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On Collectionwise Hausdorff Bitopological Spaces

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ABSTRACT

In this work, we introduce the concept of collectionwise Hausdorff bitopological spaces by using p_1–open sets. Further, we also study the relations of collectionwise Hausdorff spaces with some separation axioms and paralindelöff bitopological spaces.

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1. INTRODUCTION AND PRELIMINARIES

Let X be a topological space, a subset A of X is said to be a discrete set if the subspace A is discrete space, i.e., A is discrete set in X if for each point $x \in X$, there is an open set U in X containing x , s.t. $U \cap A = \{x\}$. A family of discrete points D (equivalently, a closed discrete set) of a topological space X is separated if there is a disjoint family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = \{d\}$ for all $d \in D$, see Balasubramaniam (1982).

A closed discrete set is widely used to study the structural of some topological properties as Hausdorff, regular and normal spaces with finite sets. In addition, it is used to introduce the idea of collectionwise Hausdorff. The topological space is said to be collectionwise Hausdorff if given any closed discrete collection of points in the topological space, there is a collection of pairwise disjoint open sets containing the points. Then it follows that every T_1 space which is collectionwise Hausdorff becomes a Hausdorff.

The notion of collectionwise Hausdorff (CwH) spaces has played an increasingly important role in topology since the introduction of this concept by Bing (1951). After that several authors worked on the collectionwise Hausdorff space and its applications. For instance, Tall (1969, 1976) and Navy (1981) have studied the relations between collectionwise Hausdorff and various covering axioms as paralindelöfness and paracompactness.

In literature there are also several generalizations of the notion of Lindelöf spaces and were studied separately for different reasons and purposes. For instance, Balasubramaniam (1982) introduced and studied the notion of nearly Lindelöf spaces. Then in 1996, Cammaroto and Santoro (1966) studied and gave further new results about these spaces which are considered as one of the main generalizations of Lindelöf spaces. However we note that the generalization need not straight forward, see Steen and Seebach (1978).

In order to generalize the topological concepts further topological spaces, bitopological spaces were first initiated by Kelly (1963) and thereafter several researchers have been carrying to study similar topological concepts into bitopological settings, see Dvalishvili (2005), Fora and Hdeib (1983) and Konstadilaki-Savopoulou and Reilly (1981). Recently, various topological properties related to Lindelöf spaces were extended and studied into the bitopological spaces. For example, the pairwise almost Lindelöfness in bitopological spaces and subspaces as well as the subsets and some of their properties, see Fora and Hdeib (1983) and Kilicman and Salleh (2007), pairwise weakly regular-Lindelöf spaces in Kilicman and Salleh (2008), pairwise almost regular-Lindelöf spaces in Kilicman and Salleh (2009), mapping and continuity concept in bitopological spaces, see Kilicman and Salleh (2009) and product properties among pairwise Lindelöf spaces in Kilicman and Salleh (2011).

The collectionwise Hausdorff spaces were studied in Boone (1974) and introduced the relation with separation axioms, see Nyikos and Porter (2008). The purpose of this paper is to study the concept of collectionwise Hausdorff property in bitopological spaces. In this paper we consider two kinds of collectionwise Hausdorff bitopological spaces namely known as p_1-collectionwise Hausdorff and collectionwise Hausdorff. Throughout this paper, CwH will be the abbreviation of collectionwise Hausdorff space and (X,τ_1,τ_2) or simply X represents a bitopological space on which no separation axioms are assumed unless otherwise mentioned.

2. COLLECTIONWISE HAUSDORFF (CWH) BITOPOLOGICAL SPACES

In this section, we shall introduce the concept of p_1-CwH and strongly p_1-CwH spaces. Before that we need the following definition.

Definition 2.1 (Kilicman and Salleh (2009)). Let (X,τ_1,τ_2) be a bitopological space.

(i) A subset F of X is p_1-open set if $F \in \tau_1 \cup \tau_2$.
(ii) A subset F of X is p_1-closed set if it is the complement of p_1-open set, that is, a subset F of X is p_1-closed in X if $= F_1 \cap F_2$, where$F_i \in co\tau_i , i = 1,2$.

Definition 2.2 A bitopological space (X,τ_1,τ_2) is said to be CwH bitopological space if every closed discrete collection of points has expansion of disjoint collection of p_1-open sets in X,i.e., if $D = \{d_\alpha : \alpha \in \Delta\}$ is the closed discrete collection of points, there is a disjoint collection $\{U_\alpha : \alpha \in \Delta\}$ of p_1-open sets such that $d_\alpha \in U_\alpha$ for all $\alpha \in \Delta$.

We will denote CwH bitopological space in the above definition by p_1-CwH.

Theorem 2.3 Every p_1-closed subset of p_1-CwH is p_1-CwH.

Proof. Let (X,τ_1,τ_2) be p_1-CwH and let Y be a p_1-closed subset of X. If $\{x_\alpha : \alpha \in \Delta\}$ is a closed discrete collection of points in, then it is also in X. Since X is p_1-cwH, there exists a disjoint family of p_1-open subsets $V = \{V_\alpha : \alpha \in \Delta\} \cup \{X - Y\}$ of X such that each x_α is contained in an element in V. But Y and $X - Y$ are disjoint, hence the subcollection $\{U_\alpha = V_\alpha \cap Y : \alpha \in \Delta\}$ of p_1-open subsets belongs to Y. Then $U = \{U_\alpha : \alpha \in \Delta\}$ is a disjoint family of p_1-open subsets in Y such that $x_\alpha \in U_\alpha$ for all $\alpha \in \Delta$. Therefore Y is p_1-CwH.

Definition 2.4 A bitopological space (X,τ_1,τ_2) is said to be strongly p_1-CwH space if every closed discrete collection of points has expansion of discrete collection of p_1-open sets in X,i.e., if $D = \{d_\alpha : \alpha \in \Delta\}$ is the closed discrete collection of points, then there exists a discrete collection $\{U_\alpha : \alpha \in \Delta\}$ of p_1-open sets such that $d_\alpha \in U_\alpha$ for all $\alpha \in \Delta$.

Definition 2.5 (Kilicman and Salleh (2009)). A bitopological space (X,τ_1,τ_2) is said to be p_1-normal if for any two disjoint p_1-closed subsets A and B of X, there are two disjoint p_1-open sets U, V such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$

Theorem 2.6 Every p_1-normal, p_1-CwH space is strongly p_1-CwH.

Proof. Suppose X is p_1-normal space. Let A and B be two disjoint p_1-closed subsets of X, with A discrete. Since X is p_1-CwH, there exists an expansion of A by p_1-open subsets of X, $\{U_a : a \in A\}$. By normality of X, there is p_1-open set , V , such that $A \subset V$ and $p_1 - cl(V) = \cup \{U_a : a \in A\}$, then $\{V \cap U_a : a \in A\}$ is a discrete p_1-open expansion of A. Then X is strongly p_1-CwH.

3. p_1-CwH AND p_1-PARALINDELÖFF

We first give the necessary definitions.

Definition 3.1 (Konstadilaki-Savopoulou and Reilly (1981)). Let (X,τ_1,τ_2) be a bitopological space.

(i) A cover U of a space X is called weakly pairwise open (p_1-open) if it consists of either τ_1- open sets or τ_2- open sets or both (i.e. $U \subset \tau_1 \cup \tau_2$).

(ii) If U is a weaker pairwise open (p_1-open) cover of X , then the weakly pairwise open (p_1-open) cover V of X is a parallel refinement of U if every τ_1- open set in V is contained in some τ_1- open sets of U and every τ_2- open set in V is contained in some τ_2- open sets of U.

(iii) A refinement V of a weakly pairwise open (p_1-open) cover of X is said to be pairwise locally countable if every point $x \in X$ has τ_1- open (τ_2- open) neighbourhood of x which meets at most countably many of the τ_1- open (τ_2- open) sets of V, and these form a refinement of the family of τ_1- open (τ_2- open) sets of U.

Now, we shall define a weaker form of paralindelöff bitopological space, by abbreviation p_1-paralindelöff, as following.

Definition 3.2 A bitopological space (X, τ_1, τ_2) is said to be p_1-paralindelöff if every weakly pairwise open (p_1-open) cover U has weakly pairwise open (p_1-open) refinement V which is a pairwise locally countable.

Lemma 3.3 Let H and K be subsets of a bitopological space X. For any $h \in H$ and $k \in K$, let $h * k$ and $k * h$ elements in $H \cup K$ provided $h \neq k$. Let $S(x)$ be p_1-open neighbourhood of x for each $x \in H \cup K$. Suppose that for each $x \in H \cup K$, there is no element $\acute{x} \in H \cup K$ such that $x * \acute{x}$ and $x' \in p_1\text{-}cl(S(x))$. Suppose also that for each $x \in H \cup K$, there are only countable many points $\acute{x} \in H \cup K$ with $x * \acute{x}$ for which $S(x) \cap S(\acute{x}) \neq \emptyset$. Then, each $S(x)$ can be refined to p_1-open neighbourhood $R(x)$ of x. So that the collection $\{R(x) : x \in H \cup K\}$ satisfies the following: For each $R(x)$, there is no set $R(\acute{x})$ such that $x \sim \acute{x}$ and $R(x) \cap R(\acute{x}) \neq \emptyset$.

Proof. Let $\sim$ be the equivalence relation of $H \cup K$ generated by the rule $x \sim \acute{x}$ if $x * \acute{x}$ and $S(x) \cap S(\acute{x}) \neq \emptyset$. Let $E = (H \cup K)/\sim$ be the set of $\sim$-equivalence classes on $H \cup K$. By the assumption, every such equivalence class is countable. From E, let e be a class with the countable members as $x_{e,0}, x_{e,1}, x_{e,2}, \ldots$(finitly many or ω-many as needed). For each $x_{e,n}$, let $R(x_{e,n}) = S(x_{e,n}) - \cup \{p_1 - cl(S(x_{e,j}) : j < n \text{ and } x_{e,j} * x_{e,n}\}$.

Definition 3.4 A p_1-Hausdorff space is p_1-regular (or p_1–T_3) if each $x \in X$ and p_1-closed set A, such that $x \notin A$, there are two disjoint p_1-open sets U, V such that $x \in U, A \subset V, U \cap V = \emptyset$.

Theorem 3.5 Every p_1-paralindelöff, p_1–T_3 space is p_1-CwH.

Proof. Let $X_0 = \{x_\alpha : \alpha \in \Delta\}$ be a discrete collection of points of X. By the regularity of X, for each $\alpha \in \Delta$, let U_α be a p_1-open neighbourhood of x_α such that $p_1 - cl(U_\alpha) \cap X_0 = \{x_\alpha\}$. Then the family $U = \{U_\alpha; \alpha \in \Delta\} \cup \{X - X_0\}$ forms a weakly p_1-open cover of X. Since X is p_1-paralindelöff, U has a pairwise locally countable p_1-open refinement $V = \{V_\alpha : \alpha \in \Delta\} \cup \{X - X_0\}$. For each $\alpha \in \Delta$, let V_α be a p_1-open neighbourhood of x_α belong to V, i.e., $x_\alpha \in V_\alpha \subset V$ for all $\alpha \in \Delta$.

Due to $V_0 \subset V, V_0 = \{V_\alpha : \alpha \in \Delta\}$ is also a pairwise locally countable. So for each $\alpha \in \Delta$, let $\acute{V}_\alpha$ be a p_1-open subset of V_α which witnesses this at x_α, i.e., $x_\alpha \in \acute{V}_\alpha$ and $\acute{V}_\alpha$ meets at most countable many members of V_0.

Thus the collection of p_1-open sets $\acute{V}_0 = \{\acute{V}_\alpha : \alpha \in \Delta\}$ is star countable, i.e., $st(\acute{V}_0, V_0) = \cup \{\acute{V}_\alpha \cap V_0 : \acute{V}_\alpha \in \acute{V}_0\}$ is countable.

Now, by applying the Lemma 3.3, let $H = K = X_0$ and $S(x_\alpha) = \acute{V}_\alpha$ for each $\alpha \in \Delta$; let the sets $R(x_\alpha) \subset S(x_\alpha)$ satisfy the lemma's conclusion.

Let $W_\alpha = R(x_\alpha)$ for all $\alpha \in \Delta$, then $W_0 = \{W_\alpha : \alpha \in \Delta\}$ is a collection of disjoint p_1-open sets with $x_\alpha \in W_\alpha$ for all $\alpha \in \Delta$. Therefore, X is p_1-CwH.

4. THE IMPLICATION BETWEEN COLLECTIONWISE HAUSDORFF AND HAUSDORFF IN BITOPOLOGICAL SPACES

In this section, from the idea of definition of Lindelöf spaces, see Kilicman and Salleh (2009, 2011), we shall introduce the new concept of Hausdorff space in bitopological setting.

Definition 4.1 A bitopological space (X, τ_1, τ_2) is said to be i-Hausdorff space if the topological space (X, i) is Hausdorff space. X is said to be Hausdorff if it is i-Hausdorff for each $i = 1, 2$. Equivalently, (X, τ_1, τ_2) is Hausdorff if every two distinct points $x, y \in X$, there exist two i −open sets U, V such that $x \in U, y \in V, U \cap V = \emptyset$ for each $i = 1, 2$.

Now, we shall generalize the definition of Hausdorff bitopological space to a new definition of collectionwise Hausdorff bitopological space as following.

Definition 4.2 A bitopological space (X, τ_1, τ_2) is said to be i-collectionwise Hausdorff space if the topological space (X, i) is collectionwise Hausdorff space. X is said to be collectionwise Hausdorff if it is i-collectionwise Hausdorff for each $i = 1, 2$. Equivalently, (X, τ_1, τ_2) is collectionwise Hausdorff if every closed discrete collection of points has expansion of disjoint collection of i-open sets in X for each $i = 1,2$.

Every collectionwise Hausdorff space must be Hausdorff space, since every finite subset of a T_i space is discrete. Thus, in particular, every two point subsets are discrete, but the converse is not true in general as in the following example.

Example 4.3 Let X be a set of real numbers, τ_1 be the discrete topology, and τ_2 be the usual topology. Then it is clear that τ_1 is both Hausdorff and collectionwise Hausdorff space and τ_2 is Hausdorff but it is not collectionwise Hausdorff space.

Remark 4.4 Comparing the Definition 4.2 with the Definition 2.2, we can note that every collectionwise Hausdorff is p_1-CwH, but the converse is not true in general as in Example 4.3. (X, τ_1, τ_2) is p_1-CwH, but it is not collectionwise Hausdorff space because (X, τ_2) is not a collectionwise Hausdorff space.

Definition 4.5 A bitopological space (X, τ_1, τ_2) is said to be $i-P$-space if any countable intersection of arbitrary collection of i -open sets is i-open set. X is said to be P-space if it is $i-P$-space for each $i = 1,2$. Equivalently, (X, τ_1, τ_2) is P-space if any countable intersection of arbitrary collection of i–open sets is i–open set for each $i = 1,2$.

Theorem 4.6 If (X, τ_1, τ_2) is an i-Hausdorff space and $i-P$-space, then X is i-collectionwise Hausdorff space for $i, j = 1,2, i \neq j$.

Proof. When the collection of points is finite, it is easy to prove, so we omit it.

For an arbitrary case, let $\{x_\alpha : \alpha \in \mathcal{A}\}$ be an arbitrary discrete collection of points. For $\alpha, \beta \in \mathcal{A}, \alpha \neq \beta$, let $V_{\alpha\beta}$ and $V_{\beta\alpha}$ be disjoint i-open sets in X such that $x_\alpha \in V_{\alpha\beta}$, $x_\beta \in V_{\beta\alpha}$.

Now, consider $U_\alpha = \cap_{\beta\in\mathcal{A}} V_{\alpha\beta}$, $U_\beta = \cap_{\alpha\in\mathcal{A}} V_{\beta\alpha}$, $\alpha \neq \beta$. Since X is $i-P$-space, U_α is i–open set, $x_\alpha \in U_\alpha$ since $x_\alpha \in V_{\alpha\beta}$ for all $\alpha \neq \beta$.

Now, we want to show that all $U_\alpha, \alpha \in \mathcal{A}$, is disjoint i–open set. Since for all α and β then $U_\alpha \subset V_{\alpha\beta}$, $U_\beta \subset V_{\beta\alpha}$ with $V_{\alpha\beta} \cap V_{\beta\alpha} = \emptyset$.

So, $U_\alpha \cap V_{\alpha\beta} = \emptyset, \alpha, \beta \in \mathcal{A}$. Therefore X is an i-collectionwise Hausdorff space.

In example 4.3, the usual topology is not a collectionwise Hausdorff space because it does not satisfy the condition of P –space.

Corollary 4.7 If (X, τ_1, τ_2) is Hausdorff space and P –space, then X is collectionwise Hausdorff space.

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