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The Gompertz Flexible Weibull Distribution and its Applications

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ABSTRACT

This paper introduces the Gompertz flexible Weibull distribution as an extension of the flexible Weibull distribution. Its various statistical properties are obtained and established while the method of maximum likelihood estimation is used in estimating the unknown model parameters. The application of Gompertz flexible Weibull distribution is illustrated by making use of three real life data sets, this is done to demonstrate its potentials over some other important distributions like the Gompertz Weibull, Gompertz Burr type XII, Gompertz Lomax, exponentiated flexible Weibull, exponentiated flexible Weibull extension and Kumaraswamy flexible Weibull distributions. Simulation studies were also conducted and the behavior of the Gompertz flexible Weibull parameters were investigated.

Keywords: Family of distributions,flexible Weibull, generalization, Gompertz distribution, mathematical statistics, maximum likelihood estimation, statistical properties.

1. Introduction

Weibull distribution is one of the popular standard distributions in statistics, engineering and medicine. It has the Rayleigh and exponential distributions as sub-models. It has a monotonic failure rate and it is suitable for modeling real life phenomena with monotonic failure rates (Ahmad and Iqbal, 2017). Further details about some statistical properties and real life applications of the Weibull distribution are available in Oguntunde et al. (2014).

Despite the attractive properties of the Weibull distribution, it has the disadvantage of not being able to handle data sets with non-monotonic failure rate. To this end, various modifications and extensions of the Weibull distribution have been proposed and introduced in recent years. For instance, the Kumaraswamy Weibull distribution (Cordeiro et al., 2010), beta Weibull distribution (Famoye et al., 2005), transmuted Weibull distribution (Aryal and Tsokos, 2011), flexible Weibull distribution (Bebbington et al., 2007) and several others are notable examples and generalizations aimed at obtaining non-monotonic failure rates.

Out of all the various modifications of the Weibull distribution, of interest in this research is the flexible Weibull distribution because it has many applications in applied statistics, life testing experiments, clinical studies and reliability analysis (Bebbington et al., 2007, El-Desouky et al., 2017b). Besides, it has increasing, decreasing or bathtub shaped failure rate.

Interests in some recent researches have been shifted to generalizing the flexible Weibull distribution. All these attempts were aimed at increasing the modeling capability of the flexible Weibull distribution. For instance, the beta flexible Weibull distribution (El-Desouky et al., 2017a), exponential flexible Weibull extension distribution (El-Desouky et al., 2017b), exponentiated flexible Weibull extension distribution (El-Gohary et al., 2015), exponentiated generalized flexible Weibull extension distribution (Mustafa et al., 2016), generalized flexible Weibull extension distribution (Ahmad and Iqbal, 2017) and transmuted flexible Weibull extension distribution (Ahmad and Hussain, 2017) are remarkable ones in the literature.

Also, it is of interest in this research to extend the flexible Weibull distribution with the aim of developing a more flexible and versatile compound distribution but with a relatively new generalized family of distribution; the Gompertz family of distribution. It was developed recently by Alizadeh et al. (2017) and it has also been used by Oguntunde et al. (2017b) to illustrate the superiority of the Gompertz Lomax distribution over the Weibull Lomax, beta Lomax and Kumaraswamy Lomax distributions; this however is a good selling point for the Gompertz family of distribution. Besides, its ability to develop a compound distribution that can perform better than its counterpart compound distributions developed from beta, Weibull and Kumaraswamy families of distributions is a great motivation for this research. The various properties of the Gompertz flexible Weibull distribution will be derived and three different real life data sets will be used to demonstrate the potentials of the distribution.

The major motivation are the study of modelling and analyses of life time data are important and crucial. However, the quality of statistical analyses depends heavily on the fitness of the assumed life time distribution. The Gompertz family can be applied more effectively on censored incomplete data because it is more tractable than some families. Therefore, the new model can analyze continuous univariate and multivariate sets. Moreover, to generate distributions with reversed-J, right-skewed, symmetric and left-skewed shaped.

The remaining part of this paper is written in the following manner; the densities of the four-parameter Gompertz flexible Weibull distribution are obtained in Section 2 including its various statistical properties and estimation of model parameters while real life applications are provided in Section 3 and simulation studies are provided in Section 4. The conclusion are provided in Sections 5.

2. The Gompertz Flexible Weibull (GoFW) Distribution

Let T denote a random variable, the densities of the flexible Weibull distribution are:

$$G(t) = 1 - \exp\left(-e^{\alpha t - \frac{\eta}{t}}\right) \qquad ; \quad t > 0, \alpha > 0, \eta > 0 \tag{1}$$

and

$$g(t) = \left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \exp\left(-e^{\alpha t - \frac{\eta}{t}}\right) \qquad ; \quad t > 0, \alpha > 0, \eta > 0 \tag{2}$$

respectively, where $G(t)$ is the cumulative distribution function (cdf) and $g(t)$ is the probability density function (pdf).

Also, the cdf and pdf of the Gompertz generalized family of distribution are:

$$F(t) = 1 - e^{\left(\frac{\theta}{\gamma}\right)\left\{1-[1-G(t)]^{-\gamma}\right\}} \tag{3}$$

and

$$f(t) = \theta g(t)\left[1 - G(t)\right]^{-\gamma-1} e^{\left(\frac{\theta}{\gamma}\right)\left\{1-[1-G(t)]^{-\gamma}\right\}} \tag{4}$$

respectively, where θ and γ are additional shape parameters.

Therefore, the cdf of the Gompertz flexible Weibull (GoFW) distribution is obtained by substituting the expression in (1) into (3) as follows:

$$F(t) = 1 - e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \tag{5}$$

Its corresponding pdf is obtained by substituting (1) and (2) into (4) to give:

$$f(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t-\frac{\eta}{t}} \left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \tag{6}$$

for α, η, θ and $\gamma > 0$.

Possible plots for the pdf of the GoFW distribution at varied parameters are as shown in Figure 1:

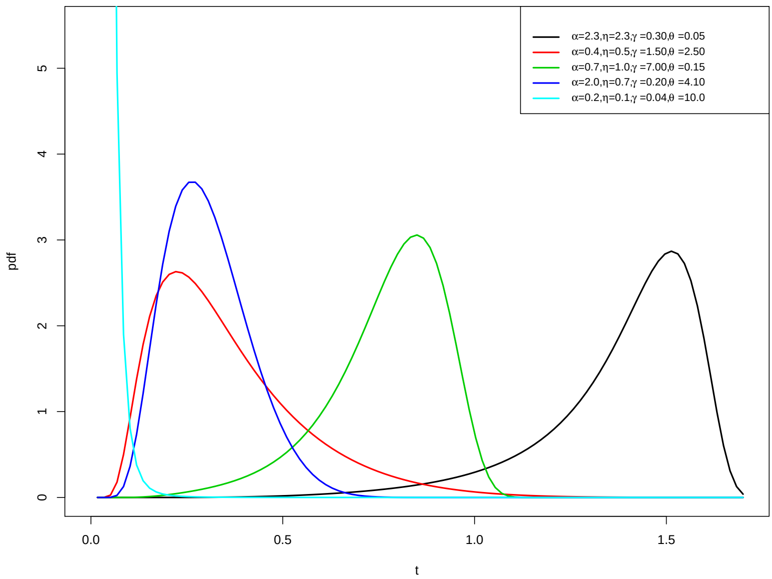

Figure 1: Plot for the pdf of the Gompertz flexible Weibull distribution

From Figure 1, it can be seen that the shape of the GoFW distribution could either be decreasing or unimodal (inverted bathtub) based on the parameter values.

2.1 Expansion for the pdf

$$f(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \tag{7}$$

By making use of the series expansion; $e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}$,

$$f(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} \times$$
$$\sum_{i=0}^{\infty} \frac{\left(\frac{\theta}{\gamma}\right)^i}{i!} \left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}^i$$

Using binomial expansion,

$$f(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} \sum_{i=0}^{\infty} \frac{\left(\frac{\theta}{\gamma}\right)^i}{i!} \times$$
$$\sum_{k=0}^{\infty} (-1)^k \binom{i}{k} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma k}$$

then,

$$f(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{i!}\left(\frac{\theta}{\gamma}\right)^i \binom{i}{k} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma(k+1)}$$

Using the series expansion of $\left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma(k+1)}$,

$$f(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k}{i!}\left(\frac{\theta}{\gamma}\right)^i \binom{i}{k} \times$$
$$\sum_{j=0}^{\infty} \frac{[\gamma(k+1)]^j}{j!} e^{j(\alpha t - \frac{\eta}{t})}$$
$$= \left(\alpha + \frac{\eta}{t^2}\right) \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^k[\gamma(k+1)]^j \theta^{i+1}}{i!\,j!} \left(\frac{1}{\gamma}\right)^i \binom{i}{k} \times$$
$$e^{(j+1)(\alpha t)} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} [\eta(j+1)]^l t^{-l}$$

which reduces to give:

$$f(t) = \left(\alpha + \frac{\eta}{t^2}\right) \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{k+l}[\gamma(k+1)]^j \theta^{i+1}}{i!\,j!\,l!} \times$$
$$\left(\frac{1}{\gamma}\right)^i \binom{i}{k} e^{(j+1)(\alpha t)} [\eta(j+1)]^l t^{-l} \quad (8)$$

The expression in equation (8) can however be used to derive expressions for the moment, moment generating function (mgf), entropy and many other properties.

2.2 Reliability Analysis

The reliability analysis of the GoFW distribution which include the reliability function, failure rate, reversed hazard function and odds function are obtained as follows:

Survival Function:
Survival (or reliability) function is given as:

$$S(t) = 1 - F(t) \tag{9}$$

Therefore, the survival function of the GoFW distribution is obtained as:

$$S(t) = e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \quad ; \quad \alpha, \eta, \theta, \gamma > 0 \tag{10}$$

Hazard Function(or Failure Rate):
Hazard function is obtained from:

$$h(t) = \frac{f(t)}{S(t)} \tag{11}$$

Hence, the failure rate for the GoFW distribution is obtained as:

$$h(t) = \theta\left(\alpha + \frac{\eta}{t^2}\right)e^{\alpha t-\frac{\eta}{t}}\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma} \quad ; \quad \alpha, \eta, \theta, \gamma > 0 \tag{12}$$

The plots for the failure rate of the GoFW distribution using varying parameter values and are presented in Figure 2.

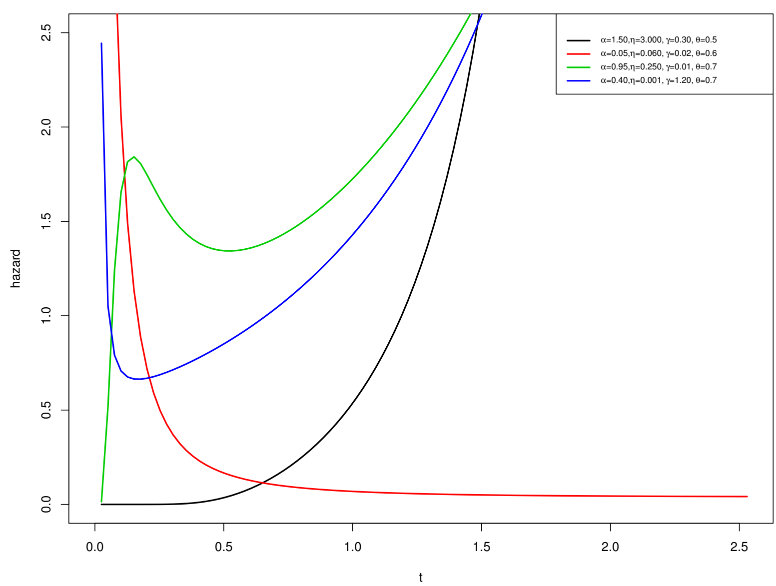

Figure 2: Plot for the failure rate of the Gompertz flexible Weibull distribution

From Figure 2, it can be deduced that the failure rate has shape(s) that could either be uni-antimodal, unimodal, increasing and decreasing (these depend on the values of the parameters).

The Reversed Hazard Function:
This is derived from:

$$r(t) = \frac{f(t)}{F(t)} \tag{13}$$

The reversed hazard function for the GoFW distribution is thus obtained as:

$$r(t) = \frac{\theta\left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}}{1 - e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}} \quad ; \quad \alpha, \eta, \theta, \gamma > 0 \tag{14}$$

The Odds Function:
This is obtained using:

$$O(t) = \frac{F(t)}{1 - F(t)} \tag{15}$$

Thus, the odds function of the GoFW distribution is obtained as:

$$O(t) = \frac{1 - e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}}{e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}} \quad ; \quad \alpha, \eta, \theta, \gamma > 0 \tag{16}$$

2.3 The Quantile Function and Median

This is derived as the inverse cdf and it can be represented by:

$$Q(u) = F^{-1}(u) \tag{17}$$

Thus, for GoFW distribution, the quantile function is obtained as:

$$Q(u) = \frac{1}{2\alpha}\left\{\ln\left\{-\ln\left[1 - \left(\frac{\gamma}{\theta}\ln(1-u)\right)\right]^{-\frac{1}{\gamma}}\right\} + \left[\left(\ln\left\{-\ln\left[1 - \left(\frac{\gamma}{\theta}\ln(1-u)\right)\right]^{-\frac{1}{\gamma}}\right\}\right)^2 + 4\,\alpha\,\eta\right]^{\frac{1}{2}}\right\} \tag{18}$$

where $u \sim Uniform(0, 1)$.

This implies that random samples can be generated from the GoFW distribution using the expression:

$$x_q = \frac{1}{2\alpha}\left\{\ln\left\{-\ln\left[1-\left(\frac{\gamma}{\theta}\ln(1-u)\right)\right]^{-\frac{1}{\gamma}}\right\}+\left[\left(\ln\left\{-\ln\left[1-\left(\frac{\gamma}{\theta}\ln(1-u)\right)\right]^{-\frac{1}{\gamma}}\right\}\right)^2+4\,\alpha\,\eta\right]^{\frac{1}{2}}\right\} \tag{19}$$

The median can be obtained by substituting $u = 0.5$ in equation (18) as follows:

$$\text{Median} = \frac{1}{2\alpha}\left\{\ln\left\{-\ln\left[1-\left(\frac{\gamma}{\theta}\ln(0.5)\right)\right]^{-\frac{1}{\gamma}}\right\}+\left[\left(\ln\left\{-\ln\left[1-\left(\frac{\gamma}{\theta}\ln(0.5)\right)\right]^{-\frac{1}{\gamma}}\right\}\right)^2+4\,\alpha\,\eta\right]^{\frac{1}{2}}\right\} \tag{20}$$

Other quantiles can also be obtained when appropriate value(s) of u are substituted.

2.4 Skewness and Kurtosis

The coefficient of skewness and kurtosis of the GoFW distribution can be obtained by making use of the quantile measures. Following Kenney and Keeping (1962), the Bowely's skewness is given by:

$$\text{Skewness} = \frac{Q_{0.75} - 2Q_{0.5} + Q_{0.25}}{Q_{0.75} - Q_{0.25}} \tag{21}$$

Also, Moors (1988) gave the Moors kurtosis as:

$$\text{Kurtosis} = \frac{Q_{0.875} - Q_{0.625} - Q_{0.375} + -Q_{0.125}}{Q_{0.75} - Q_{0.25}} \tag{22}$$

where $Q_{(\cdot)}$ are the quantiles which can be obtained from the expression in (18) after substituting the corresponding value(s) for u.

2.5 Moments

The r^{th} moments of the GoFW distribution are derived as follows:

$$\mu_r = E\left(T^r\right) = \int_0^\infty t^r f(t)dt \tag{23}$$

Then,

$$\mu_r = \int_0^\infty t^r \theta \left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} dt$$

However, the series expansion for $f(t)$ has been given in equation (8). Therefore, the r^{th} moment of the GoFW distribution is the solution of:

$$\begin{aligned}\mu_r = & \sum_{i=0}^{m}\sum_{k=0}^{n}\sum_{j=0}^{d}\sum_{l=0}^{s} \frac{(-1)^{k+l}[\gamma\,(k+1)]^j \theta^{i+1}}{i!\,j!\,l!}\left(\frac{1}{\gamma}\right)^i \binom{i}{k} [\eta(j+1)]^l \times \\ & \int_0^\infty t^{r-l}\left(\alpha + \frac{\eta}{t^2}\right) e^{(j+1)(\alpha t)} dt\end{aligned} \tag{24}$$

which is,

$$\begin{aligned}\mu_r = & \sum_{i=0}^{m}\sum_{k=0}^{n}\sum_{j=0}^{d}\sum_{l=0}^{s} \frac{(-1)^{k+l}[\gamma\,(k+1)]^j \theta^{i+1}}{i!\,j!\,l!}\left(\frac{1}{\gamma}\right)^i \binom{i}{k} [\eta(j+1)]^l \times \\ & \left[\int_0^\infty \alpha t^{r-l} e^{(j+1)(\alpha t)} dt + \int_0^\infty \eta t^{r-l-2} e^{(j+1)(\alpha t)} dt\right] \\ = & \sum_{i=0}^{m}\sum_{k=0}^{n}\sum_{j=0}^{d}\sum_{l=0}^{s} \frac{(-1)^{k+l}[\gamma\,(k+1)]^j \theta^{i+1}}{i!\,j!\,l!}\left(\frac{1}{\gamma}\right)^i \binom{i}{k} [\eta(j+1)]^l \times \\ & \left[\frac{\Gamma(r-l+1)}{\alpha^{r-l}\,(j+1)^{r-l+1}} + \frac{\eta\Gamma(r-l-1)}{\alpha^{r-l-1}\,(j+1)^{r-l-1}}\right]\end{aligned} \tag{25}$$

2.6 Moment Generating Function

Let Y denote a random variable, the moment generating function (mgf) is given by:

$$M_Y(t) = \int_0^\infty e^{ty} f(y)dy \tag{26}$$

Using series expansion for e^{ty},

$$M_Y(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \int_0^{\infty} y^r f(y) dy = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r \tag{27}$$

Therefore from (25), the mgf of the GoFW distribution is derived as:

$$M_Y(t) = \sum_{i=0}^{m} \sum_{k=0}^{n} \sum_{j=0}^{d} \sum_{l=0}^{s} \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{(-1)^{k+l} [\gamma (k+1)]^j \theta^{i+1}}{i!\, j!\, l!} \left(\frac{1}{\gamma}\right)^i \binom{i}{k} \times$$
$$[\eta(j+1)]^l \left[\frac{\Gamma(r-l+1)}{\alpha^{r-l} (j+1)^{r-l+1}} + \frac{\eta \Gamma(r-l-1)}{\alpha^{r-l-1} (j+1)^{r-l-1}} \right] \tag{28}$$

2.7 Distribution of Order Statistics

If $t_1, t_2, ..., t_n$ are random samples arranged in ascending order from a cdf and pdf distributed according to the GoFW distribution, then the pdf of the k^{th} order statistics of the GoFW distribution is derived as follows:

$$f_{k:n}(t) = \frac{n!}{(k-1)!(n-k)!} f(t) \left[F(t)\right]^{k-1} \left[1 - F(t)\right]^{n-k} \tag{29}$$

$$= \frac{n!}{(k-1)!(n-k)!} \theta \left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} \times$$
$$e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \left[1 - e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}\right]^{k-1} \times$$
$$\left[e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}\right]^{n-k} \tag{30}$$

The distribution of both minimum and maximum order statistics of the GoFW are therefore obtained as:

$$f_{1:n}(t) = n\theta \left(\alpha + \frac{\eta}{t^2}\right) e^{\alpha t - \frac{\eta}{t}} \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \times$$
$$\left[e^{\left(\frac{\theta}{\gamma}\right)\left\{1 - \left[\exp\left(-e^{\alpha t - \frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}\right]^{n-1} \tag{31}$$

and

$$f_{n:n}(t) = n\theta\left(\alpha+\frac{\eta}{t^2}\right)e^{\alpha t-\frac{\eta}{t}}\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma} e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma}\right\}} \times \left[1-e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t-\frac{\eta}{t}}\right)\right]^{-\gamma}\right\}}\right]^{n-1} \tag{32}$$

respectively.

2.8 Estimation of Model Parameters

The unknown parameters of the GoFW distribution can be obtained by the method of maximum likelihood estimation (MLE). Let us denote $t_1, t_2, ..., t_n$ as random samples from the pdf of the GoFW distribution, the likelihood function is thus obtained as:

$$f\left(t_1, t_2, ..., t_n; \alpha, \eta, \gamma, \theta\right) = \prod_{i=1}^{n}\left[\theta\left(\alpha+\frac{\eta}{t_i^2}\right)e^{\alpha t_i-\frac{\eta}{t_i}}\left[\exp\left(-e^{\alpha t_i-\frac{\eta}{t_i}}\right)\right]^{-\gamma} \times e^{\left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t_i-\frac{\eta}{t_i}}\right)\right]^{-\gamma}\right\}}\right] \tag{33}$$

Let $L = \log f\left(t_1, t_2, ..., t_n; \alpha, \eta, \gamma, \theta\right)$ denote the log-likelihood function, then:

$$L = n\log(\theta) + \sum_{i=1}^{n}\log\left(\alpha+\frac{\eta}{t_i^2}\right) + \sum_{i=1}^{n}\left(\alpha t_i-\frac{\eta}{t_i}\right) + \gamma\sum_{i=1}^{n}e^{\alpha t_i-\frac{\eta}{t_i}} + \left(\frac{\theta}{\gamma}\right)\left\{1-\left[\exp\left(-e^{\alpha t_i-\frac{\eta}{t_i}}\right)\right]^{-\gamma}\right\} \tag{34}$$

The maximum likelihood estimates of parameters α, η, γ and θ can be derived by solving the resulting nonlinear equation of $\frac{dL}{d\alpha} = 0$, $\frac{dL}{d\eta} = 0$, $\frac{dL}{d\gamma} = 0$ and $\frac{dL}{d\theta} = 0$.

However, the solution could not be obtained analytically but it can be computed numerically when data sets are available. Particularly, R software was adopted in this research to compute the estimates.

3. Real Life Application

To demonstrate the potentials of the GoFW distribution, it was applied to three (3) real life data sets and it was compared with the Gompertz Weibull (GoW), Gompertz Burr type XII (GoBXII), Gompertz Lomax (GoLo), exponentiated flexible Weibull (EFW), exponentiated flexible Weibull extension (EFWE) and Kumaraswamy flexible Weibull (KuFW) distributions. The log likelihood (l) value, Akaike information criterion (AIC), Bayesian information criterion (BIC), corrected Akaike information criterion ($AICC$) and Hannan Quinn information criterion ($HQIC$) are used as selection criteria. Their statistics are defined as follows:

$$AIC = -2l + 2k,$$
$$AICC = AIC + \frac{2k\,(k+1)}{n-k-1},$$

and

$$BIC = -2l + k\,log(n),$$

where k is the number of parameters in the model, n is the sample size and $l(.)$ represents the maximized value of the log-likelihood function. Smaller values of these statistics indicate a better fit.

First data set:
This data set represents the waiting times (in minutes) of 100 bank customers before service is being rendered. The data has been used previously by Ghitany et al. (2008) and Oguntunde et al. (2017a). The summary of the data set is given in Table 1:

Table 1: Data summary on 100 bank customers

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
100	0.800	38.500	8.100	9.877	52.37411	1.472765	5.540292

The results of the model fitting are however listed in Table 2:

Table 2: The ML estimates, l, AIC, $AICC$, and BIC for the first data set

Distribution	**ML estimate** $(\widehat{\alpha}, \widehat{\eta}, \widehat{\gamma}, \widehat{\theta})$	l	AIC	$AICC$	BIC	$HQIC$
GoFW	$(0.1223, 4.5967, 0.0060, 2.9837)$	-317.10	642.21	642.63	652.63	646.43
GoW	$(1.1742, -0.1589, 0.0893, 1.6221)$	-317.90	643.80	644.22	654.22	648.02
GoBXII	$(0.1056, 2.4125, 0.1618, 3.4292)$	-317.34	642.68	643.10	653.10	646.90
GoLo	$(0.0162, 3.0939, 3.0601, 0.4791)$	-319.33	646.67	647.09	657.09	650.88
EFW	$(0.3030, 0.0552, 19.6163, ---)$	-320.57	647.13	647.38	654.95	650.29
EFWE	$(0.2830, 0.0349, 2.7840, ---)$	-365.65	737.30	737.55	745.12	740.46
KuFW	$(3.8723, 3.0574, 0.0347, 1.6352)$	-321.51	651.02	651.45	661.45	655.24

From Table 2, the GoFW model has the lowest AIC, $AICC$, BIC, and $HQIC$ values. The difference between the AIC values of GoFW, GoW and GoBXII models are less than 2 which indicate no substantial difference between the models. Note that GoFW model yield the highest log-likelihood value. Hence, we may conclude that the GoFW model provides a better fit compared to the other competing models; Gompertz Weibull (GoW), Gompertz Burr type XII (GoBXII), Gompertz Lomax (GoLo), exponentiated flexible Weibull (EFW), exponentiated flexible Weibull extension (EFWE) and Kumaraswamy flexible Weibull (KuFW) distributions.

The histogram of the data set with the estimated pdfs, cdfs for the competing models are presented in Figures 3 and 4 respectively.

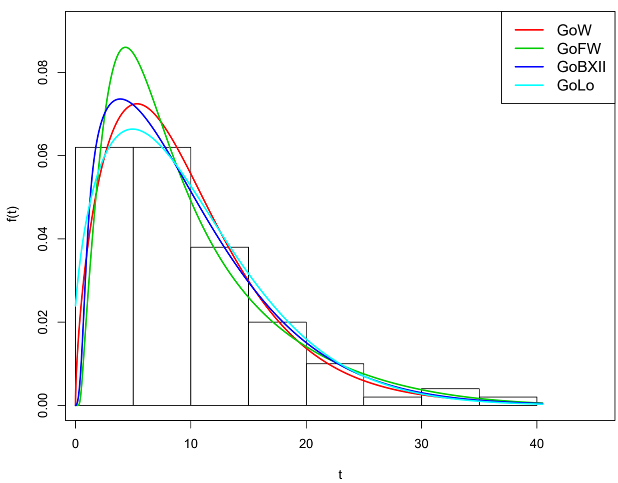

Figure 3: Histogram of the first data set with the estimated pdf of the fitted models

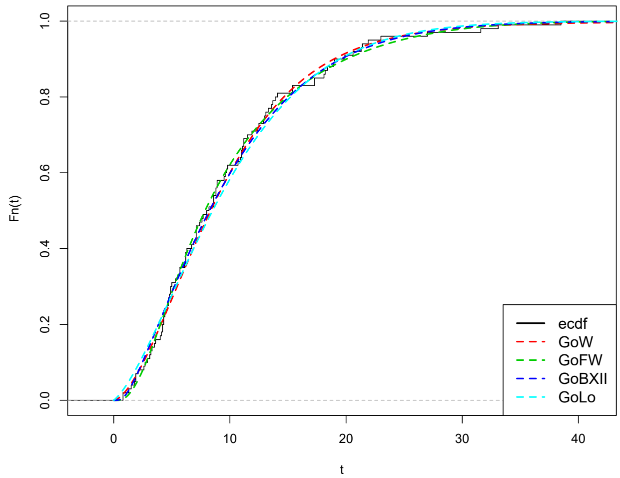

Figure 4: Plot for the estimated cdf of the fitted models

It is evident from these figures that the GoFW provides a better fit to the histogram than the other competing models.

Second data set:
This data set represents the lifetime of 50 devices. It was given by Aarset (1987) and it has been widely reported in some other literature (for example, see Lai et al. (2003), Silva et al. (2010), Wang et al. (2015)). The data summary is as given in Table 3.

Table 3: Data summary on lifetime of 50 devices

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
50	0.10	86.00	48.50	45.69	1078.153	-0.137827	1.413863

The results of the model fitting are presented in Table 4.

Table 4: The ML estimates, l, AIC, $AICC$, BIC and $HQIC$ for the second data set

Distribution	**ML estimate** $(\widehat{\alpha}, \widehat{\eta}, \widehat{\gamma}, \widehat{\theta})$	l	AIC	$AICC$	BIC	$HQIC$
GoFW	$(0.0533, 2.1597, 0.0088, 0.3237)$	-220.38	448.76	449.65	456.41	451.67
GoW	$(0.0376, 0.3132, 0.5288, 0.5865)$	-231.64	471.29	472.18	478.94	474.20
GoBXII	$(0.0087, 2.3870, 1.3760, 0.3922)$	-238.42	484.85	485.74	492.50	487.76
GoLo	$(1.6028, 3.6817, 0.0025, 2.4966)$	-235.90	479.80	480.69	487.45	482.72
EFW	$(4.2196, 0.0147, 0.1331, ---)$	-226.98	459.97	460.50	465.71	462.16
EFWE	$(0.0797, 0.0145, 0.3869, ---)$	-224.62	455.25	455.77	460.99	457.44
KuFW	$(0.1566, 0.0800, 0.0389, 1.7064)$	-221.33	450.66	451.55	458.31	453.58

From Table 4, although the AIC statistic show no substantial difference between the GoFW and KuFW models, the GoFW distribution has the lowest AIC, $AICC$, BIC and $HQIC$ values including the highest log-likelihood value, then it can also be concluded that the GoFW distribution provides a better fit than its counterparts.

The histogram of the second data set with the estimated pdfs, cdfs for the competing models are presented in Figures 5 and 6.

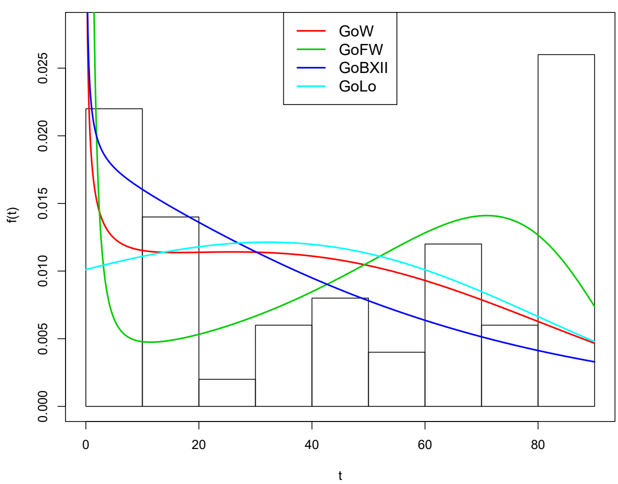

Figure 5: Histogram of the second data set with the estimated pdf of the fitted models

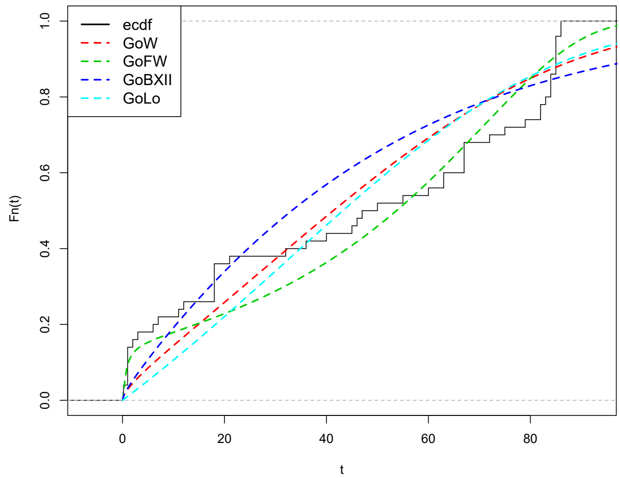

Figure 6: Plot for the estimated cdf of the fitted models

From Figures 5 and 6, it can be seen that the GoFW distribution provides a better fit to the histogram than the other competing models, it can therefore be selected as the best among the other competing models.

Third data set:
This data represents the data set on strength of 1.5 cm glass fibres obtained by workers at the UK National Physical Laboratory. The data set can also be found in the works of Bourguignon et al. (2014), Merovci et al. (2016), Smith and Naylor (1987). The data summary is as presented in Table 5.

Table 5: Data summary on glass fibres

n	Min.	Max.	Median	Mean	Var.	Skewness	Kurtosis
63	0.550	2.240	1.590	1.507	0.10505	-0.87858	3.923761

The results obtained for all the competing models are as presented in Table 6.

Table 6: The ML estimates, l, AIC, $AICC$, BIC and $HQIC$ for the third data set

Distribution	**ML estimate** $(\widehat{\alpha}, \widehat{\eta}, \widehat{\gamma}, \widehat{\theta})$	l	AIC	$AICC$	BIC	$HQIC$
GoFW	$(0.6551, 24.7586, -0.0862, 2.9009)$	-13.82	35.64	36.33	44.21	39.01
GoW	$(0.0097, 3.4978, 1.0184, 1.0020)$	-14.83	37.66	38.35	46.24	41.04
GoBXII	$(0.0482, 2.4765, 0.8928, 3.2441)$	-14.00	36.00	36.69	44.57	39.37
GoLo	$(0.0045, 8.1790, 0.5069, 1.5158)$	-14.50	37.01	37.69	45.57	40.37
EFW	$(0.5845, 2.1768, 6.6145, ---)$	-15.46	36.93	37.34	43.36	39.46
EFWE	$(0.0291, 0.8491, 0.2739, ---)$	-21.75	49.50	49.90	55.93	52.03
KuFW	$(0.5563, 0.2331, 2.4998, 4.6446)$	-14.12	36.23	36.92	44.81	39.61

From Table 6, AIC statistic indicate no substantial difference between GoFW GoBXII, GoLo, EFW and KuFW models. We observed that the GoFW distribution has the lowest AIC, $AICC$, BIC and $HQIC$ values including the highest log-likelihood value, this means that the GoFW distribution provides a better fit to the data set as compared to the other competing models.

The histogram of the third data set and the estimated pdfs, cdfs for the competing models are presented in Figures 7 and 8.

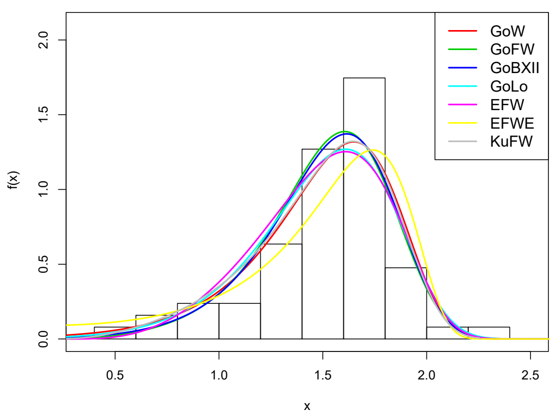

Figure 7: Histogram of the third data set with the estimated pdf of the fitted models

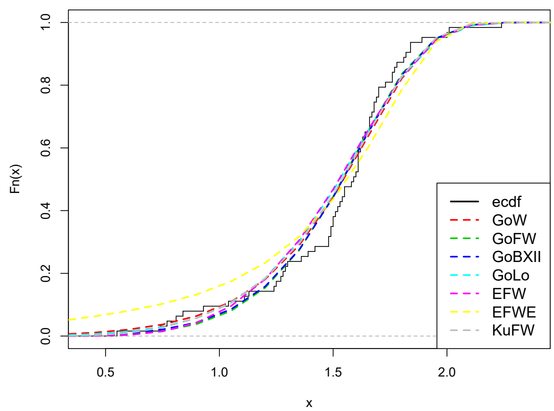

Figure 8: Plot for the estimated cdf of the fitted models

From Figures 7 and 8, it shows that the GoFW provides a better fit to the histogram than the other competing models. It can therefore be chosen as the best distribution.

4. Simulation Study

To investigate the behavior of the GoFW parameters, data sets were generated from the GoFW distribution with a replication number of $k = 1,000$; random samples of sizes $n = 50, 100, 200$ and 300 were then further selected. In this simulation study, three (3) different cases were considered. For the first case, the true parameter values considered are; $\alpha = 1, \eta = 1, \theta = 1, \gamma = 1$ while $\alpha = 2, \eta = 2, \theta = 2, \gamma = 2$ and $\alpha = 0.5, \eta = 0.5, \theta = 0.5, \gamma = 0.5$ were considered for the second and third cases respectively. Using R software, the MLE, bias and root mean square error (RMSE) were obtained, the results are however displayed in Tables 7 to 9.

Table 7: Simulation study at $\alpha = 1, \eta = 1, \theta = 1, \gamma = 1$

n	$Mean$	$Bias$	$RMSE$
50	(1.0583, 1.0365, 1.0706, 0.9803)	(0.0583, 0.0365, 0.0706, −0.0197)	(0.2006, 0.2181, 0.4568, 0.3695)
100	(1.0263, 1.0284, 1.0571, 0.9913)	(0.0263, 0.0284, 0.0571, −0.0087)	(0.1421, 0.1611, 0.3362, 0.2915)
200	(1.0007, 1.0156, 1.0421, 1.0204)	(0.0007, 0.0156, 0.0421, 0.0204)	(0.0980, 0.1154, 0.2640, 0.2128)
300	(0.9977, 1.0064, 1.0201, 1.0218)	(−0.0023, 0.0064, 0.0201, 0.0218)	(0.0835, 0.0932, 0.20430.1914)

Table 8: Simulation study at $\alpha = 2, \eta = 2, \theta = 2, \gamma = 2$

n	$Mean$	$Bias$	$RMSE$
50	$(2.0770, 2.0414, 2.0601, 2.0259)$	$(0.0770, 0.0414, 0.0601, 0.0259)$	$(0.3446, 0.2246, 0.6360, 0.6605)$
100	$(2.0416, 2.0391, 2.0930, 2.0253)$	$(0.0416, 0.0391, 0.0930, 0.0253)$	$(0.2538, 0.1783, 0.5291, 0.5641)$
200	$(2.0109, 2.0300, 2.1022, 2.0479)$	$(0.0109, 0.0300, 0.1022, 0.0479)$	$(0.1795, 0.1379, 0.42810.4498)$
300	$(1.9965, 2.0212, 2.0871, 2.0563)$	$(-0.0035, 0.0212, 0.0871, 0.0563)$	$(0.1457, 0.1128, 0.3805, 0.3736)$

Table 9: Simulation study at $\alpha = 0.5, \eta = 0.5, \theta = 0.5, \gamma = 0.5$

n	$Mean$	$Bias$	$RMSE$
50	$(0.5146, 0.5314, 0.5298, 0.5007)$	$(0.0146, 0.0314, 0.0298, 0.0007)$	$(0.0690, 0.1559, 0.1822, 0.1418)$
100	$(0.5069, 0.5195, 0.5167, 0.4988)$	$(0.0069, 0.0195, 0.0167, -0.0012)$	$(0.0484, 0.1024, 0.1211, 0.1051)$
200	$(0.5006, 0.5078, 0.5068, 0.5080)$	$(0.0006, 0.0078, 0.0068, 0.0080)$	$(0.0332, 0.0689, 0.0831, 0.0782)$
300	$(0.4992, 0.5037, 0.5032, 0.5082)$	$(-0.0008, 0.0037, 0.0032, 0.0082)$	$(0.0271, 0.0549, 0.0672, 0.0638)$

From Tables 7 to 9, the means get closer to the true parameter values as sample size increases from 50 to 300, the absolute bias, root mean square error (RMSE) also reduces for all the selected parameter values as the sample size increases.

5. Conclusion

The Gompertz flexible Weibull (GoFW) distribution has been successfully derived in this paper and its various statistical properties have been established. The model's shape could be unimodal or decreasing. The maximum likelihood method of estimation is proposed in estimating the unknown model parameters. The model exhibits uni-antimodal, unimodal, increasing and decreasing failure rates, hence, it can be used to describe and model real life phenomena with inverted bathtub, bathtub, increasing and decreasing failure rates. Applications to three real life data sets reveals that the GoFW distribution is an improvement and a better choice over the Gompertz Weibull (GoW), Gompertz Burr type XII (GoBXII), Gompertz Lomax (GoLo), exponentiated flexible Weibull (EFW), exponentiated flexible Weibull extension (EFWE) and Kumaraswamy flexible Weibull (KuFW) distributions. From the simulation study, it can be deduced that the GoFW parameters are stable.

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