ISSN: 0128-7680
Pertanika J. Sci. & Technol. 12(1): 69 - 80 (2004)

Trajectories of Random Quadratic Operators of the Random Mendelian Model of Heredity

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Received: 24 November 2001

ABSTRAK

Model rawak Mendelian bagi keturunan adalah apabila operator kuadratik stokastik yang menakrifkan model ini mengambil dua nilai V_{α} dan V_{β}, iaitu $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. Kertas ini memberikan keterangan lengkap perilaku trajektori bagi operator kuadratik rawak.

ABSTRACT

It is considered random Mendelian model of heredity, when the random quadratic stochastic operators, which define this model, admit two values and V_{β}, where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$. This paper provides a full description of the behaviour of random trajectories of random quadratic operators.

Keywords: Random Mendelian model, model of heredity, stochastic operator

INTRODUCTION

Genetic Motivation

Before we discuss the mathematics of genetics, we need to acquaint ourselves with some necessary language from biology. A vague, but nevertheless informative definition of a gene is that it is simply a unit of hereditary information. The genetic code of an organism is carried in its chromosomes. Each gene in a chromosome has a different form that it can take. These forms are called alleles. Alleles A of the so-called normal or wild type in a population are most widespread. Any other forms are considered as mutable alleles a. For simplicity, we will assume that there are two forms $\{A,a\}$ of a gene, that is, the mutable allele a is unique.

Let us consider a population of N organisms with a single gene and let N_1 of them have allele A and N_2 of them have allele a, where $N = N_1 + N_2$. Assume that $\frac{N_1}{N} = X_A$ is a fraction of organisms while allele A and $\frac{N_2}{N} = X_a$ is the fraction of the organisms with allele a. Clearly, $X_A \geq 0$, $X_a \geq 0$ and $X_A + X_a = 1$.

Now assume $p_{AA,A}$ is the probability that parents with allele A interbreed to produce organisms with allele A. By simple Mendelian inheritance, we have

$$
\begin{aligned}
&p_{AA,A}=1 \quad p_{Aa,A}=p_{aA,A}=\frac{1}{2} \quad p_{aa,A}=0\\
&p_{AA,a}=0 \quad p_{Aa,a}=p_{aA,a}=\frac{1}{2} \quad p_{aa,A}=1
\end{aligned} \tag{1}
$$

These numbers are called coefficients of heredity. In random interbreeding (panmixia) the fractions of the organisms with alleles A and a in the next generation are found from the formulae of *complete probability*, i.e. the parent pairs with alleles A and A (respectively A and a, a and a) arise for a fixed state x_A, x_A with probability $x_A x_A = x_A^2$ (respectively $x_A x_a = x_a x_a$ and $x_a x_a = x_a^2$), so

$$
\begin{aligned}
X_A' &= p_{AA,A}X_A^2 + 2p_{Aa,A}X_A X_a + p_{aa,A}X_a^2\\
X_a' &= p_{AA,a}X_A^2 + 2p_{Aa,a}X_A X_a + p_{aa,a}X_a^2
\end{aligned} \tag{2}
$$

Substituting (1) in (2), we then have

$$
\begin{aligned}
X_A' &= X_A^2 + X_A X_a = X_A(X_A + X_a)\\
X_a' &= X_A X_a + X_A^2 = X_a(X_A + X_a)
\end{aligned}
$$

that is, under simple Mendelian inheritance, the fractions of organisms with alleles A and a are invariant. The same goes for all new generations.

We will consider the generalised Mendelian model of heredity, which is defined by the following coefficients of heredity:

$$
\begin{aligned}
&p_{AA,A}=1 \quad p_{Aa,A}=p_{aA,A}=\alpha \quad p_{aa,A}=0\\
&p_{AA,a}=0 \quad p_{Aa,a}=p_{aA,a}=1-\alpha \quad p_{aa,A}=1
\end{aligned} \tag{3}
$$

where $0 \le \alpha \le 1$. If $\alpha = \frac{1}{2}$, the generalised Mendelian model of heredity reduces to simple model (2).

For simplicity, let us enumerate $A \to 1$ and $a \to 2$. The set

$$
S^1 = \{(X_1, X_2): X_1 \ge 0, X_2 \ge 0, X_1 + X_2 = 1\}
$$

is called a one dimensional simplex (see *Fig. 1*).

A transformation $V_\alpha : S^1 \to S^1$ which is defined as follows:

$$
\begin{aligned}
X_1' &= X_1^2 + 2\alpha\, X_1 X_2\\
X_2' &= X_2^2 + 2(1-\alpha)\, X_1 X_2
\end{aligned} \tag{4}
$$

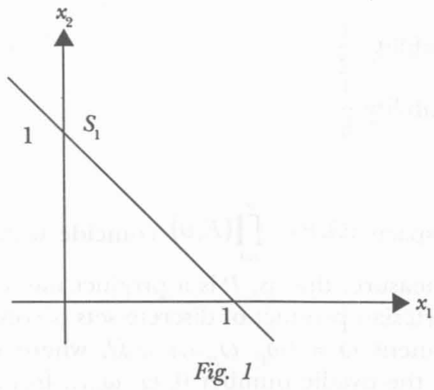

Fig. 1

where $0 \le \alpha \le 1$ is called a quadratic stochastic operator (q.s.o), which is determined by the coefficients of heredity (3).

The trajectory of a point $\underline{X}^{(0)} \in S^1$, where $\underline{X}^{(t)} = (\underline{X}_1^{(t)}, \underline{X}_1^{(t)})$, $t = 0,1, \ldots, n$, is defined as a sequence of iterations $\underline{X}^{(1)} = V_\alpha \underline{X}^{(0)}$, $\underline{X}^{(2)} = V_\alpha \underline{X}^{(1)}, \ldots, \underline{X}^{(n)} = V_\alpha \underline{X}^{(n-1)}$ for $n > 2$. One of the main problems in mathematical biology involves the study of the asymptotic behaviour of the trajectories of the q.s.o. [1:3]. The fixed points of the q.s.o. are called equilibrium states of the population. From the definition of q.s.o. (4), we obtain the following:

Theorem 1

Let V_α be a q.s.o. on S^1, where $0 \le \alpha \le 1$.

(i) If $0 \le \alpha < \frac{1}{2}$, then (0,1) and (1,0) are fixed points of q.s.o. (4) and for any $\underline{X}^{(0)} \in \text{Int } S^1$, the trajectory of $\underline{X}^{(0)}$ converges to (0,1);

(ii) If $\frac{1}{2} < \alpha \le 1$, then (0,1) and (1,0) are fixed points of q.s.o. (4) and for any $\underline{X}^{(0)} \in \text{Int } S^1$, the trajectory of $\underline{X}^{(0)}$ converges to (1,0);

(iii) If $\alpha = \frac{1}{2}$, V_α is an identity transformation.

The proof to Theorem 1 is trivial, following from the definition of q.s.o. (4).

The Random Models

Let $T_o = V_\alpha$ and $T_1 = V_\beta$ be two q.s.o. (4). Let us consider the probability space (E, υ) where $E = \{0,1\}$ and $\upsilon(0) = \upsilon(1) = \frac{1}{2}$. A random quadratic stochastic operator T is defined as follows:

$$T = \begin{cases} T_o \text{ with probability } \frac{1}{2} \\ T_1 \text{ with probability } \frac{1}{2} \end{cases} \tag{5}$$

Let the probabilistic space $(\Omega, P) = \prod_{i=1}^{\infty}(E, \upsilon)$ coincide with segment [0,1) with the usual Lebesque measure, that is, P is a product measure. It is well known that the countable Cartesian product of discrete sets is continuous. In our case, $E = \{0,1\}$ and an element $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, where $\omega_n \in \{0,1\}$, so every $\omega \in \Omega$ corresponds to the dyadic number $0, \omega_1, \omega_2, \ldots$ from [0,1). Then, for an arbitrary $\omega = (\omega_1, \omega_2, \ldots) \in \Omega$, we assume

$$\{T_n(\omega)\}_0^{\infty} = \left(T_{\omega_0}, T_{\omega_1}, T_{\omega_2}, \ldots\right), \tag{6}$$

where for any i, $\omega_i \in \{0,1\}$.

Let $\underline{X}^{(0)} \in S^1$ be any arbitrary initial distribution. The random trajectory of this point is defined in the following way:

$$\underline{X}^{(n+1)} = T_n(\omega)\underline{X}^{(n)} \qquad \text{where } n = 0,1,2,\ldots .$$

Now we give a full description of the random trajectories of the random q.s.o. (5) and (6).

Theorem 2

Let T be a random q.s.o (5) and (6). Then (0,1) and (1,0) are the fixed points of this operator and

(i) if $\alpha + \beta < 1$, where $0 \le \alpha < \frac{1}{2}$ and $0 \le \beta < \frac{1}{2}$, and $0 \le \alpha < \frac{1}{2}$ and $\beta = \frac{1}{2}$ or $\alpha = \frac{1}{2}$ and $0 \le \beta < \frac{1}{2}$, then for arbitrary initial distribution $\underline{X}^{(0)} \in \text{Int } S^1$, almost all random trajectories of this point converge to (0,1).

(ii) if $\alpha + \beta < 1$, $\frac{1}{2} < \alpha \le 1$ and $\frac{1}{2} < \beta \le 1$, and $\frac{1}{2} < \alpha \le 1$ and $\beta = \frac{1}{2}$ or $\alpha = \frac{1}{2}$ and $\frac{1}{2} < \beta \le 1$, then for arbitrary initial distribution $\underline{X}^{(0)} \in \text{Int } S^1$, almost all random trajectories of this point converge to (1,0).

(iii) if $\alpha + \beta = 1$, where $\alpha = \beta$, then arbitrary initial distribution $\underline{X}^{(0)} \in \text{Int } S^1$, is a fixed point.

Proof: Let us consider the following *Fig. 2.*

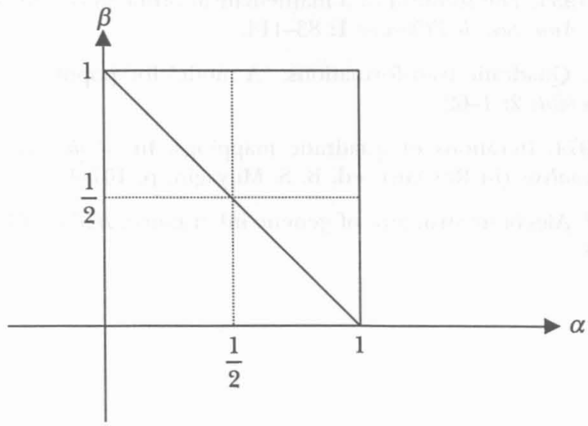

Fig. 2

(i) The proof of Theorem 2(i) follows from Theorem 1.
(ii) The proof of Theorem 2(ii) also follows from Theorem 1.
(iii) The case of $\alpha + \beta = 1$ when $\alpha = \beta = \frac{1}{2}$, is a trivial case.

We have carried out extensive numerical simulations (more than 10000 iterations for each case, please refer to appendix) and we observed that:

(i) the result of Theorem 2(i) also holds for $\left\{(\alpha,\beta): 0 \le \alpha < \frac{1}{2} \text{ and } \frac{1}{2} < \beta \le 1\right\}$, or $\left\{(\alpha,\beta): \frac{1}{2} < \alpha \le 1 \text{ and } 0 \le \beta < \frac{1}{2}\right\}$.

(ii) the result of Theorem 2(iii) also holds for $\left\{(\alpha,\beta): 0 \le \alpha < \frac{1}{2} \text{ and } \frac{1}{2} < \beta \le 1\right\}$, or $\left\{(\alpha,\beta): \frac{1}{2} < \alpha \le 1 \text{ and } 0 \le \beta < \frac{1}{2}\right\}$.

(iii) if $\alpha + \beta = 1$, where $\alpha \neq \beta$, then for arbitrary initial distribution $\underline{X}^{(0)} \in \text{Int } S^1$, almost all random trajectories of this point do not converge.

CONCLUSION

In this paper, we have obtained the behaviour of random trajectories of a random q.s.o.. For all (α,β) which are in the neighbourhood of $\alpha + \beta = 1$, we observe that the random trajectories of an arbitrary initial point may either converge very slowly or may not converge at all. A sample of the numerical results for all the cases mentioned above, are shown in the appendix.

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APPENDIX

The numerical simulations for each of the case mentioned on page 7 were carried out by selecting, arbitrarily, 100 different combinations of the values of α and β. This is to ensure that the results obtained are representative of the search space under consideration. In the simulation, for each combination of α and β, 10 arbitrary starting points $(x_1^{(0)}, x_2^{(0)})$ were chosen. They were selected such that they represent different areas of the search space.

We run the simulations for 50 iterations although larger numbers of iterations were also examined. However, we observed that 50 iterations are sufficient to examine the convergence of each starting point for all the cases considered.

Selected numerical results for all the cases that cannot be proven analytically are given below. We note that for presentation purposes we have run the programme for 20 iterations and for three selected starting points ((0.3000,0.7000), (0.9500,0.0500), (0.5000,0.5000)).

(I) $\alpha + \beta < 1$

(i) $0 \leq \alpha < \frac{1}{2};\ \frac{1}{2} < \beta \leq 1$

n	$\alpha = 0.4;\ \beta = 0.5$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.3000	0.7000	0.9500	0.0500	0.5250	0.4750
2	0.2580	0.7420	0.9500	0.0500	0.4751	0.5249
3	0.2580	0.7420	0.9500	0.0500	0.5001	0.4999
4	0.2197	0.7803	0.9500	0.0500	0.4501	0.5499
5	0.2197	0.7803	0.9405	0.0595	0.4006	0.5994
6	0.2197	0.7803	0.9405	0.0595	0.3525	0.6475
7	0.2197	0.7803	0.9293	0.0707	0.3754	0.6246
8	0.1854	0.8146	0.9162	0.0838	0.3988	0.6012
9	0.1552	0.8448	0.9162	0.0838	0.4228	0.5772
10	0.1290	0.8710	0.9162	0.0838	0.4472	0.5528
11	0.1065	0.8935	0.9008	0.0992	0.3977	0.6023
12	0.1065	0.8935	0.8829	0.1171	0.3498	0.6502
13	0.0875	0.9125	0.8829	0.1171	0.3726	0.6274
14	0.0875	0.9125	0.8623	0.1377	0.3960	0.6040
15	0.0715	0.9285	0.8385	0.1615	0.3481	0.6519
16	0.0715	0.9285	0.8385	0.1615	0.3708	0.6292
17	0.0582	0.9418	0.8114	0.1886	0.3942	0.6058
18	0.0582	0.9418	0.7808	0.2192	0.3464	0.6536
19	0.0582	0.9418	0.7466	0.2534	0.3690	0.6310
20	0.0582	0.9418	0.7466	0.2534	0.3225	0.6775

n	$\alpha = 0.1;\ \beta = 0.7$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.3840	0.6160	0.9690	0.0310	0.6000	0.4000
2	0.1948	0.8052	0.9810	0.0190	0.4080	0.5920
3	0.2575	0.7425	0.9885	0.0115	0.5046	0.4954
4	0.1045	0.8955	0.9930	0.0070	0.3046	0.6954
5	0.1420	0.8580	0.9875	0.0125	0.1352	0.8648
6	0.1907	0.8093	0.9924	0.0076	0.0416	0.9584
7	0.2525	0.7475	0.9864	0.0136	0.0576	0.9424
8	0.1015	0.8985	0.9757	0.0243	0.0793	0.9207
9	0.0285	0.9715	0.9852	0.0148	0.1085	0.8915
10	0.0064	0.9936	0.9910	0.0090	0.1473	0.8527
11	0.0013	0.9987	0.9839	0.0161	0.0468	0.9532
12	0.0018	0.9982	0.9712	0.0288	0.0111	0.9889
13	0.0004	0.9996	0.9824	0.0176	0.0155	0.9845
14	0.0005	0.9995	0.9686	0.0314	0.0216	0.9784
15	0.0001	0.9999	0.9442	0.0558	0.0047	0.9953
16	0.0001	0.9999	0.9653	0.0347	0.0066	0.9934
17	0.0000	1.0000	0.9385	0.0615	0.0092	0.9908
18	0.0000	1.0000	0.8923	0.1077	0.0019	0.9981
19	0.0001	0.9999	0.8155	0.1845	0.0027	0.9973
20	0.0001	0.9999	0.8757	0.1243	0.0005	0.9995

(ii) $\frac{1}{2} < \alpha < 1;\ 0 \leq \beta \leq \frac{1}{2}$

n	$\alpha = 0.65;\ \beta = 0.3$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.2160	0.7840	0.9310	0.0690	0.5750	0.4250
2	0.2668	0.7332	0.9503	0.0497	0.6483	0.3517
3	0.3255	0.6745	0.9314	0.0686	0.5571	0.4429
4	0.3914	0.6086	0.9058	0.0942	0.6311	0.3689
5	0.2961	0.7039	0.9314	0.0686	0.5380	0.4620
6	0.3586	0.6414	0.9506	0.0494	0.4386	0.5614
7	0.4276	0.5724	0.9318	0.0682	0.3401	0.6599
8	0.3297	0.6703	0.9063	0.0937	0.2503	0.7497
9	0.2413	0.7587	0.9318	0.0682	0.3066	0.6934
10	0.1681	0.8319	0.9064	0.0936	0.3704	0.6296
11	0.2100	0.7900	0.9318	0.0682	0.4404	0.5596
12	0.2598	0.7402	0.9509	0.0491	0.3418	0.6582
13	0.3175	0.6825	0.9649	0.0351	0.4093	0.5907
14	0.3825	0.6175	0.9751	0.0249	0.3126	0.6874
15	0.4533	0.5467	0.9824	0.0176	0.2266	0.7734
16	0.3542	0.6458	0.9876	0.0124	0.2792	0.7208
17	0.2627	0.7373	0.9826	0.0174	0.1987	0.8013
18	0.1852	0.8148	0.9878	0.0122	0.2465	0.7535
19	0.2305	0.7695	0.9914	0.0086	0.3022	0.6978
20	0.1596	0.8404	0.9880	0.0120	0.2178	0.7822

n	$\alpha = 0.6;\ \beta = 0.1$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.1320	0.8680	0.9120	0.0880	0.5500	0.4500
2	0.1549	0.8451	0.9281	0.0719	0.5995	0.4005
3	0.1811	0.8189	0.8746	0.1254	0.4074	0.5926
4	0.2108	0.7892	0.7869	0.2131	0.4557	0.5443
5	0.0777	0.9223	0.8204	0.1796	0.2573	0.7427
6	0.0920	0.9080	0.8499	0.1501	0.1044	0.8956
7	0.1087	0.8913	0.7479	0.2521	0.0296	0.9704
8	0.0312	0.9688	0.5970	0.4030	0.0066	0.9934
9	0.0070	0.9930	0.6451	0.3549	0.0079	0.9921
10	0.0014	0.9986	0.4620	0.5380	0.0095	0.9905
11	0.0017	0.9983	0.5117	0.4883	0.0114	0.9886
12	0.0021	0.9979	0.5617	0.4383	0.0024	0.9976
13	0.0025	0.9975	0.6109	0.3891	0.0029	0.9971
14	0.0030	0.9970	0.6584	0.3416	0.0006	0.9994
15	0.0036	0.9964	0.7034	0.2966	0.0001	0.9999
16	0.0007	0.9993	0.7451	0.2549	0.0001	0.9999
17	0.0001	0.9999	0.5932	0.4068	0.0000	1.0000
18	0.0000	1.0000	0.6415	0.3585	0.0000	1.0000
19	0.0000	1.0000	0.6875	0.3125	0.0000	1.0000
20	0.0000	1.0000	0.5156	0.4844	0.0000	1.0000

(II) $\alpha + \beta > 1$

(i) $0 \leq \alpha < \frac{1}{2};\ \frac{1}{2} < \beta \leq 1$

n	$\alpha = 0.3;\ \beta = 0.75$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.4050	0.5950	0.9737	0.0262	0.6250	0.3750
2	0.3086	0.6914	0.9865	0.0135	0.5313	0.4688
3	0.4153	0.5847	0.9932	0.0068	0.6558	0.3442
4	0.3182	0.6818	0.9966	0.0034	0.5655	0.4345
5	0.4266	0.5734	0.9952	0.0048	0.4672	0.5328
6	0.5489	0.4511	0.9976	0.0024	0.3676	0.6324
7	0.6727	0.3273	0.9966	0.0034	0.4838	0.5162
8	0.5847	0.4153	0.9953	0.0047	0.6087	0.3913
9	0.4875	0.5125	0.9976	0.0024	0.7278	0.2722
10	0.3876	0.6124	0.9988	0.0012	0.8269	0.1731
11	0.2927	0.7073	0.9983	0.0017	0.7696	0.2304
12	0.3962	0.6038	0.9977	0.0023	0.6987	0.3013
13	0.3005	0.6995	0.9988	0.0012	0.8039	0.1961
14	0.4056	0.5944	0.9984	0.0016	0.8827	0.1173
15	0.3091	0.6909	0.9977	0.0023	0.8413	0.1587
16	0.4159	0.5841	0.9989	0.0011	0.9081	0.0919
17	0.3188	0.6812	0.9984	0.0016	0.9498	0.0502
18	0.4273	0.5727	0.9978	0.0022	0.9308	0.0692
19	0.5497	0.4503	0.9969	0.0031	0.9630	0.0370
20	0.6735	0.3265	0.9984	0.0016	0.9487	0.0513

n	$\alpha = 0.4;\ \beta = 0.8$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.4260	0.5740	0.9785	0.0215	0.6500	0.3500
2	0.3771	0.6229	0.9911	0.0089	0.6045	0.3955
3	0.5180	0.4820	0.9964	0.0036	0.7479	0.2521
4	0.4681	0.5319	0.9986	0.0014	0.7102	0.2898
5	0.6175	0.3825	0.9983	0.0017	0.6691	0.3309
6	0.7592	0.2408	0.9993	0.0007	0.6248	0.3752
7	0.8689	0.1311	0.9992	0.0008	0.7655	0.2345
8	0.8461	0.1539	0.9990	0.0010	0.8732	0.1268
9	0.8201	0.1799	0.9996	0.0004	0.9396	0.0604
10	0.7906	0.2094	0.9998	0.0002	0.9737	0.0263
11	0.7574	0.2426	0.9998	0.0002	0.9685	0.0315
12	0.8677	0.1323	0.9998	0.0002	0.9624	0.0376
13	0.8447	0.1553	0.9999	0.0001	0.9841	0.0159
14	0.9234	0.0766	0.9999	0.0001	0.9935	0.0065
15	0.9093	0.0907	0.9999	0.0001	0.9922	0.0078
16	0.9588	0.0412	0.9999	0.0001	0.9968	0.0032
17	0.9509	0.0491	0.9999	0.0001	0.9987	0.0013
18	0.9789	0.0211	0.9999	0.0001	0.9985	0.0015
19	0.9913	0.0087	0.9999	0.0001	0.9994	0.0006
20	0.9965	0.0035	1.0000	0.0000	0.9993	0.0007

(ii) $\frac{1}{2} < \alpha < 1;\ 0 \le \beta \le \frac{1}{2}$

n	$\alpha = 0.95;\ \beta = 0.1$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.1320	0.8680	0.9120	0.0880	0.7250	0.2750
2	0.2351	0.7649	0.9842	0.0158	0.9044	0.0956
3	0.3970	0.6030	0.9718	0.0282	0.8353	0.1647
4	0.6124	0.3876	0.9499	0.0501	0.9591	0.0409
5	0.4225	0.5775	0.9927	0.0073	0.9277	0.0723
6	0.6421	0.3579	0.9992	0.0008	0.8741	0.1259
7	0.8489	0.1511	0.9986	0.0014	0.7861	0.2139
8	0.7464	0.2536	0.9975	0.0025	0.6516	0.3484
9	0.5949	0.4051	0.9997	0.0003	0.8559	0.1441
10	0.4021	0.5979	0.9995	0.0005	0.9669	0.0331
11	0.6185	0.3815	1.0000	0.0000	0.9957	0.0043
12	0.8309	0.1691	1.0000	0.0000	0.9923	0.0077
13	0.9573	0.0427	1.0000	0.0000	0.9992	0.0008
14	0.9941	0.0059	1.0000	0.0000	0.9985	0.0015
15	0.9994	0.0006	1.0000	0.0000	0.9973	0.0027
16	0.9989	0.0011	1.0000	0.0000	0.9997	0.0003
17	0.9980	0.0020	1.0000	0.0000	0.9995	0.0005
18	0.9964	0.0036	1.0000	0.0000	1.0000	0.0000
19	0.9996	0.0004	1.0000	0.0000	1.0000	0.0000
20	0.9993	0.0007	1.0000	0.0000	1.0000	0.0000

n	$\alpha = 0.85;\ \beta = 0.5$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.3000	0.7000	0.9500	0.0500	0.6750	0.3250
2	0.4470	0.5530	0.9832	0.0167	0.8286	0.1714
3	0.6200	0.3800	0.9832	0.0167	0.8286	0.1714
4	0.7849	0.2151	0.9832	0.0167	0.9280	0.0720
5	0.7849	0.2151	0.9948	0.0052	0.9280	0.0720
6	0.9031	0.0969	0.9984	0.0016	0.9280	0.0720
7	0.9644	0.0356	0.9984	0.0016	0.9280	0.0720
8	0.9644	0.0356	0.9984	0.0016	0.9280	0.0720
9	0.9644	0.0356	0.9995	0.0005	0.9748	0.0252
10	0.9644	0.0356	0.9995	0.0005	0.9920	0.0080
11	0.9884	0.0116	0.9999	0.0001	0.9976	0.0024
12	0.9964	0.0036	1.0000	0.0000	0.9976	0.0024
13	0.9989	0.0011	1.0000	0.0000	0.9993	0.0007
14	0.9997	0.0003	1.0000	0.0000	0.9993	0.0007
15	0.9999	0.0001	1.0000	0.0000	0.9993	0.0007
16	0.9999	0.0001	1.0000	0.0000	0.9998	0.0002
17	0.9999	0.0001	1.0000	0.0000	0.9998	0.0002
18	0.9999	0.0001	1.0000	0.0000	0.9999	0.0001
19	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
20	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000

(III) $\alpha + \beta = 1;\quad \alpha \neq \beta$

(i) $0 \leq \alpha < \frac{1}{2};\ \frac{1}{2} < \beta \leq 1$

n	$\alpha = 0.3;\ \beta = 0.75$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.3630	0.6370	0.9642	0.0357	0.5750	0.4250
2	0.2936	0.7064	0.9746	0.0254	0.5017	0.4983
3	0.3559	0.6441	0.9820	0.0180	0.5767	0.4233
4	0.2871	0.7129	0.9873	0.0127	0.5035	0.4965
5	0.3485	0.6515	0.9836	0.0164	0.4285	0.5715
6	0.4166	0.5834	0.9884	0.0116	0.3550	0.6450
7	0.4895	0.5105	0.9850	0.0150	0.4237	0.5763
8	0.4145	0.5855	0.9805	0.0195	0.4969	0.5031
9	0.3417	0.6583	0.9863	0.0137	0.5719	0.4281
10	0.2743	0.7257	0.9903	0.0097	0.6454	0.3546
11	0.2145	0.7855	0.9875	0.0125	0.5767	0.4233
12	0.2651	0.7349	0.9837	0.0163	0.5035	0.4965
13	0.2066	0.7934	0.9885	0.0115	0.5785	0.4215
14	0.2558	0.7442	0.9851	0.0149	0.6516	0.3484
15	0.1987	0.8013	0.9807	0.0193	0.5835	0.4165
16	0.2465	0.7535	0.9864	0.0136	0.6564	0.3436
17	0.1908	0.8092	0.9824	0.0176	0.7241	0.2759
18	0.2371	0.7629	0.9772	0.0228	0.6642	0.3358
19	0.2913	0.7087	0.9705	0.0295	0.7311	0.2689
20	0.3533	0.6467	0.9791	0.0209	0.6721	0.3279

(ii) $\frac{1}{2} < \alpha \leq 1;\ 0 \leq \beta < \frac{1}{2}$

n	$\alpha = 0.85;\ \beta = 0.15$					
	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$	$x_1^{(n)}$	$x_2^{(n)}$
0	0.3000	0.7000	0.9500	0.0500	0.5000	0.5000
1	0.1530	0.8470	0.9167	0.0833	0.6750	0.3250
2	0.2437	0.7563	0.9702	0.0298	0.8286	0.1714
3	0.3727	0.6273	0.9499	0.0501	0.7291	0.2709
4	0.5364	0.4636	0.9166	0.0834	0.8674	0.1326
5	0.3623	0.6377	0.9701	0.0299	0.7869	0.2131
6	0.5241	0.4759	0.9904	0.0096	0.6695	0.3305
7	0.6987	0.3013	0.9838	0.0162	0.5146	0.4854
8	0.5513	0.4487	0.9726	0.0274	0.3397	0.6603
9	0.3781	0.6219	0.9912	0.0088	0.4967	0.5033
10	0.2135	0.7865	0.9852	0.0148	0.6717	0.3283
11	0.3311	0.6689	0.9954	0.0046	0.8261	0.1739
12	0.4861	0.5139	0.9986	0.0014	0.7255	0.2745
13	0.6610	0.3390	0.9996	0.0004	0.8649	0.1351
14	0.8178	0.1822	0.9999	0.0001	0.7831	0.2169
15	0.9221	0.0779	1.0000	0.0000	0.6642	0.3358
16	0.8718	0.1282	1.0000	0.0000	0.8203	0.1797
17	0.7936	0.2064	1.0000	0.0000	0.7172	0.2828
18	0.6790	0.3210	1.0000	0.0000	0.8592	0.1408
19	0.8315	0.1685	1.0000	0.0000	0.9439	0.0561
20	0.7335	0.2665	1.0000	0.0000	0.9068	0.0932