+k+1} \equiv f(x_{n+k+1}, \overline{y}^{(n)}_{n+k+1})$.

(4) Computing y_{n+k} from (3) in the form

$$y_{n+k} - h\beta_k f_{n+k} = -\sum_{j=0}^{k-1} \alpha_j y_{n+j} + h\beta_{k+1}\overline{f}_{n+k+1} \tag{6}$$

In this paper, a similar idea to that in Cash (1980) is applied to the 2-point BBDF method

$$\sum_{j=0}^{3} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i} f_{n+k}, \qquad k = i = 1, 2. \tag{7}$$

developed in Ibrahim (2007), to develop a new implicit block formula we shall call a 2–point block extended backward differentiation formula (BEBDF). The strategy employed involved adding a future point to (7) to generate a formula of the form:

$$\sum_{j=0}^{3} \alpha_{j,i} y_{n+j-1} = h\beta_{k,i} f_{n+k} + h\beta_{k+1,i} f_{n+k+1}, \qquad k = 1, 2. \tag{8}$$

The interpolation points involved in the derivation of the formula (8) is shown below.

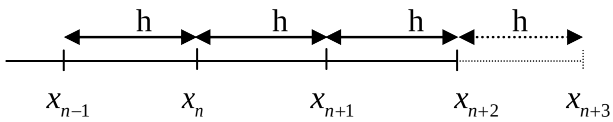

Figure 1: Interpolation points involved in the 2-point BEBDF method

2. DERIVATION OF THE 2–POINT BEBDF

Given the previous values at x_{n-1} and x_n, we shall derive a formula that will compute the solution at x_{n+1} and x_{n+2} simultaneously. The point x_{n+3} in Figure 1 is the "super future" point.

An implicit 2 – point BEBDF is constructed using a linear operator defined by:

$$L_i[y(x_n),h]:\alpha_{0,i}y_{n-1}+\alpha_{1,i}y_n+\alpha_{2,i}y_{n+1}+\alpha_{3,i}y_{n+2}-\beta_{1,i}hf_{n+k,i}-\beta_{2,i}hf_{n+k+1,i}=0 \tag{9}$$

where $k=i=1,2.$

$k=i=1$ k=i=1 corresponds to the first point while k=i=2 corresponds to the second point.

To derive the first point y_{n+1}, let k=i=1 and define the operator L_1 by:

$$L_1[y(x_n),h]:\alpha_{0,1}y_{n-1}+\alpha_{1,1}y_n+\alpha_{2,1}y_{n+1}+\alpha_{3,1}y_{n+2}-\beta_{1,1}hf_{n+k,1}-\beta_{2,1}hf_{n+k+1,1}=0 \tag{10}$$

The associated approximate relationship for (10) can be written as

$$\begin{aligned}&\alpha_{0,1}y(x_n-h)+\alpha_{1,1}y(x_n)+\alpha_{2,1}y(x_n+h)+\alpha_{3,1}y(x_n+2h)\\&-\beta_{1,1}hy'(x_n+h)-\beta_{2,1}y'(x_n+2h)=0\end{aligned} \tag{11}$$

Expanding $y(x_n-h), y(x_n), y(x_n+h), (x_n+2h), y'(x_n+h)$ and $y'(x_n+2h)$ as Taylor series about x_n and collecting like terms gives

$$C_{0,1}y(x_n)+C_{1,1}hy'(x_n)+C_{2,1}h^2y''(x_n)+C_{3,1}h^3y'''(x_n)+\ldots=0 \tag{12}$$

where

$$\left.\begin{aligned}&C_{0,1}=\alpha_{0,1}+\alpha_{1,1}+\alpha_{2,1}+\alpha_{3,1}=0\\&C_{1,1}=-\alpha_{0,1}+\alpha_{2,1}+2\alpha_{3,1}-\beta_{1,1}-\beta_{2,1}=0\\&C_{2,1}=\frac{1}{2}\alpha_{0,1}+\frac{1}{2}\alpha_{2,1}+2\alpha_{3,1}-\beta_{1,1}-2\beta_{2,1}=0\\&C_{3,1}=-\frac{1}{6}\alpha_{0,1}+\frac{1}{6}\alpha_{2,1}+\frac{4}{3}\alpha_{3,1}-\frac{1}{2}\beta_{1,1}-2\beta_{2,1}=0\\&C_{4,1}=\frac{1}{24}\alpha_{0,1}+\frac{1}{24}\alpha_{2,1}+\frac{2}{3}\alpha_{3,1}-\frac{1}{6}\beta_{1,1}-\frac{4}{3}\beta_{2,1}=0\\&\vdots\end{aligned}\right\} \tag{13}$$

The coefficient of y_{n+1} is normalized to 1. Solving the simultaneous equations thus formed for $\alpha_{ji's}$ and $\beta_{ji's}$ gives the formula for y_{n+1} as

$$y_{n+1}=\frac{1}{9}y_{n-1}-y_n+\frac{17}{9}y_{n+2}-2hf_{n+1}-\frac{2}{3}hf_{n+2} \tag{14}$$

The second point y_{n+2} is derived using the linear operator

$$L_2[y(x_n),h]:\alpha_{0,2}y_{n-1}+\alpha_{1,2}y_n+\alpha_{2,2}y_{n+1}+\alpha_{3,2}y_{n+2}-\beta_{2,2}hf_{n+2}-\beta_{3,2}hf_{n+3}=0 \tag{15}$$

By adopting a similar procedure in deriving the first point (14), we obtain the formula for the second point as

$$y_{n+2}=\frac{17}{197}y_{n-1}-\frac{99}{197}y_n+\frac{279}{197}y_{n+1}+\frac{150}{197}hf_{n+2}-\frac{18}{197}hf_{n+3} \tag{16}$$

The 2–point BEBDF is therefore given by

$$\left.\begin{aligned} y_{n+1}&=\frac{1}{9}y_{n-1}-y_n+\frac{17}{9}y_{n+2}-2hf_{n+1}-\frac{2}{3}hf_{n+2}\\ y_{n+2}&=\frac{17}{197}y_{n-1}-\frac{99}{197}y_n+\frac{279}{197}y_{n+1}+\frac{150}{197}hf_{n+2}-\frac{18}{197}hf_{n+3}\end{aligned}\right\} \tag{17}$$

3. ORDER OF THE METHOD

Define the method (17) in general matrix form as

$$\sum_{j=0}^{1}A_j^*Y_{m+j-1}=h\sum_{j=0}^{2}B_{j-1}^*F_{m+j-1} \tag{18}$$

where

$$A_0^*=\begin{pmatrix}-\frac{1}{9} & 1\\ -\frac{17}{197} & \frac{99}{197}\end{pmatrix},\qquad A_1^*=\begin{pmatrix}1 & -\frac{17}{9}\\ -\frac{279}{197} & 1\end{pmatrix}$$

$$B_{-1}^*=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},\qquad B_0^*=\begin{pmatrix}-2 & -\frac{2}{3}\\ 0 & \frac{150}{197}\end{pmatrix},$$

$$B_1^*=\begin{pmatrix}0 & 0\\ -\frac{18}{197} & 0\end{pmatrix},\qquad Y_m=\begin{pmatrix}y_{n+1}\\ y_{n+2}\end{pmatrix}=\begin{pmatrix}y_{2m+1}\\ y_{2m+2}\end{pmatrix},$$

$$Y_{m-1}=\begin{pmatrix}y_{n-1}\\ y_n\end{pmatrix}=\begin{pmatrix}y_{2m-1}\\ y_{2m}\end{pmatrix}=\begin{pmatrix}y_{2(m-1)+1}\\ y_{2(m-1)+2}\end{pmatrix},$$

$$F_{m-1} = \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} f_{2m-1} \\ f_{2m} \end{pmatrix} = \begin{pmatrix} f_{2(m-1)+1} \\ f_{2(m-1)+2} \end{pmatrix},$$

$$F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} = \begin{pmatrix} f_{2m+1} \\ f_{2m+2} \end{pmatrix}, \qquad F_{m+1} = \begin{pmatrix} f_{n+3} \\ f_{n+4} \end{pmatrix} = \begin{pmatrix} f_{2m+3} \\ f_{2m+4} \end{pmatrix} = \begin{pmatrix} f_{2(m+1)+1} \\ f_{2(m+1)+2} \end{pmatrix}$$

and $n = 2m.$

Equation (18) can be written as

$$\begin{pmatrix} -\frac{1}{9} & 1 \\ -\frac{17}{197} & \frac{99}{197} \end{pmatrix} \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} 1 & -\frac{17}{9} \\ -\frac{279}{197} & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = h \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_{n-1} \\ f_n \end{pmatrix}$$
$$+ h \begin{pmatrix} -2 & -\frac{2}{3} \\ 0 & \frac{150}{197} \end{pmatrix} + h \begin{pmatrix} 0 & 0 \\ -\frac{18}{197} & 0 \end{pmatrix} \begin{pmatrix} f_{n+3} \\ f_{n+4} \end{pmatrix} \tag{19}$$

Let $A_0^*, A_1^*, B_{-1}^*, B_0^*$ and B_1^* be block matrices defined by

$$A_0^* = \begin{pmatrix} A_0 & A_1 \end{pmatrix}, \qquad A_1^* = \begin{pmatrix} A_2 & A_3 \end{pmatrix}, \qquad B_{-1}^* = \begin{pmatrix} B_0 & B_1 \end{pmatrix},$$
$$B_0^* = \begin{pmatrix} B_2 & B_3 \end{pmatrix}, \qquad B_1^* = \begin{pmatrix} B_3 & B_4 \end{pmatrix},$$

where

$$A_0 = \begin{pmatrix} -\frac{1}{9} \\ -\frac{17}{197} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 \\ \frac{99}{197} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 \\ -\frac{270}{197} \end{pmatrix}, \quad A_3 = \begin{pmatrix} -\frac{17}{9} \\ 1 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; B_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{150}{197} \end{pmatrix},$$

$$B_4 = \begin{pmatrix} 0 \\ -\frac{18}{197} \end{pmatrix}, \quad B_5 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Definition 3.1

The order of the block method (18) and its associated linear operator given by:

$$L\left[y(x);h\right]=\sum_{j=0}^{k=3}\left[A_j y(x+jh)\right]-h\sum_{j=0}^{k+1}\left[B_j y'(x+jh)\right] \quad (20)$$

is a unique integer p such that $C_q=0, q=0(1)p,$ and $C_{p+1}\neq 0$; where the C_q are constant (column) matrices defined by:

$$C_0=A_0+A_1+\cdots+A_k$$
$$C_1=A_1+2A_2+\cdots+kA_k-\left(B_0+B_1+B_2+\cdots+B_k\right)$$
$$C_q=\frac{1}{q!}\left(A_1+2^qA_2+\cdots+k^qA_k\right)-\frac{1}{(q-1)!}\left(B_1+2^{q-1}B_2+\cdots+(k+1)^{q-1}B_{k+1}\right) \quad (21)$$

For q=0(1)5, we have

$$C_0=A_0+A_1+A_2+A_3=0$$
$$C_1=\left(A_1+2A_2+3A_3\right)-\left(B_0+B_1+B_2+B_3+B_4\right)$$
$$C_2=\frac{1}{2!}\left(A_1+2^2A_2+3^2A_3\right)-\frac{1}{1!}\left(B_1+2^1B_2+3^1B_3+4^1B_4\right)$$
$$C_3=\frac{1}{3!}\left(A_1+2^3A_2+3^3A_3\right)-\frac{1}{2!}\left(B_1+2^2B_2+3^2B_3+4^2B_4\right)$$
$$C_4=\frac{1}{4!}\left(A_1+2^4A_2+3^4A_3\right)-\frac{1}{3!}\left(B_1+2^3B_2+3^3B_3+4^3B_4\right)$$
$$C_5=\frac{1}{5!}\left(A_1+2^5A_2+3^5A_3\right)-\frac{1}{4!}\left(B_1+2^4B_2+3^4B_3+4^4B_4\right)=\begin{pmatrix}\frac{1}{30}\\ \frac{111}{1970}\end{pmatrix}\neq 0 \quad (22)$$

Therefore, the formula (17) is of order 4, with error constant

$$\begin{pmatrix}\frac{1}{30}\\ \frac{111}{1970}\end{pmatrix}$$

4. STABILITY OF THE METHOD

To analyze the method (17) for stability, we apply the scalar test equation

$$y' = \lambda y \qquad \lambda \in \mathbb{C} \qquad \Re(\lambda) < 0 \tag{23}$$

The method (17) can be written in matrix form as

$$\begin{aligned}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{17}{9} \\ \frac{279}{197} & 0 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} + \begin{pmatrix} \frac{1}{9} & -1 \\ \frac{17}{197} & -\frac{99}{197} \end{pmatrix}\begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} \\ &+ h\begin{pmatrix} -2 & -\frac{2}{3} \\ 0 & \frac{150}{197} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} + h\begin{pmatrix} 0 & 0 \\ -\frac{18}{197} & 0 \end{pmatrix}\begin{pmatrix} f_{n+3} \\ f_{n+4} \end{pmatrix}\end{aligned} \tag{24}$$

which is equivalent to

$$\begin{aligned}\begin{pmatrix} 1 & -\frac{17}{9} \\ -\frac{279}{197} & 1 \end{pmatrix}\begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} &= \begin{pmatrix} \frac{1}{9} & -1 \\ \frac{17}{197} & -\frac{99}{197} \end{pmatrix}\begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} \\ &+ h\begin{pmatrix} -2 & -\frac{2}{3} \\ 0 & \frac{150}{197} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} + h\begin{pmatrix} 0 & 0 \\ -\frac{18}{197} & 0 \end{pmatrix}\begin{pmatrix} f_{n+3} \\ f_{n+4} \end{pmatrix}\end{aligned} \tag{25}$$

We defined the k - block, r-point method (25) in general matrix form as

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_m + B_1 F_{m+1}) \tag{26}$$

where $r = 2$ and $n = 2m,\ m = 0,1,2,\ldots$

$$A_0 = \begin{pmatrix} 1 & -\dfrac{17}{9} \\ -\dfrac{279}{197} & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \dfrac{1}{9} & -1 \\ \dfrac{17}{197} & -\dfrac{99}{197} \end{pmatrix},$$

$$B_0 = \begin{pmatrix} -2 & -\dfrac{2}{3} \\ 0 & \dfrac{150}{197} \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 \\ -\dfrac{18}{197} & 0 \end{pmatrix},$$

$$Y_m = \begin{pmatrix} y_{n+1} \\ y_{n+2} \end{pmatrix} = \begin{pmatrix} y_{2m+1} \\ y_{2m+2} \end{pmatrix}, \qquad Y_{m-1} = \begin{pmatrix} y_{n-1} \\ y_n \end{pmatrix} = \begin{pmatrix} y_{2m-1} \\ y_{2m} \end{pmatrix} = \begin{pmatrix} y_{2(m-1)+1} \\ y_{2(m-1)+2} \end{pmatrix},,$$

$$F_m = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} = \begin{pmatrix} f_{2m+1} \\ f_{2m+2} \end{pmatrix}, \qquad F_{m+1} = \begin{pmatrix} f_{n+3} \\ f_{n+4} \end{pmatrix} = \begin{pmatrix} f_{2m+3} \\ f_{2m+4} \end{pmatrix} = \begin{pmatrix} f_{2(m+1)+1} \\ f_{2(m+1)+2} \end{pmatrix}.$$

Substituting the scalar complex test equation (23) into (26) and letting $\lambda h = \bar{h}$ gives

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_m + B_1 Y_{m+1}) \tag{27}$$

where $A_0, A_1, B_0, B_1, Y_m, Y_{m-1}$ are as previously defined and

$$Y_{m+1} = \begin{pmatrix} y_{n+3} \\ y_{n+4} \end{pmatrix} = \begin{pmatrix} y_{2(m+1)+1} \\ y_{2(m+1)+2} \end{pmatrix}$$

Let denote the determinant. Then solving $\det\left|t(A_0 - B_0 - B_1) - A_1\right| = 0$ gives the following stability polynomial

$$R(t,\bar{h}) = \frac{6}{197} + \frac{324}{197}t + \frac{224}{197}\bar{h}\,t - \frac{330}{197}t^2 + \frac{448}{197}\bar{h}\,t^2 - \frac{264}{197}\bar{h}^2\,t^2 = 0 \tag{28}$$

For zero stability, we set $\bar{h} = 0$ in (28) and solve for t. Hence we have

$$\frac{6}{197} + \frac{324}{197}t - \frac{330}{197}t^2 = 0 \tag{29}$$

Solving (29) for t we obtain

$$t = -\frac{1}{55} \text{ and } t = 1.$$

which shows that the method is zero stable.

We plot the region of absolute stability of the 2–point BEBDF in Figure 2. The region exterior to the circle is the stability region which shows that the method is A–stable and suitable for solving stiff initial value problems.

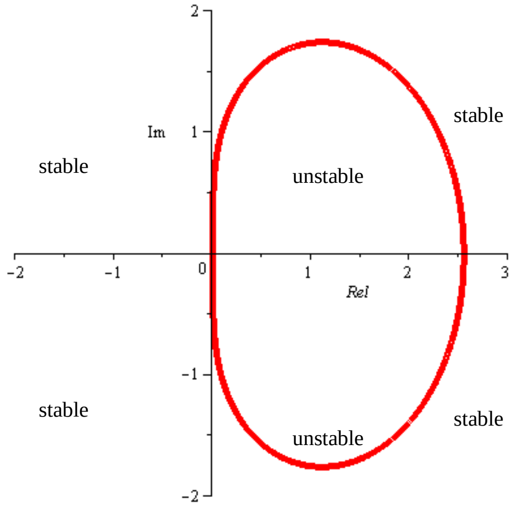

Figure 2: Stability Region of the 2 - point BEBDF

5. IMPLEMENTATION OF THE METHOD

The method is implemented in a Newton's like iteration.
We denote $y_{n+j}^{(i+1)}$ as the $(i+1)^{th}$ iteration and

$$e_{n+j}^{(i+1)} = y_{n+j}^{(i+j)} - y_{n+j}^{(i+j)}, \qquad j = 1,2 \tag{30}$$

Define

$$\left.\begin{aligned} F_1 &= y_{n+1} - \frac{17}{9} y_{n+2} + 2hf_{n+1} + \frac{2}{3} hf_{n+2} - \frac{1}{9} y_{n-1} + y_n \\ F_2 &= y_{n+2} - \frac{279}{197} y_{n+1} - \frac{150}{197} hf_{n+2} + \frac{18}{197} hf_{n+3} - \frac{17}{197} y_{n-1} + \frac{99}{197} y_n \end{aligned}\right\} \tag{31}$$

The Newton's iteration for the 2–point formula therefore takes the form

$$y_{n+j}^{(i+1)} - y_{n+j}^{(i)} = -\left[F_j^{'}(y_{n+j}^{(i)})\right]^{-1}\left[F_j(y_{n+j}^{(i)})\right], \qquad j = 1, 2. \tag{32}$$

which can be written in the form

$$F_j^{'}(y_{n+j}^{(i)})e_{n+j}^{(i+1)} = -F_j(y_{n+j}^{(i)}) \tag{33}$$

and in matrix form, (33) is equivalent to

$$\begin{pmatrix} 1+2h\dfrac{df_{n+1}}{dy_{n+1}} & -\dfrac{17}{9}+\dfrac{2}{3}h\dfrac{df_{n+2}}{dy_{n+2}} \\ -\dfrac{279}{197} & 1-\dfrac{150}{197}h\dfrac{df_{n+2}}{dy_{n+2}} \end{pmatrix}\begin{pmatrix} e_{n+1}^{i+1} \\ e_{n+2}^{i+1} \end{pmatrix} =$$

$$\begin{pmatrix} -1 & \dfrac{17}{9} \\ \dfrac{279}{197} & -1 \end{pmatrix}\begin{pmatrix} y_{n+1}^{i} \\ y_{n+2}^{i} \end{pmatrix} + h\begin{pmatrix} -2 & -\dfrac{2}{3} & 0 \\ 0 & \dfrac{150}{197} & -\dfrac{18}{197} \end{pmatrix}\begin{pmatrix} f_{n+1}^{i} \\ f_{n+2}^{i} \\ f_{n+3}^{i} \end{pmatrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \tag{34}$$

where ξ_1 and ξ_2 are the back values.

6. PROBLEMS TESTED

The following problems are used to test the performance of the method. They consist of both linear and non-linear problems.

(1) $\quad y' = \dfrac{y(1-y)}{2y-1} \qquad y(0) = \dfrac{5}{6} \qquad 0 \le x \le 1$

Exact solution

$$y(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{36}e^{-x}}$$

Source: Alvarez and Rojo (2002)

(2) $y' = \frac{50}{y} - 50y \quad y(0) = \sqrt{2} \qquad 0 \le x \le 1$

Exact solution:

$$y(x) = (1 + e^{-100x})^{\frac{1}{2}}$$

Source: Burden and Faires (2001)

(3) $y' = -100(y - 1) \qquad y(0) = 2 \qquad 0 \le x \le 20$

Exact solution:

$$y(x) = e^{-100x} + 1$$

Source: Artificial problem

(4) $\begin{array}{lll} y_1' = y_2 & y_1(0) = 1 & \\ y_2' = -y_1 - \frac{26}{5} y_2 & y_2(0) = 1 & 0 \le x \le 2 \end{array}$

Exact Solution:

$$y_1(x) = -\frac{1}{4} e^{-5x} + \frac{5}{4} e^{-\frac{x}{5}}$$

$$y_2(x) = \frac{5}{4} e^{-5x} - \frac{1}{4} e^{-\frac{x}{5}}$$

Source: Dormand (1996).

This is linear system $y'' + cy' + ky = 0$ reduced to a system of linear equations, modelling the damped simple harmonic motion, described as a vibrating spring whose motion is restricted by a force proportional to the velocity. y is the displacement of a unit mass attached to the spring, $c = \frac{26}{5}$ is the damping constant and $k = 1$ is the stiffness constant for the spring. The past transient term e^{-5x} decays more rapidly except for small values of x. On the other hand, the slower component on its own provides a fairly good approximation for the complete solution when $x > 1$.

(5) $\begin{array}{lll} y_1' = y_2 & y_1(0) = 1 & \\ y_2' = -200y_1 - 20y_2 & y_2(0) = -10 & 0 \le x \le 10 \end{array}$

Exact solution

$$y_1(x) = e^{-10x}\cos 10x$$
$$y_2(x) = -10e^{-10x}(\cos 10x + \sin 10x)$$

Source: Lambert (1973).

(6) $$y_1' = -20y_1 - 19y_2 \qquad y_1(0) = 2 \qquad 0 \le x \le 20$$
$$y_2' = -19y_1 - 20y_2 \qquad y_2(0) = 0$$

Exact solution

$$y_1(x) = e^{-39x} + e^{-x}$$
$$y_2(x) = e^{-39x} - e^{-x}$$

Source: Cheney (2012).

7. NUMERICAL RESULTS

Numerical results that compare the performance of the method with the 2–point BBDF with a given fixed step length are given in Tables 1–6 below. The maximum error and the time taken to complete the integration are also given. The results show that the method derived has better accuracy than BBDF.

The following notations are used in the tables:

h	Step size
MAXE	Maximum Error
TIME	Time in seconds
2BBDF	2 – point BBDF
2BEBDF	2–point BEBDF

To give a more visual impact, the graph of $Log_{10}(MAXE)$ over Time for the problems solved were plotted. Given below are the graphs of the scaled maximum error problem by problem. The hidden edge (dash) line indicates BEBDF method while the visible edge (thick) line indicates BBDF method.

TABLE 1: Numerical results for problem 1

h	Method	MAXE	Time
10^{-2}	2BBDF	1.47086e-03	2.46000e-04
	2BEBDF	6.64937e-04	5.91040e-04
10^{-3}	2BBDF	1.52651e-04	1.50300e-03
	2BEBDF	7.05780e-05	2.95367e-03
10^{-4}	2BBDF	1.53220e-05	1.39009e-02
	2BEBDF	7.10123e-06	3.14080e-02
10^{-5}	2BBDF	1.53277e-06	1.39695e-01
	2BEBDF	7.10560e-07	2.79435e-01
10^{-6}	2BBDF	1.53305e-07	1.37724e+00
	2BEBDF	7.10611e-08	2.81880e+00

TABLE 2: Numerical results for problem 2

h	Method	MAXE	Time
10^{-2}	2BBDF	1.44729e-01	2.41416e-04
	2BEBDF	9.24961e-03	4.40750e-04
10^{-3}	2BBDF	2.15168e-02	1.59720e-03
	2BEBDF	7.96762e-03	2.94533e-03
10^{-4}	2BBDF	2.55682e-03	1.388375e-02
	2BEBDF	1.07245e-03	3.03819e-02
10^{-5}	2BBDF	2.59686e-04	1.48665e-01
	2BEBDF	1.10428e-04	2.82938e-01
10^{-6}	2BBDF	2.60086e-05	1.36387e+00
	2BEBDF	1.10751e-05	2.81364e+00

TABLE 3: Numerical results for problem 3

h	Method	MAXE	Time
10^{-2}	2BBDF	1.83156e-02	3.34792e-03
	2BEBDF	1.83156e-02	3.34792e-03
10^{-3}	2BBDF	5.67155e-02	3.22815e-02
	2BEBDF	5.97499e-02	3.42977e-02
10^{-4}	2BBDF	7.18323e-03	3.16316e-01
	2BEBDF	4.36785e-04	3.42515e-01
10^{-5}	2BBDF	7.34012e-04	3.19415e+00
	2BEBDF	3.23640e-05	3.40134e+00
10^{-6}	2BBDF	7.35584e-05	3.16101e+01
	2BEBDF	3.47615e-06	3.41661e+01

TABLE 4: Numerical results for problem 4

h	Method	MAXE	Time
10^{-2}	2BBDF	4.05485e-02	6.66750e-04
	2BEBDF	1.54095e-02	7.42000e-04
10^{-3}	2BBDF	4.54013e-03	5.97775e-03
	2BEBDF	4.07357e-04	6.69608e-03
10^{-4}	2BBDF	4.58919e-04	5.91634e-02
	2BEBDF	2.38486e-05	6.55754e-02
10^{-5}	2BBDF	4.59411e-05	5.91698e-01
	2BEBDF	2.20771e-06	7.31862e-01
10^{-6}	2BBDF	4.59459e-06	5.93144e+00
	2BEBDF	2.18989e-07	7.32775e+00

TABLE 5: Numerical results for problem 5

h	Method	MAXE	Time
10^{-2}	2BBDF	1.61785e-01	3.65725e-03
	2BEBDF	1.67366e-01	3.91558e-03
10^{-3}	2BBDF	1.45948e-01	3.53500e-02
	2BEBDF	1.82997e-02	3.78459e-02
10^{-4}	2BBDF	1.44490e-02	3.52383e-01
	2BEBDF	7.63068e-04	3.74330e-01
10^{-5}	2BBDF	1.44347e-03	3.51502e+00
	2BEBDF	6.93925e-05	3.76159e+00
10^{-6}	2BBDF	1.44332e-04	3.52527e+01
	2BEBDF	6.87941e-06	3.74746e+01

TABLE 6: Numerical results for problem 6

h	Method	MAXE	Time
10^{-2}	2BBDF	6.29433e-02	7.29025e-03
	2BEBDF	6.41545e-02	1.11905e-03
10^{-3}	2BBDF	2.61104e-02	7.11481e-02
	2BEBDF	8.33432e-03	7.66378e-02
10^{-4}	2BBDF	2.84789e-03	7.10616e-01
	2BEBDF	2.87015e-04	7.71190e-01
10^{-5}	2BBDF	2.87180e-04	7.11981e+00
	2BEBDF	2.19722e-05	7.65406e+00
10^{-6}	2BBDF	2.87420e-05	7.12867e+01
	2BEBDF	2.13643e-06	7.73116e+01

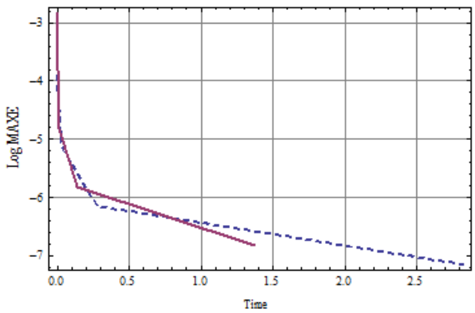

Figure 3: Graph of $Log_{10}(MAXE)$ vs TIME for problem 1

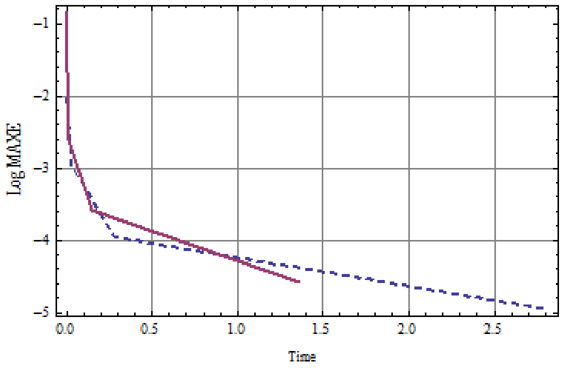

Figure 4: Graph of $Log_{10}(MAXE)$ vs TIME for problem 2

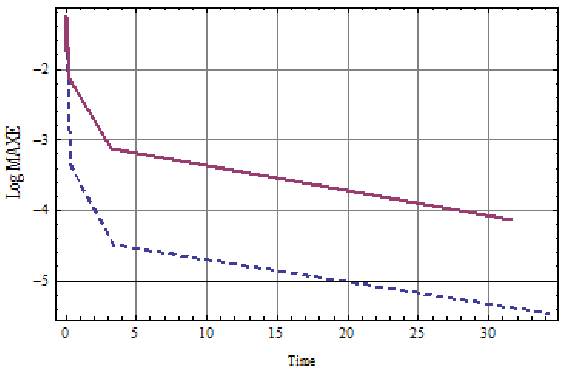

Figure 5: Graph of $Log_{10}(MAXE)$ vs TIME for problem 3

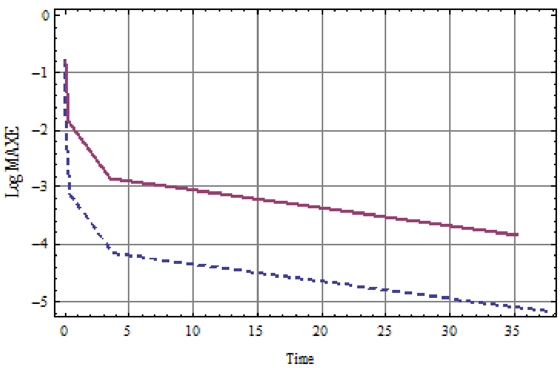

Figure 6: Graph of $Log_{10}(MAXE)$ vs TIME for problem 4

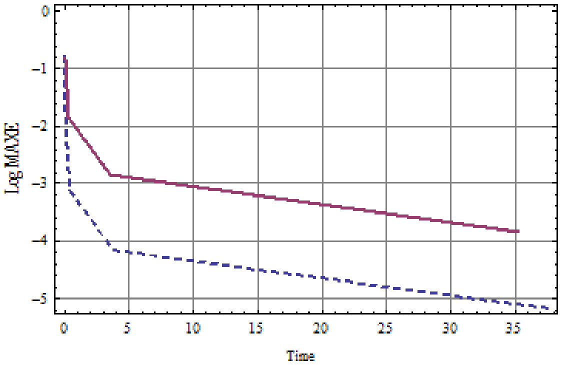

Figure 7: Graph of $Log_{10}(MAXE)$ vs TIME for problem 5

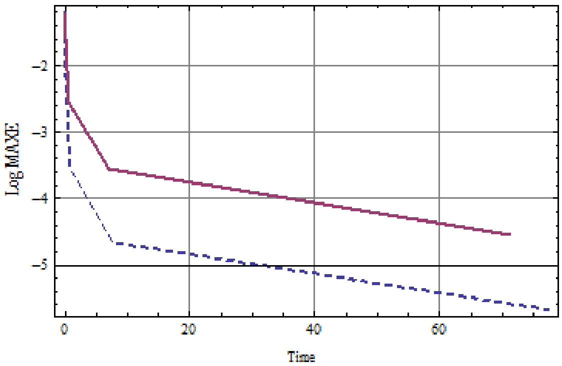

Figure 8: Graph of $Log_{10}(MAXE)$ vs TIME for problem 6

The results from the tables show that for almost all the problems solved, the method derived is better in terms of accuracy than the 2BBDF. However, in terms of computation time, the time taken to complete the integration using the 2BBDF method is better than that in our method; even though the differences are marginal. Also, both methods took the same number of integration steps to complete the integration.

From the given figures, for almost all the problems and for a particular step size h, the scaled errors of the method 2BEBDF are smaller compared to 2BBDF method. This shows the error growth is also smaller for the extended method, hence more stable and accurate compared to the 2BBDF.

8. CONCLUSION

This paper applied the technique of adding a 'super future point' to the 2–point BBDF to derive a new block method called a 2–point block extended backward differentiation formula. The method derived computes the solution of stiff IVPs at two points simultaneously. The order of the method is 4 and the plot of the stability region showed that it is A–stable. The formula is implemented using Newton's like iteration. A comparison of the results obtained for solving some stiff IVPs shows its advantage in accuracy over the BBDF. The computation time for the new method is seen to be competitive.

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