Pertanika J. Sci. & Technol. **10(1)**: 111 - 118 (2002)

ISSN: 0128-7680

On Solutions of Some Polynomial-Functional Equations

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Received: 16 September 1998

ABSTRAK

Sistem persamaan polinomial-fungsian tertentu yang dijanakan daripada sifat-sifat biasa pembezaan diperhatikan. Penyelesaian sistem ini ke atas medan dan bentuk kanonik mereka relatif terhadap kesetaraan natural diberikan.

ABSTRACT

Systems of some polynomial-functional equations, which are derived from the usual properties of derivative, are considered. Solutions of these systems over a field and their canonical forms relative to natural equivalences are given.

Keywords: Algebra, derivative, polynomial

INTRODUCTION

Let R be a commutative, associative algebra over a field F of characteristic zero, $d : R \rightarrow R$ be a derivative i.e. d is an additive map and $d(ab) = d(a)b + ad(b)$ for any $a,b \in R$. The last equality means that to know the derivative of a product it is enough to know the derivatives of the factors because of polynomialness $d(ab)$ in a, b, $d(a)$ and $d(b)$ over F. If for a while we denote $d(xy)$ as a polynomial $f[x, y, d(x), d(y)]$ then the equality $d((xy)z) = d(x(yz))$ draws in the equality

$$f[xy, z, f[x, y, d(x), d(y)], d(z)] = f[x, yz, d(x), f[y, z, d(y), d(z)]],$$

the equality $d(xy) = d(yx)$ draws in the equality

$$f[x, y, d(x), d(y)] = f[y, x, d(y), d(x)]$$

Therefore the following problem is natural: Let $x_1, x_2, x_3, x_4, x_5, x_6$ be indeterminates over the field F. Find all such polinomials $f[x_1, x_2, x_3, x_4]$ over F for which the following system of polynomial-functional equations is valid.

$$\begin{cases} f[x_1x_2, x_3, f[x_1, x_2, x_4, x_5], x_6] = f[x_1, x_2x_3, x_4, f[x_2, x_3, x_5, x_6]] \\ f[x_1x_2, x_3, x_4] = f[x_2, x_1, x_4, x_3] \end{cases} \quad (1)$$

The paper deals with this problem. To follow the proofs of the results below one needs only some basic notions on polynomials, which can be found nearly in any text book on algebra e.g. in (Waerden 1991).

Lemma. The first polynomial-functional equation of system (1) has only the following solutions in $F[x_1,x_2,x_3,x_4]$:

$$f = \sum_{i=0}^{k} a_i (x_1 x_2)^i \text{ , where } k \in N \cup \{0\} \text{ and } a_i \in F;$$

$$f = x_1^n x_4 + \sum_{i=0}^{k} a_i (x_1^n - x_1^i) x_2^i, \text{ where } k,\ n \in N \cup \{0\} \text{ and } a_i \in F;$$

$$f = x_2^n x_3 + \sum_{i=0}^{k} a_i x_1^i (x_2^n - x_2^i) x_2^i \text{ where } k,\ n \in N \cup \{0\} \text{ and } a_i \in F;$$

$$f = x_1^m x_4 + x_2^n x_3 + a x_1^m x_2^n, \text{ where } m,n \in N \cup \{0\},\ m \neq n \text{ and } a \in F;$$

$$f = x^n{}_1 x_4 + x^n{}_2 x_3 + \sum_{i=0}^{k} a_i (x_1^i x_2^n + x_1^n x_2^i - x_1^i x_2^i),$$

where $k,\ n \in N \cup \{0\}$ and $a_i \in F$;

$$f = g[x_1] x_4 + g[x_2] x_3 + a x_3 x_3 + \frac{1}{a}(g[x_1] g[x_2] - g[x_1 x_2] - g[x_1 x_2])$$

where $g[x_1] \in F[x_1]$ *and* $a \in F^* = F \setminus \{0\}$.

Proof of lemma. Let $f \in F[x_1,x_2,x_3,x_4]$ be a polynomial for which the first equality of (1) is valid. Comparing degrees on the left and the right sides of this equality in x_4 (x_6) as polynomials over $F[x_1,x_2,x_3,x_5,x_6]$ (corresp. $F[x_1,x_2,x_3,x_4,x_5]$) one has inequality

$$\deg x_3 \, f[x_1,x_2,x_3,x_4] \leq 1 \text{ (corresp. } \deg x_4 \, f[x_1,x_2,x_3,x_4] \leq 1)$$

Due to this

$$f[x_1,x_2,x_3,x_4] = x_3 x_4 f_0[x_1,x_2] + x_3 f_1[x_1,x_2] + x_4 f_2[x_1,x_2] + f_3[x_1,x_2], \qquad (2)$$

where $f_i[x_1,x_2] \in F[x_1, x_2]$. Considering the first equality of (1) as an equality of polynomials in x_4, x_5, x_6 over $F[x_1,x_2,x_3]$ it can be written as the following system of equalities (3)

$$\begin{cases} f[x_1,x_2] f_0[x_1x_2,x_3] = f_0[x_2,x_3] f_0[x_1,x_2x_3] \\ f_1[x_1x_2] f_0[x_1x_2,x_3] = f_2[x_2,x_3] f_0[x_1,x_2x_3] \\ f_2[x_1,x_2] f_0[x_1x_2,x_3] = f_0[x_2,x_3] f_2[x_1,x_2x_3] \\ f_0[x_1,x_2] f_1[x_1x_2,x_3] = f_0[x_1,x_2x_3] f_1[x_2,x_3] \\ f_1[x_1,x_2] f_1[x_1x_2,x_3] = f_0[x_1,x_2x_3] f_3[x_2,x_3] + f_1[x_1,x_2x_3] \\ f_1[x_1x_2,x_3] f_2[x_1,x_2] = f_1[x_2,x_3] f_2[x_1,x_2x_3] \\ f_0[x_1x_2,x_3] f_3[x_1,x_2] + f_2[x_1x_2,x_3] = f_2[x_2,x_3] f_2[x_1,x_2x_3] \\ f_3[x_1,x_2] f_1[x_1x_2,x_3] + f_3[x_1x_2,x_3] = f_2[x_1,x_2x_3] f_3[x_2,x_3] + f_3[x_1,x_2x_3] \end{cases}$$

Comparing degrees on the left and the right sides of the first equality of this system in x_1 and x_3 one gets $f_0 \in F$.

Let us consider first the $f_0 = 0$ case. In this case the 5th (7th) equality of the above system shows that $f_2[x_1, x_2]$ (corresp. $f_2[x_1, x_2]$) does not depend on x_1 (corresp.x_2) i.e. $f_1[x_1, x_2] = f_1[x_2]$ (corresp. $f_2[x_1,x_2] = f_2[x_1]$). Moreover $f_1[x_2] = ax_2^n$ ({corresp.$f_2[x_1] = bx_1^m$), where a (corresp. b) is 0 or 1 and n (corresp. m) $\in N \cup \{0\}$. In this case system (3) reduces to the equality

$$ax_3^n f_3[x_1,x_2] + f_3[x_1x_2, x_3] = bx_1^m f_3[x_2,x_3] + f_3[x_1,x_2x_3] \tag{4}$$

If $a = b = 0$ then equation (4) has only the following solutions

$$f_3[x_1,x_2] = \sum_{i=0}^{k} a(x_1x_2)^i \text{ where } k \in N \cup \{0\} \text{ and } a_i \in F$$

and therefore in this case due to (2) one has

$$f = f_3[x_1,x_2] \sum_{i=0}^{k} a(x_1x_2)^i, \text{ where } k \in N \cup \{0\} \text{ and } a_i \in F.$$

If $a = 1, b = 0$ then equation (4) has only the following solutions

$$f_3[x_1,x_2] = \sum_{i=o}^{k} a_i x_1^i (x_2^n - x_2^i), \text{ where } k \in N \cup \{0\} \text{ and } a_i \in F.$$

and therefore in this case due to (2) one has

$$f = x_2^n x_3 + \sum_{i=o}^{k} a_i x_1^i (x_2^n - x_2^i), \text{ where } k,\ n \in N \cup \{0\}\ a_i \in F.$$

If a = 0, b = 1 then equation (4) has only the following solutions

$$f_3[x_1, x_2] = \sum_{i=0}^{k} a_i (x_1^m - x_1^i) x_2^i, \text{ where } k_\in NU\{0\} \text{ and } a_{i\in}F^1$$

and therefore in this case due to (2) one has

$$f = x_1^m x_4 + \sum_{i=0}^{k} a_i (x_1^m - x_1^i) x_2^i, \text{ where } K,\ m_\in NU\{0\} \text{ and } a_{i\in}F^1$$

$$f_3[x_1,x_2]=\sum_{i=0}^{k}a_i(x_1^m-x_1^i)x_2^i \text{, where}$$

If $a = b = 1$ and $n \neq m$ then equation (4) has only the following solutions

$$f_3[x_1,x_2] = cx_1^m x_2^n, \text{ where } c \in F.$$

and therefore in this case due to (2) one has

$$f = x_1^m x_4 + x_2^n x_3 + cx_1^m x_2^n, \text{ where } m,\ n \in N \cup \{0\}, \text{ and } c \in F.$$

If $a = b = 1$ and $n = m$ then equation (4) has only the following solutions

$$f_3[x_1,x_2] = \sum_{i=0}^{k}a_i(x_1^i x_2^n + x_1^n x_2^i - x_1^i x_2^i),$$

where $k \in N \cup \{0\}$ and $a_i \in F$ and therefore in this case due to (2) one has

$$f = x_1^n x_4 + x_2^n x^3 + \sum_{i=0}^{k}a_i(x_1^i x_2^n + x_1^n x_2^i - x_1^i x_2^i),$$

where $k,n \in N \cup \{0\}$ and $a_i \in F$.

Let us now consider the $f_0 \neq 0$ case. In this case the second equality of system (3) shows that $f_1[x_1,x_2] = f_2[x_2,x_3]$ i.e. $g[x_2] = f_1[x_1,x_2] = f_2[x_2,x_3]$. Moreover due to the 5th equality of (3) one has

$$f_3[x_2,\ x_3]= \frac{1}{f_0}\ (g[x_2]g[x_3] - g[x_2x_3])$$

and after this fact the other equalities of (3) become identities. Thus in this case one has

$$f = g[x_1]x_4 + g[x_2]x_3 + ax_3x_4 + \frac{1}{a}\ (g[x_1]g[x_2] - g[x_1x_2]), \text{ where } g[x] \in F[x] \text{ and}$$

$a \in F^*$. The proof of lemma is completed.

Now it is easy to list all solutions of system (1). To do this it is enough to choose only those polynomials from the list presented in the lemma for which the equality

$$f\,[x_1,\ x_2,\ x_3,\ x_4] = f\,[x_2,\ x_1,\ x_4,\ x_3]$$

holds. Let us present the result as the following theorem.

Theorem 1.

System (1) has only the following solutions in $F[x_1, x_2, x_3, x_4]$:

$$f = \sum_{i=0}^{k} a(x_1x_2)^i, \text{ where } k \in N \cup \{0\} \text{ and } a_i \in F,$$

$$f = x_1^n x_4 + x_2^n x_3 + \sum_{i=0}^{k} a_i(x_1^i x_2^n + x_1^n x_2^i - x_1^i x_2^i),$$

where $k,n \in N \cup \{0\}$ and $a_i \in F$,

$$f = g[x_1]x_4 + g[x_2]x_3 + ax_3x_4 + \frac{1}{a}(g[x_1]g[x_2] - g[x_1x_2]),$$

where $g[x] \in F[x]$ and $a \in F^*$.

One can check by direct calculation or use Theorem 1 be sure that if f is a solution for system (1) then

$$\overline{f}[x_1, x_2, x_3, x_4] = af[x_1, x_2, a^{-1}(x_3 - p[x_1]), a^{-1}(x_4 - p[x_2])] + p[x_1 x_2]]$$

is also solution for system (1)for any $a \in F^*$ and $p[x] \in F[x]$. In other words the set of all solutions of system (1) is invariant with respect to the following action of the group $G_1 = F^* \ltimes F[x]$ on it:

$$((a,p[x]), f[x_1, x_2, x_3, x_4]) \mapsto a f[x_1, x_2, a^{-1}(x_3 - p[x_1]), a^{-1}(x_4 - p[x_2])] + p[x_1x_2]$$

It correspondence to the fact that if $d: R \to R$ is such an operation that $d(xy)$ is a polynomial in x, y, dx, dy over F and $a \in F^*$, $p[x] \in F[x]$ then $\delta(xy)$ is also a polynomial in x, y, δx, δy over F, where $\delta = ad + p$ i.e. $\delta(x) = ad(x) + p[x]$ by definition.

The following result gives the "simplest" forms of $d(x_1x_2)$ with respect to that action.

Theorem 1'.

With an accuracy of the above action of group G_1 there are only the following equalities:

1) $d(x_1x_2) = o$, which corresponds to $f = 0$,
2) $d(x_1x_2) = d(x_1)x_2^n + x_1^n d(x_2)$, where n in $N \cup \{0\}$, which corresponds to $f = x_3x_2^n + x_1^n x_4$,
3) $d(x_1x_2) = d(x_1)d(x_2)$, which corresponds to $f = x_3x_4$.

Proof of Theorem 1'. If

$$d(x_1x_2) = \sum_{i=0}^{k} a_i(x_1x_2)^i \text{ , where } k \in N \cup \{0\} \text{ and } a_i \in F$$

then for $\delta = d + p$, where $p[x]= \sum_{i=0}^{k} a_i x^i$ one has $\delta\,(x_1x_2) = 0$.

If

$$d(x_1x_2) = d(x_1)x_2^n + x_1^n d(x_2) + \sum_{i=0}^{k} a_i(x_1^i x_2^n + x_1^n x_2^i - x_1^i x_2^i),$$

where k, $n \in N \cup \{0\}$ and $a_i \in F$, then for $\delta = d + p$, where $p[x]= \sum_{i=0}^{k} a_i x^i$,

one has $\delta(x_1x_2) = \delta(x_1)x_2^n + x_1^n\,\delta(x_2)$.

If

$$f = g[x_1]x_4 + g[x_2]x_3 + ax_3x_4 + \frac{1}{a}(g[x_1]g[x_2] - g[x_1x_2]),$$

where $g[x] \in F[x]$ and $a \in F^*$, then for $\delta = ad + p$, where $p[x] = g[x]$, one has $\delta\,(x_1x_2) = \delta\,(x_1)\,\delta\,(x_2)$. This completes the proof of theorem 1'.

If in addition the operation d: $R \to R$ is an additive map then the equality $d((x+y)z) = d(xz) + d(yz)$ draws in equality

$$f\,[x+y,\ z,\ d(x)+ d(y),\ d(z)]= f\,[x,\ z,\ d(x),\ d(z)]+ f[y,\ z,\ d(y),\ d(z)].$$

Therefore finding the solutions of the following system

$$\begin{cases} f[x_1x_2, x_3, f[x_1, x_2, x_4, x_5], x_6] = f[x_1, x_2x_3, x_4, f[x_2, x_3, x_5, x_6]] \\ f[x_1, x_2, x_3, x_4] = f[x_2, x_1, x_4, x_3] \\ f[x_1 + x_2, x_3, x_4 + x_5, x_6] = f[x_1, x_3, x_4, x_6] + f[x_2, x_3, x_5, x_6] \end{cases} \quad (5)$$

of polynomial-functional equations is natural.

Theorem 2.

System (5) has only the following solutions in $F\,[x_1,\ x_2,\ x_3,\ x_4]$:

$f = ax_1x_2$, *where* $a \in F$

$f = x_1x_4 + x_2x_3 + ax_1x_2$, *where* $a \in F$

$f = ax_3x_4 + b(x_1x_4 + x_2x_3) + \dfrac{b(b-1)}{a}x_1x_2$, where $a \in F^*$, $b \in F$

Proof of Theorem 2. To prove this theorem the obtained solutions of system (1) can be used. But here we offer an easier way: Indeed, due to (2) the 3rd equality of system (5) can be written in the following equivalent form:

$$\begin{cases} f_0[x_1+x_2,x_3] = f_0[x_1,x_3] \\ f_1[x_1+x_2,x_3] = f_1[x_1,x_3] = f_1[x_2,x_3] \\ f_2[x_1+x_2,x_3] = f_2[x_1,x_3] + f_2[x_2,x_3] \\ f_3[x_1+x_2,x_3] = f_3[x_1,x_3] + f_3[x_2,x_3] \end{cases}$$

The first (second; third; fourth) equality of this system means that $f_0[x_1,x_2]= g_0[x_2]$ for some $g_0[x] \in F[x]$ (corresp. $f_1[x_1,x_2] = g_1[x_2]$ for some $g_1[x] \in F[x]$; $f_2[x_1,x_2] = x_1g_2[x_2]$ for some $g_2[x] \in F[x]$; $f_3[x_1,x_2] = x_1g_3[x_2]$ for some $g_3[x] \in F[x]$). In other words one has

$$f[x_1,x_2,x_3,x_4] = x_3x_4g_0[x_2] + x_3g_1[x_2] + x_4g_2[x_2] + x_1g_3[x_2].$$

Afterwards due to the 2nd equality of system (5) one has

$$\begin{cases} g_0[x_1] = g_0[x_2] \text{ i.e. } g_0[x] = a \in F \\ g_1[x_2] = x_2g_2[x_1] \\ x_1g_2[x_2] = g_1[x_1] \\ x_1g_3[x_2] = x_2g_3[x] \ i.e.\ g_3[x] = \text{ for some } c \in F \end{cases}$$

The 2nd and 3rd equalities of the last system imply $g^2[x]= b \in F$, $g_1[x] = bx$. Therefore

$$f\,[x_1,x_2,x_3,x_4] = ax_3x_4 + b(x_3x_2 + x_4x_2) + cx_1x_2$$

Now it is easy to see that for such $f\,[x_1,x_2,x_3,x_4]$ the 1st equality of system (5) is valid if and only if

$$f\,[x_1,x_2,x_3,x_4] = cx_1x_2 \text{ or } f\,[x_1,x_2,x_3,x_4] = x_1x_4 + x_2x_3 + cx_1x_2 \text{ or}$$

$$f\,[x_1,x_2,x_3,x_4] = ax_3x_4 + b\,(x_1x_4 + x_2x_3) + \frac{b(b-1)}{a}x_1x_2, \text{ where } a \in F^*$$

This completes the proof of theorem 2.

One can check by direct calculation or use Theorem 2 be sure that if f is a solution for system (5) then

$$\overline{f}\,[x_1, x_2, x_3, x_4] = af\,[x_1, x_2, a^{-1}(x_3 - cx_1), a^{-1}(x_4 - cx_2)] + cx_1x_2$$

is also solution for system (5) for any $a \in F^*$ and $c \in F$. In other words the set of all solutions of system (5) is invariant with respect to the following action of the group $G_2 = F^* \ltimes F$ on it:

$$((a,c), f[x_1, x_2, x_3, x_4]) \mapsto af[x_1, x_2, a^{-1}(x_3 - cx_1), a^{-1}(x_4 - cx_2)] + cx_1 x_2.$$

It correspends to the fact that if $d: R \to$ R is such an additive operation that $d(xy)$ is a polynomial in x, y, dx, dy over F and $a \in F^*$, $c \in F$ then $\delta(xy)$ is also a polynomial in x, y, δx, δy over F, where $\delta = ad + c$ i.e. $\delta(x) = ad(x) + cx$ by definition, moreover δ is also an additive operation.

The following result gives the "simplest" forms of $d(x_1 x_2)$ with respect to that action.

Theorem 2'.

With an accuracy of the above action of group G_2 there are only the following equalities:

1) $d(x_1 x_2) = o$, which corresponds to $f = 0$,
2) $d(x_1 x_2) = d(x_1)x_2 + x_1 d(x_2)$, which corresponds to $f = x_3 x_2 + x_1 x_4$,
3) $d(x_1 x_2) = d(x_1)d(x_2)$, which corresponds to $f = x_3 x_4$.

Proof of Theorem 2'. If $d(x_1, x_2) = cx_1 x_2$ where $c \in F$ then for $S = d-c$, one has $\delta(x_1 x_2) = 0$.

If $d(x_1 x_2) = d(x_1)x_2 + x_1 d(x_2) + cx_1 x_2$, where $c \in F$ then for $\delta = d + c$ one has $\delta(x_1 x_2) = \delta(x_1)x_2 + x_1\delta(x_2)$

If $f[x_1, x_2, x_3, x_4] = ax_3 x_4 + b(x_1 x_4 + x_2 x_3) + \frac{b(b-1)}{a} x_1 x_2$, where a $\in F^*$ then for $\delta = ad + b$, one has $\delta(x_1 x_2) = \delta(x_1)\,\delta(x_2)$. This completes the proof of theorem 2'.

Theorem 2' can be considered as a confirmation of special positions of differential operators and homomorphisms in theory of commutative associative rings. Roughly speaking, Theorem 2' says that they exhaust all additive maps d:R $\to$ R for which $d(xy)$ is polynomial in $x, y, d(x), d(y)$.

Of course an analogical problem can be considered for other types of algebras, for example, associative algebras, Lie algebras or, in general, polynomial algebras (Procesi 1973). It would be interesting to investigate this problem for associative algebras.

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