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An Iterative Solution for Second Order Linear Fredholm Integro-Differential Equations

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ABSTRACT

The objective of this paper is to analyze the application of the quarter-sweep iterative concept on Quadrature-Difference schemes namely central difference (CD)-composite trapezoidal (CT) with the Gauss-Seidel iterative method to solve second order linear Fredholm integro-differential equations. The formulation and implementation of the Full-, Half- and Quarter-Sweep Gauss-Seidel methods namely FSGS, HSGS and QSGS are presented for performance comparison. Furthermore, computational complexity and percentage reduction calculations are also presented with several numerical simulations. The numerical results show that the proposed QSGS method with the corresponding discretization schemes is superior compared to the FSGS and HSGS methods.

Keywords: Linear Fredholm, second order integro-differential equations, quarter-sweep iterations, Gauss-Seidel method, second order central difference scheme, trapezoidal scheme.

1. INTRODUCTION

Consider the linear second order Fredholm integro-differential equations

$$\sum_{i=0}^{2} a_i(x) y^{(i)}(x) = g(x) + P(x)y(x) + \int_0^1 K(x,t) y(t) dt, \quad x \in (0,1) \tag{1}$$

subject to the two-point boundary conditions

$$y(0) = y_0, \; y(1) = y_1,$$

where a_i, $i = 0, 1, 2$. $K(x,t) \in L^2(0,1) \times (0,1)$, $P(x) \in L^2(0,1)$ and $g(x) \in L^2(0,1)$ are given functions and $y(x)$ is the unknown function to be determined (Lakestani *et al.* (2006)). The conditions for existence and uniqueness of solution of such problems have been investigated by Agarwal (1983, 1986).

Solutions of linear Fredholm integro-differential equations (LFIDEs) have been studied by many authors. Many studies have been carried out with Quadrature schemes by Zhao and Corless (2006), Aruchunan and Sulaiman (2010, 2011a, 2011b, 2013a, 2013b). Besides that, methods such as wavelet-Galerkin (El-Sayed and Abdel-Aziz (2003)), Adomian's (Deeba *et al.* (2000)), Tau (Hosseini and Shahmorad (2003, 2005)) and Sinc collocation (Rishidinia and Zarebinia (2005)) are also analysed in solving LFIDEs. However, these methods are lead to dense linear systems and can be prohibitively expensive to solve n-th order linear systems. Moreover, these methods are based on the standard or full-sweep iterative methods which are more expensive in terms of computation time. Therefore, in this paper, a discretization scheme namely quarter-sweep central difference-composite trapezoidal (QSCD-QSCT) scheme is applied to discretize Eq. (1) to generate a system of linear equations.

The remaining of this paper is as follows. In Section 2, explanation of the full-, half- and quarter-sweep iteration concepts and the details of the formulation of QSCD-QSCT discretization schemes are elaborated with approximation equations. In Section 3, formulations of the FSGS, HSGS and QSGS iterative methods are shown with the development of a numerical algorithm. In Section 4, several numerical tests are conducted to validate the efficiency of the methods. Furthermore, analysis on computational complexity is given in Section 5 followed by conclusion in Section 6.

2. COMPLEXITY REDUCTION APPROCHES

Basically, the proposed HSGS method is inspired by the concept of half-sweep iteration which as introduced by Abdullah (1991) via the Explicit Decoupled Group (EDG) iterative method to solve two-dimensional Poisson equations. The applications of half-sweep iterative methods have been implemented by Sulaiman *et al*. (2004a), Muthuvalu and Sulaiman (2008) and Aruchunan and Sulaiman (2012a, 2012b). Othman and Abdullah (2000) extended the concept of half-sweep iteration by establishing the quarter-sweep iteration concept via the Modified Explicit Group (MEG) method to solve two-dimensional Poisson equations. Further studies to verify the effectiveness of the quarter-sweep iteration concept have also been carried out by Sulaiman *et al*. (2004b) and Akhir *et al*. (2012). The quarter-sweep iteration inherits the characteristic of the half-sweep iteration in which its implementation process will only consider nearly quarter of all interior nodes of the solution domain. Figure 1(a), 1(b) and 1(b) show full-, half- and quarter-sweep iteration concepts.

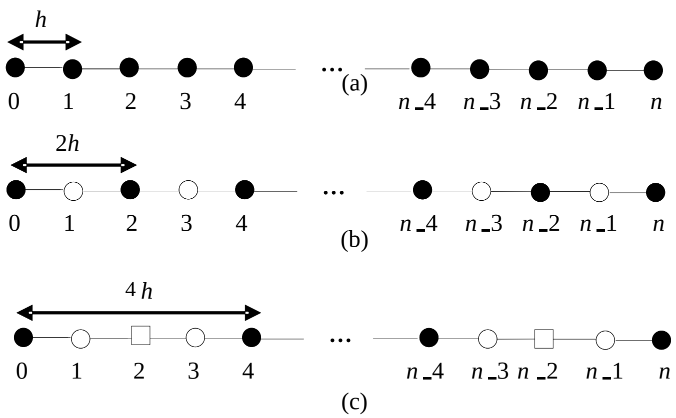

Figure 1: a), b) and c) show distribution of uniform node points for the full-, half- and quarter-sweep cases respectively.

Based on Fig. 1, the full-, half- and quarter-sweep iterative methods will compute approximate values only at the solid nodes ● until the convergence criterion is reached. Then, approximate solutions at the remaining points nodes (nodes of type ○ and □) can be calculated using the direct method as given in Sulaiman *et al*. (2009).

2.1 Formulation of Quarter-Sweep Quadrature-Difference Schemes

In this section, central difference (CD) and composite trapezoidal (CT) discretization schemes will be reformulated by applying the full-, half- and quarter-sweep iteration concept in order to discretize the differential and integral terms in Eq. (1) to form the approximation equations. The full-, half- and quarter-sweep CD and CT formula can be written as follows

$$y''(x_i) = \frac{y(x_{i+p}) - 2y(x_i) + y(x_{i-p})}{(ph)^2} + O\left((ph)^2\right) \tag{2}$$

for $i = 1, 2, \ldots, n-1$ and

$$\int_a^b y(t)dt = \sum_{j=p}^{n-p} A_j y(t_j) + \varepsilon_n(y) \tag{3}$$

where

$$A_j = \begin{cases} \frac{1}{2} ph, & j = 0, n \\ ph, & otherwise \end{cases}$$

in which t_j $(j = p, 2p, \ldots, n-p)$ are the abscissas of the partition points of the integration interval (a, b) or quadrature (interpolation) nodes; A_j $(j = 0, 1, 2, \ldots, n)$ are numerical coefficients that do not depend on the function $y(t)$; h is the constant step length between the node points as defined below

$$h = \frac{b-a}{n}$$

where a and b is the lower and upper limit of the integral term in Eq. (1) and n is the number of subinterval in (a,b); $O\left((ph)^2\right)$ and $\varepsilon_n(y)$ are the truncation errors of Eqs. (2) and (3) which are not considered in the calculations. Meanwhile, the value of p (1, 2 and 4) corresponds respectively to the full- half- and quarter-sweep iterative methods.

By substituting Eqs. (2) and (3) into Eq. (1), a system of linear algebraic equations are obtained for the approximation values of $y(x)$ at the nodes $x_1, x_2 \ldots, x_{n-1}$. Therefore, the full- half- and quarter-sweep iteration concepts, together with the CD and CT approximation schemes, yield

$$\frac{y_{i+p} - 2y_i + y_{i-p}}{(ph)^2} = g_i + P_i + \sum_{j=p,2p,3p}^{n-p} A_j K(x,t) y_j \tag{4}$$

for $i = p, 2p, \ldots n-p$,

where, $p = 1, 2$ and 4 are respectively for the full-, half- and quarter-sweep approach.

The linear system generated either by the full-, half- and quarter-sweep approximation equation can be expressed by

$$E \underset{\sim}{y} = \underset{\sim}{f} \tag{5}$$

where

$$E = \begin{bmatrix} \sigma_{p,p} & \varsigma_{p,2p} & \tau_{p,3p} & \cdots & \tau_{p,n-3p} & \tau_{p,n-2p} & \tau_{p,N-p} \\ \varsigma_{2p,p} & \sigma_{2p,2p} & \varsigma_{2p,3p} & \cdots & \tau_{2p,n-3p} & \tau_{2p,n-2p} & \tau_{2p,N-p} \\ \tau_{3p,p} & \varsigma_{3p,2p} & \sigma_{3p,3p} & \cdots & \tau_{3p,n-3p} & \tau_{3p,n-2p} & \tau_{3p,N-p} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \tau_{n-3p,p} & \tau_{n-3p,2p} & \tau_{n-3p,3p} & \cdots & \sigma_{n-3p,n-3p} & \varsigma_{n-3p,n-2p} & \tau_{n-2p,n-p} \\ \tau_{n-2p,p} & \tau_{n-2p,2p} & \tau_{n-2p,3p} & \cdots & \varsigma_{n-2p,n-3p} & \sigma_{n-2p,n-2p} & \varsigma_{n-p,n} \\ \tau_{n-p,p} & \tau_{n-p,2p} & \tau_{n-p,3p} & \cdots & \tau_{n-p,n-3p} & \varsigma_{n-p,n-2p} & \sigma_{n-p,n-p} \end{bmatrix}_{\left(\frac{n}{p}-1\right)\times\left(\frac{n}{p}-1\right)}$$

in which

$$\sigma_{i,i} = -2 - h^2 P_i - h^2 A_i K_{i,i}\,,\ \varsigma_{i,j} = 1 - h^2 A_j K_{i,j} \text{ and } \tau_{i,j} = -h^2 A_j K_{i,j}$$

$$\underset{\sim}{f} = \begin{bmatrix} h^2 g_p + \left(1 + h^2 A_p K_{p,0}\right) y_0 + \left(h^2 A_n K_{p,n}\right) y_n \\ h^2 g_{2p} + \left(h^2 A_p K_{2p,0}\right) y_0 + \left(h^2 A_n K_{2p,n}\right) y_n \\ h^2 g_{3p} + \left(h^2 A_p K_{3p,0}\right) y_0 + \left(h^2 A_n K_{3p,n}\right) y_n \\ \vdots \\ h^2 g_{n-3p} + \left(h^2 A_p K_{n-3,0}\right) y_0 + \left(h^2 A_n K_{n-3p,n}\right) y_n \\ h^2 g_{n-2p} + \left(h^2 A_p K_{n-2,0}\right) y_0 + \left(h^2 A_n K_{n-2p,0}\right) y_n \\ h^2 g_{n-p} + \left(h^2 A_p K_{n-p,0}\right) y_0 + \left(-1 + h^2 A_n K_{n-p,0}\right) y_n \end{bmatrix} \text{ and}$$

$$\underset{\sim}{y} = \begin{bmatrix} y(x_p) \\ y(x_{2p}) \\ y(x_{3p}) \\ \vdots \\ y(x_{n-3p}) \\ y(x_{n-2p}) \\ y(x_{n-1}) \end{bmatrix}.$$

obviously, E is a dense coefficient matrix. From Equation (5), it is noticeable that applications of the half- and quarter-sweep iteration concepts reduce the coefficient matrix, E from order $(n-1)$ to $\left(\frac{n}{2}-1\right)$ and $\left(\frac{n}{4}-1\right)$ respectively.

3. FORMULATION OF FAMILY OF GAUSS-SEIDEL ITERATIVE METHODS

The standard GS iterative method is also called the Full-Sweep Gauss-Seidel (FSGS) method. Combinations of the GS method with half- and quarter-sweep iterations are known as Half-Sweep Gauss-Seidel (HSGS) and Quarter-Sweep Gauss-Seidel (QSGS) methods respectively (Aruchunan and Sulaiman (2011b)). As mentioned above, the generated linear systems of Eq. (1) as simplified in Eq. (5) will be solved by using the FSGS, HSGS and QSGS iterative methods. Let the coefficient matrix, E , be decomposed into

$$E = D - L - U \tag{6}$$

where D, $-L$ and $-U$ are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Therefore, the general scheme for the FSGS, HSGS and QSGS iterative methods can be written as

$$\underset{\sim}{y}^{(k+1)} = (D - L)^{-1}\left(U \underset{\sim}{y}^{(k)} + \underset{\sim}{f}\right). \tag{7}$$

As a matter of fact, the iterative methods attempt to find a solution to the system of linear equations by repeatedly were solving the linear system using approximations to the vector $\underset{\sim}{y}$ for solving Eq. (1). Iterations for FSGS, HSGS and QSGS methods continue until the solution is within a predetermined acceptable loop on the error. By determining the values of matrices D, $-L$ and $-U$ as stated in Equation (6), the general algorithm solving Eq. (1) using the FSGS, HSGS and QSGS iterative methods and the Gauss-Seidel method, is as follows

Full-, Half- and Quarter-sweep Gauss-Seidel Algorithm

Step 1 : Initialize all the parameters. Set et $k = 0$.

Step 2 : for $i = p, 2p, \ldots, n-2p, n-p$, Compute

$$\underset{\sim}{y}^{(k+1)} = \frac{1}{E_{i,i}}\left(f_i - \sum_{j=p,2p,3p}^{i-p} E_{i,j}\, \underset{\sim}{y}^{(k+1)} - \sum_{j=i+p,i+2p,i+3p}^{n-p} E_{i,j}\, \underset{\sim}{y}^{(k)}\right)$$

Step 3 : Check the convergence

If the error, $\left\| \underset{\sim}{y}_i^{(k+1)} - \underset{\sim}{y}_i^{(k)} \right\| \le \varepsilon = 10^{-10}$, is satisfied, iteration is terminated and go to Step 4; otherwise, repeat the iteration sequence (i.e., go to Step 2)

Step 4 : Stop.

4. NUMERICAL EXPERIMENT

In this section, two well-posed problems are carried out to validate the effectiveness of the proposed method. Three parameters such as number of iterations, execution time and maximum absolute error are considered as measurements to evaluate the performance of the methods. The FSGS method was used as the control of comparison of numerical results. Throughout the numerical simulations, the convergence test was carried out with tolerance error of $\varepsilon = 10^{-10}$ with several mesh sizes such as 60, 120, 240, 480 and 960. The results of numerical simulations, which were obtained from implementations of the FSGS, HSGS and QSGS iterative methods for problems 1 and 2 are shown recorded in Tables 1 and 2 respectively. The percentage reduction of number of iterations and execution time for the HSGS and QSGS methods relative to the FSGS method is summarized in Table 3.

Problem 1 (Delves and Mohammed (1985))

Consider the second order linear FIDE

$$y''(x) = x - 2 + \int_0^1 60(x-t)y(t)\,dt\,, \quad 0 < x < 1 \tag{8}$$

with two point boundary conditions,

$$y(0) = 0 \text{ and } y(1) = 0\,.$$

The exact solution is

$$y(x) = x\,.$$

Problems 2 (Amaal and Sudad (2010))

Consider the second order linear FIDE

$$y''(x) = e^x - (e-1)x - 1 + \int_0^1 (x+t)y(t)\,dt\,, \quad 0 < x < 1 \tag{9}$$

with two point boundary conditions,

$$y(0) = 1 \text{ and } y(1) = e$$

The exact solution is

$$y(x) = e^x\,.$$

TABLE 1: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 1)

	Number of iterations				
Methods	Mesh Sizes				
	60	120	240	480	960
FSGS	3251	12278	45129	162727	576449
HSGS	813	3251	12278	45129	162727
QSGS	198	813	3251	12278	45129
	Execution time (seconds)				
Methods	Mesh Sizes				
	60	120	240	480	960
FSGS	0.80	4.53	45.50	543.06	7929.97
HSGS	0.37	0.82	4.59	46.82	566.91
QSGS	0.17	0.38	0.81	4.55	45.68
	Maximum absolute error				
Methods	Mesh Sizes				
	60	120	240	480	960
FSGS	7.449E-5	1.854E-5	4.332E-6	4.910E-6	5.248E-6
HSGS	5.158E-4	1.338E-4	3.408E-5	8.604E-6	2.168E-6
QSGS	1.915E-3	5.158E-4	1.338E-4	3.408E-5	8.604E-6

TABLE 2: Comparison of a number of iterations, execution time (seconds) and maximum absolute error for the iterative methods (Example 2)

	Number of iterations				
Methods	Mesh Sizes				
	60	120	240	480	960
FSGS	6064	22378	82010	298074	1072531
HSGS	1633	6064	22378	82010	298074
QSGS	437	1633	6064	22378	82010
	Execution time (seconds)				
Methods	Mesh Sizes				
	60	120	240	480	960
FSGS	1.36	20.77	139.62	2033.81	25780.26
HSGS	0.56	2.18	15.05	144.70	2096.01
QSGS	0.34	0.62	2.22	15.73	146.66
	Maximum absolute error				
Methods	Mesh Sizes				
	60	120	240	480	960
FSGS	9.684E-6	2.547E-6	1.142E-6	2.308E-6	8.668E-6
HSGS	3.688E-4	9.329E-5	2.345E-5	5.878E-6	2.308E-6
QSGS	2.841E-3	1.200E-3	5.462E-4	2.597E-5	1.260E-5

TABLE 3: Reduction percentage of the number of iterations and execution time for the HSGS and QSGS methods compared with FSGS method

Methods	Example 1	
	Number of iterations	Execution time
HSGS	71.77-74.99%	53.75-92.85%
QSGS	92.17-93.91%	78.75-99.42%
Methods	Example 2	
	Number of iterations	Execution time
HSGS	72.21-73.07%	58.82-92.89%
QSGS	92.35-92.79%	75.00-99.43%

5. COMPUTATIONAL COMPLEXITY ANALYSIS

The computational complexity of the FSGS, HSGS and QSGS iterative methods is measured based on the estimation amount of the computational work of arithmetic operations performed per iteration. Based on the full-, half- and quarter-sweep Gauss-Seidel Algorithm, it can be observed that there are $\frac{n}{p}-1$ additions/subtractions (ADD/SUB) and $\frac{n}{p}+1$ multiplications/divisions (MUL/DIV) in computing a value for each node point in the solution domain. From the order of the coefficient matrix, E in Equation (5), the total number of arithmetic operations per iteration for the FSGS, HSGS and QSGS iterative methods has been summarized in Table 4.

TABLE 4: Total number of arithmetic operations per iteration for FSGS, HSGS and QSGS methods

Methods	Arithmetic Operation	
	ADD/SUB	MUL/DIV
FSGS	$(n-1)^2$	n^2-1
HSGS	$\left(\frac{n}{2}-1\right)^2$	$\frac{n^2}{4}-1$
QSGS	$\left(\frac{n}{4}-1\right)^2$	$\frac{n^2}{16}-1$

6. CONCLUSION

In this paper, application of the quarter-sweep iteration concept on numerical schemes namely CD and CT with GS iterative method for solving dense nonsymmetric matrix equations arising from the second order integro-differential equations is examined. Through numerical solutions obtained in Tables 1 and 2, it evidently shows that applications of the half- and quarter-sweep iteration concept reduce the number of iterations and computational time significantly. Based on Table 3, the percentage reduction in number of iterations for half- and quarter-sweep concept are approximately 72% and 92% respectively, while the computational time reduces approximately 54% and 75% respectively compared to FSGS. Overall, the numerical results show that the QSGS method is a better method compared to the FSGS and HSGS methods in terms of number of the iterations and execution time. This is mainly due to the reduction in terms of computational complexity; since the QSGS method will only consider approximately quarter of all interior node points in solution domain during the iteration process (refer Table 4).

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