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Likelihood Based Estimation in the Logistic Model with Time Censored Data

Ayman S. Baklizi
Department of Statistics
Yarmouk University
Irbid-Jordan

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ABSTRAK

Prosedur-prosedur kesimpulan berdasarkan fungsi kemungkinan dipertimbangkan untuk satu pengagihan logistik dengan data tapisan masa. Pelaksanaan sampel penganggar kemungkinan maksimum yang terhad sepertimana prosedur kesimpulan kemungkinan sampel besar berdasarkan sampel Wald, Rao dan statistik nisbah kemungkinan disiasat. Hasilnya, apa yang diperolehi daripada pengagihan normal asimptot penganggar kemungkinan maksimum didapati tidak tepat. Hasil siasatan juga menunjukkan penganggaran jeda yang berdasarkan statistik Wald dan Rao memerlukan lebih banyak saiz sampel berbanding penganggaran jeda berdasarkan statistik nisbah kemungkinan untuk memperolehi ketepatan yang munasabah.

ABSTRACT

Inference procedures based on the likelihood function are considered for the one logistic distribution with time censored data. The finite sample performances of the maximum likelihood estimator as well as the large sample likelihood inferential procedures based on the Wald, the Rao, and the likelihood ratio statistics are investigated. It is found that the obtained from the asymptotic normal distribution of the maximum likelihood estimator are found no accurate. It is found also that interval estimation based on the Wald and Rao statistics need much more sample size than interval estimation based on the likelihood ratio statistics to attain reasonable accuracy.

Keywords: Confidence intervals, logistic distribution, maximum likelihood estimator, time censored data

INTRODUCTION

The logistic distribution arises in a variety of fields, for example it can be applied as a growth model in human populations and certain biological organisms (Pearl *et al.* 1940). Oliver (1964) used this distribution to model agricultural production data. It also arises in the analysis of survival data (Placket 1959) as well as the analysis of income distributions (Fisk 1961). Many other applications and motivations for this distribution are discussed in Balakrishnan (1992) and Johnson *et al.* (1994).

Inference procedures for the parameters of this distribution have been discussed in the literature. Harter and Moore (1967) and Balakrishnan (1992) discussed and investigated the properties the maximum likelihood estimator in sample of size 10 and 20, for various choices of type 2 censoring. Confidence intervals for the parameters of this models are constructed by Antle *et al.* (1970) by simulating the percentage points of some pivotal quantities based on the maximum likelihood estimator. Schaffer and Sheffield (1973) have further discussion on this problem while Bain *et al.* (1992) considered this problem when the data is type 2 censored.

In this paper I shall consider maximum likelihood estimation and asymptotic interval estimators based on inverting the likelihood based statistics when the data is time censored. These statistics are the Wald, the Rao, and the likelihood ratio (Barndorff- Nielsen and Cox 1994). These statistics are often used for interval estimation with censored data (Nelson 1990), and are known to have an asymptotic chi-squared distribution (Rao 1973). However their performances in finite samples are different and change from one model to another (Lawless 1982). Hence it is desirable to use the statistics which has a faster covergence rate to its limiting distribution and therefore is applicable for small sample sizes that may be in practice (Cox 1988).

METHOD

The Model and the Likelihood Statistics

The probability density function and the cummulative distribution function of the logistic distribution are given respectively by (Johnson *et al.* 1994)

$$f(x,\mu,\sigma)=\frac{c}{\sigma}\frac{\exp(-\frac{c(x-\mu)}{\sigma}}{\left(1+\exp(-\frac{c(x-\mu)}{\sigma}\right)^{2}} \qquad x>0,\ \sigma>0,\quad -\infty<\mu<\infty$$

where $c=\frac{\pi}{\sqrt{3}}$. Consider a random sample of size N from this distribution, of which n are less than or equal to t (some predetermined censoring time) and the remaining no observations are censored and are only known to exceed t. Thus $N = n + n_o$. Here t is fixed but n and n_0 are both random the likelihood function is given by

$$L(x,\mu,\sigma)=\left(\frac{c}{\sigma}\right)^{n}\exp\left(-\sum_{i=1}^{n}\frac{c(x_i-\mu}{\sigma}\right)\prod_{i=1}^{n}1+\exp(-\frac{c(x_i-\mu)}{\sigma}^{-2}\times$$
$$\prod_{i=1}^{n_0}\left(1-\left(1+\exp(-\frac{c(t-\mu)}{\sigma}\right)^{-1}\right)$$

The maximum likelihood estimator can be found by solving the system of first partial derivatives of the log-likelihood function, describe in the appendix. To present the Wald statistics, the Rao and the likelihood ratio statistics for this model we need the following quantities.

$$U(\mu,\sigma)=\begin{pmatrix}\dfrac{\partial l(\mu,\sigma)}{\partial\mu}\\ \dfrac{\partial l(\mu,\sigma)}{\partial\sigma}\end{pmatrix} \text{and} \quad I(\mu,\sigma)=\begin{pmatrix}\dfrac{-\partial^2 l(\mu,\sigma)}{\partial\mu^2} & \dfrac{-\partial^2 l(\mu,\sigma)}{\partial\mu\partial\sigma}\\ \dfrac{-\partial^2 l(\mu,\sigma)}{\partial\mu\partial\sigma} & \dfrac{-\partial^2 l(\mu,\sigma)}{\partial\sigma^2}\end{pmatrix}$$

The Wald statistics for μ and σ are given respectively by

$$m_1(\mu)=(\hat{\mu}-\mu)^2(I^{11}(\mu,\hat{\sigma}))^{-1}$$
$$S_1(\sigma)=(\hat{\sigma}-\sigma)^2(I^{22}(\hat{\mu},\hat{\sigma}))^{-1}$$

The Rao statistics for μ and σ are given respectively by

$$m_2(\mu)=(U_1(\mu,\tilde{\sigma}))^2 I^{11}(\mu,\tilde{\sigma}),$$
$$S_2(\sigma)=(U_2(\tilde{\mu},\sigma))^2 I^{22}(\tilde{\mu},\sigma)$$

The likelihood ratio statistics for μ and σ are given respectively by

$$m_3(\mu)=2(l(\hat{\mu},\hat{\sigma})-l(\mu,\tilde{\sigma}),$$
$$S_3(\sigma)=2(l(\hat{\mu},\hat{\sigma})-l(\tilde{\mu},\sigma))$$

Where $l(\mu,\sigma)$ = $\ln(L(\mu,\sigma))$ is the log-likelihood function, $\hat{\mu}$ and $\hat{\sigma}$ are the maximum likelihood estimator of μ and σ. $\hat{\mu}$ is the maximum likelihood estimator of μ for a given value of σ. I^{ij} is the *ij*-th element of I^{1}th element of I-1, the inverse of the information matrix.

Finite Sample Performance of the Likelihood Statistics

In this section we shall describe a simulation study conducted to investigate the finite sample behaviour of the maximum likelihood estimator, and confidence intervals based on the Wald, the Rao, and the likelihood ratio statistics.

The criteria used for the evaluation of the performance of the maximum likelihood estimator are the bias, the finite sample variance, and the adequacy of the asymptotic variance estimates (Elperin and Gertsbakh 1987). For the confidence intercals, we use the attainment of the nominal error probabilities and the symmetry of lower and upper error probabilities (Jennings 1987). Attainment of nominal error probabilities and our conclusions therefore are imprecise and can be misleading. The symmetry of lower and upper error probabilities means that of the interval fails to contain the true value of the parameter, it is equally expect this symmetry because they are using symmeteric percentiles of the approximating distributions to form their confidence intervals.

TABLE 1
Values of the bias, finite sample variance, mean squared error and asymptotic variance for the maximum likelihood estimator

		Location Parameter				Scale Parameter			
CP	N	Bias	FSV	MSE	ASV	Bias	FSV	MSE	ASV
0.0	2.0	-59	436	437	449	-308	345	354	339
	40	-40	219	219	225	176	163	166	171
	60	-27	152	152	151	-113	115	116	115
	80	6	110	110	113	-80	82	82	87
	100	8	88	88	91	-77	68	68	69
	120	11	76	76	76	-69	58	59	58
	140	13	64	64	65	-60	49	50	50
0.1	60	-26	152	152	152	-94	126	127	128
	80	6	110	110	114	-71	89	90	96
	100	9	87	87	91	-69	74	75	77
	120	11	76	76	76	-65	63	63	64
	140	13	64	64	65	-58	53	54	55
0.3	60	-13	157	157	160	-69	170	171	175
	80	23	114	114	120	-35	129	129	132
	100	21	90	90	96	-42	103	103	104
	120	23	77	78	79	-38	86	86	87
	140	24	66	66	68	-33	73	74	74
0.5	60	22	203	203	208	-41	269	269	274
	80	53	143	144	154	-5	197	197	205
	100	56	116	117	122	-2	162	162	163
	120	57	103	104	101	0	133	133	135
	140	56	87	88	86	4	113	113	115

However, symmetry of error probabilities may not occur due to the skewness of the actual sampling distribution (Jennings 1987).

In the simulations we use values of the sample size N as 60, 80, 100, 120 and 140. In each case we examined the censoring proportions 0.1, 0.3, and 0.5. For every combination of the sample size and censoring proportion we generated 2000 samples which are used to determine, the bias, the finite sample variance, the asymptotic variance of the maximum likelihood estimator, the mean squared errors of the estimators (MSE), and the error probabilities of the confidence intervals obtained from the Wald the Rao, and the likelihood ratio statistics. The levels of significance used are $\alpha= 0.05$, and 0.1. The results are given in Tables 1, 2 and 3. All values in the tables are multiplied by 10000.

FINDINGS AND CONCLUSIONS

Concerning the behaviour of the maximum likelihood estimator for the location and scale parameters, it appears that it is almost unbiased. It appears

TABLE 2
Lower and upper error probabilities of confidence intervals based on the Wald, Rao and likelihood ratio statistics. a = 0.1

		Location Parameter						Scale Parameter					
		Wald		Rao		LR		Wald		Rao		LR	
CP	N	L	U	L	U	L	U	L	U	L	U	L	U
0.0	20	300	235	100	95	255	195	995	20	0	525	435	115
	40	350	225	195	130	315	190	730	30	0	445	340	115
	60	310	290	225	200	285	265	590	50	0	430	345	140
	80	225	210	180	185	210	210	490	70	20	340	315	175
	100	225	225	190	195	220	220	500	65	55	375	335	155
	120	260	250	210	220	250	240	515	85	80	380	365	155
	140	275	275	240	235	265	260	480	110	55	395	340	180
0.1	60	310	275	215	190	290	255	585	40	0	490	310	155
	80	235	210	175	185	215	210	475	55	15	385	300	140
	100	220	225	175	200	215	220	540	65	40	390	325	140
	120	250	250	210	210	240	250	505	55	65	375	37	165
	140	290	275	240	240	270	265	450	90	55	380	340	175
0.3	60	295	210	185	255	260	225	570	25	0	570	330	185
	80	260	195	140	220	235	210	470	65	10	550	290	230
	100	260	220	165	255	235	235	510	85	20	445	310	225
	120	280	220	210	260	255	230	485	75	35	495	330	185
	140	295	220	220	270	275	245	475	120	30	485	265	230
0.5	60	365	85	45	520	295	275	610	40	0	615	295	195
	80	300	115	60	435	220	240	545	50	0	610	250	190
	100	295	110	65	425	225	215	595	90	0	560	275	235
	120	360	165	105	470	255	250	510	105	0	500	300	210
	140	340	165	105	460	235	265	460	105	0	530	270	210

also that the bias decreases when increasing the sample size, as anticipated from the asymptotic unbiasedness of maximum likelihood estimators. The bias and variance increase when increasing the censoring proportion. The variance of the maximum likelihood estimator decreases as the sample size increases, this is because of its consistency (Rao 1973). The asymptotic approximation to the variance of the maximum likelihood estimator provided by the observed information matrix seems to hold very well in this model, even for small samples with high censoring level. This also shows the high efficiency of the maximum likelihood estimator in this model.

For the interval estimation of the location parameter, it is clear that intervals based on the Wald statistics perform well. However, as the censoring proportion increases they tend to be asymmetric, especially when the sample size is small. Intervals based in the Rao statistics are generally conservative, that is, having an actual coverage probability that is less than the nominal. Moreover they tend

TABLE 3
Lower and upper error probabilities of confidence intervals based on the Wald, Rao and likelihood ratio statistics. a = 0.1

		Location Parameter						Scale Parameter					
		Wald		Rao		LR		Wald		Rao		LR	
CP	N	L	U	L	U	L	U	L	U	L	U	L	U
0.0	20	555	555	340	285	505	515	1340	100	0	735	865	310
	40	615	535	470	385	590	500	1055	115	25	635	780	270
	60	590	505	520	450	570	495	980	180	135	700	655	330
	80	480	445	445	425	475	435	800	230	180	565	580	315
	100	470	505	430	460	460	495	795	230	240	655	640	365
	120	535	490	490	460	525	475	810	275	325	640	675	380
	140	505	500	470	450	500	495	785	280	270	625	625	425
0.1	60	570	510	495	460	570	505	930	165	110	705	660	385
	80	495	440	430	430	455	440	770	190	150	605	575	340
	100	475	520	415	445	470	490	820	200	215	690	66-	375
	120	525	490	475	455	520	480	860	235	290	650	650	360
	140	490	495	460	480	485	495	775	235	270	665	635	390
0.3	60	605	440	375	515	515	470	930	170	40	820	605	400
	80	505	370	375	465	475	420	830	255	105	745	560	425
	100	465	455	405	520	445	485	820	245	160	725	590	410
	120	515	435	430	500	500	455	815	240	165	7-5	630	440
	140	535	490	475	540	525	510	785	315	155	735	580	450
0.5	60	705	300	290	740	570	460	935	145	0	955	565	510
	80	520	285	235	715	435	415	820	165	10	860	555	465
	100	545	280	255	740	470	455	885	250	40	840	63	410
	120	615	370	335	730	520	500	820	255	80	700	585	380
	140	615	385	320	745	495	530	785	285	90	790	540	440

to be asymmetric as the censoring proportion increases. On the other hand, interval based on likelihood ratio statistics tend to attain their nominal coverage probability and are symmetric in almost all situations considered.

For the scale parameter, it is clear that intervals based on the Wald and the Rao statistics are highly asymmetric, even for samples as large as 140, and hence cannot be recommended for use in practice. While intervals based on the likelihood ratio statistics still attain their nominal coverage probability and are symmetric in almost all situations considered.

The high asymmetry of Wald and Rao intervals indicates that the actual sampling distributions of the Wald and the Rao statistics are highly skewed (Jennings 1986), and they require a large sample size for the asymptotic chi-squared approximation to hold well. While the likelihood ratio statistics appears to need much lesser sample size to justify its use.

This shows that the likelihood ratio statistics converges of its limiting distribution much faster than the Wald and the Rao statistics.

The likelihood ratio statistics are applicable for one sided interval estimation and for one sided hypotheses testing because of their symmetric lower and upper error probabilities. This is, however, not the case for Wald and Rao statistics unless the sample size is high.

For all kinds of intervals considered it appears that as the sample size increases, all intervals tend to have error probabilities that are more symmetric and closer to the nominal ones. Also, larger nominal error probabilities are attained faster than smaller error probabillities.

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