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The Use of a Negative Definite State-Weighting Matrix in Linear Optimal Aircraft Stability Augmentation System Problems

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ABSTRACT

Most of the published work on the Linear Quadratic Regulator (LQR) theory states it is necessary to restrict the state-weighting matrix in the quadratic performance index to be at least positive semi-definite (P.S.D). In this paper, a method of obtaining specified closed-loop eigenvalues is described which uses a procedure that results in a corresponding state-weighting matrix which can be negative definite (N.D). The value of this method for the design of aircraft Stability Augmentation Systems (S.A.S) is that it permits a designer to use a set of specified closed-loop eigenvalues which correspond to parameters given in those aircraft flying qualities specifications published by aviation authorities. Because these flying qualities are based on low order mathematical models corresponding to particular modes of flight, the choice of appropriate closed-loop eigenvalues is direct. The control law obtained from this method not only provides considerable robustness, but also results in the prescribed closed-loop dynamics. The method is illustrated by presenting the results of two examples. The effectiveness of the method is shown from the results obtained from digital simulation of the S.A.S for both aircraft.

Keywords: Regulator, negative definite state-weighing matrix, linear quadratie

INTRODUCTION

Applying the LQR theory to any linear control problem guarantees dynamic stability as long as the basic theoretical requirements have been satisfied. The theoretical requirements commonly stated in the literature (Kalman, R. E. 1960; Athens and Falb 1966; Anderson and Moore) relate to both the mathematical model of the plant and the performance index to be minimised. The requirements are:

1) The control-weighting matrix, G, has to be positive definite (P.D).
2) The state-weighting matrix, Q, has to be at least P.S.D.
3) The pair (A,B) of the system model has to be controllable.

The designer is free to choose the matrices, Q and G. The quadratic performance index is then minimised to obtain the solution of the Riccati equation and then the optimal feedback gain matrix for the optimal control law is found.

The problem often faced by designers is that considerable experience is required to choose the most appropriate Q and G matrices to result in the dynamics of the controlled aircraft satisfying the specified flying qualities. A systematic approach to obtain these weighting matrices is required, therefore. Based on the method described by Luo and Lan (1985), it was found that the Q matrix required to produce the necessary feedback gain matrix for the optimal control law could be N.D. With the use of such N.D matrices it was still possible to obtain an optimal control law to stabilise the aircraft.

THEORY

A linear system can be described by a state-space equation

$$\dot{x} = \mathrm{A}x + \mathrm{B}u \tag{1}$$

x is the state vector $\in \mathrm{R}^n$ and u is the control vector $\in \mathrm{R}^m$. A is the state matrix of order (n x n) and B is the control matrix of order (n x m). The LQR problem is to find the feedback gain matrix, K, to minimise the performance index

$$\mathrm{J} = \frac{1}{2}\int_0^\infty \left[x^{\mathrm{T}}\mathrm{Q}x + u^{\mathrm{T}}\mathrm{G}u\right]\mathrm{dt} \tag{2}$$

Q is the state-weighting matrix, of order (n x n) and G is the control-weighting matrix of order (m x m). Note that the upper limit of the integral in Eqn (2) is infinite. This ensures that a linear time-invariant feedback control law is obtained. The optimal linear control law is defined as

$$u^{\circ} = -\mathrm{K}x \tag{3}$$

It can be shown (Kalman, R. E. 1960; Athens and Falb 1966; Anderson and Moore) that

$$\mathrm{K} = \mathrm{G}^{-1}\mathrm{B}^{\mathrm{T}}\mathrm{P} \tag{4}$$

where P is the solution to the algebraic Riccati equation (A.R.E), viz.

$$\mathrm{A}^{\mathrm{T}}\mathrm{P} + \mathrm{P}^{\mathrm{T}}\mathrm{A} - \mathrm{PBG}^{-1}\,\mathrm{B}^{\mathrm{T}}\mathrm{P} + \mathrm{Q} = 0 \tag{5}$$

The eigenvalues of the closed-loop system, ζ_i, can be found by solving the characteristic equation:

$$\det\left[\zeta_i \mathrm{I} - (\mathrm{A} - \mathrm{B.K})\right] = 0 \tag{6}$$

I is an identity matrix and i = 1,2 … n. The solution of the LQR problem involves an Hamiltonian function viz,

$$\mathrm{H} = x^{\mathrm{T}}\mathrm{Q}x + u^{\mathrm{T}}\mathrm{G}u + \lambda_x^{\mathrm{T}}(\mathrm{A}x + \mathrm{B}u) \tag{7}$$

where λ_x is the vector of Lagrangian multipliers. The solution to the LQR problem can be obtained by solving the following equations:

$$\dot{\lambda}_x = -\frac{\partial \mathrm{H}}{\partial x} = -\mathrm{A}^{\mathrm{T}}\lambda_x - \mathrm{Q}x; \qquad \lambda_x(\infty) = 0 \tag{8}$$

$$\dot{x} = \frac{\partial \mathrm{H}}{\partial \lambda_x} = \mathrm{A}x + \mathrm{B}u; \qquad x(0) = x_0 \tag{9}$$

$$\frac{\partial \mathrm{H}}{\partial u} = \mathrm{G}u + \mathrm{B}'\lambda_x = 0 \tag{10}$$

These equations can be written as:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \end{bmatrix} = \begin{bmatrix} A & -BG^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \end{bmatrix} = \overline{A} \begin{bmatrix} x \\ \lambda_x \end{bmatrix} \tag{11}$$

where $\overline{A}$ is a (2n × 2n) matrix with n of its 2n eigenvalues being the eigenvalues of the closed-loop system that satisfy Eqn (12) viz.

$$\det\left[\sigma_i I - \overline{A}\right] = 0 \tag{12}$$

σ_i denotes the specified closed-loop eigenvalues. Using Bryson's theory (1975), the P.D control-weighting matrix, G, can be chosen to have a diagonal form having elements given by

$$G_{kk} = \frac{1}{u^2_{k,max}}, \quad \text{where } G_{kj} = 0 \text{ and } k \neq j \quad \text{where } k = 1,2 \ldots m \tag{13}$$

This choice penalises each of the control input u_1, u_2 & u_m. The values $u_{1,max}$, $u_{2,max}$, & $u_{m,max}$ represent the maximum limits of each of the control input.

The weighting matrix, Q, is also assumed to have a diagonal form, with its elements given by

$$Q_{ii} = q_i, \qquad \text{where } Q_{ij} = 0,\ i \neq j \tag{14}$$

Eqn (12) can be used to determine the n elements qi of the weighting matrix when all the closed-loop eigenvalues are specified. For a specified eigenvalue, $\sigma_1 = \mu_1 + j\omega_1$, Eqn (12) provides one equation for qi. Hence,

$$f(q_1, q_2 \ldots q_n) = \det\left[(\mu_1 + j\omega_1)I - \overline{A}\right] = 0 \tag{15}$$

As a result, n algebraic equations are obtained and can be solved for the unknown elements of the Q matrix. With the resulting weighting matrices, the Riccati equation can be solved and the optimal feedback control law can be obtained using Eqn (3) and (4).

EXAMPLE

F-15 Lateral Motion

This example is taken from (Luo and Lan 1985). The lateral dynamics of the F-15 flying at Mach 1.5 at 10000ft can be expressed as a state-space equation quoted as Eqn (16).

$$\begin{bmatrix} \dot{\beta} \\ \dot{p} \\ \dot{r} \\ \dot{\phi} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} -0.493 & 0.015 & -1 & 0.02 & 0 \\ -61.176 & -7.835 & 4.991 & 0 & 0 \\ 31.804 & -0.235 & -0.994 & 0 & 0 \\ 0 & 1 & -0.015 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ p \\ r \\ \phi \\ \xi \end{bmatrix} + \begin{bmatrix} -0.002 & 0.002 \\ 8.246 & 1.849 \\ 0.246 & -0.436 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_F \\ \delta_R \end{bmatrix} \tag{16}$$

β, p, r and ϕ represent the sideslip angle, roll rate, yaw rate and roll angle whereas δ_F and δ_R are deflection angles of the flap and rudder respectively. ξ is the output of the integrator which reduces sideslip error to zero.

The control-weighting matrix G was chosen to be:

$$G = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \tag{17}$$

and the desired closed-loop eigenvalues, σ_i, were specified as:

TABLE 1
Prescribed closed-loop eigenvalues for the F-15 Model

σ_1	=	-8	(Roll mode)
σ_2	=	-0.05	(Spiral mode)
$\sigma_{3,4}$	=	-4.88 ± j3.66	(Dutch Roll mode)
σ_5	=	-0.7	(Integrator)

Note that the specified closed-loop dynamics are stable. Eqn (11) and (12) were used to obtain the n algebraic equations. The symbolic numerical software MAPLE was used to solve the n algebraic equations to find the Q matrix needed to achieve the specified closed-loop eigenvalues. The algebraic equations were solved using Newton s method. The solution obtained was:

$$Q = \begin{bmatrix} -75.795 & 0 & 0 & 0 & 0 \\ 0 & 1.6659 & 0 & 0 & 0 \\ 0 & 0 & 366.62 & 0 & 0 \\ 0 & 0 & 0 & 4.2858 & 0 \\ 0 & 0 & 0 & 0 & 13.283 \end{bmatrix} \tag{18}$$

It is important to note that this Q matrix is **negative definite**: the use of such state-weighting matrices violates the usually stated requirement that the Q matrix has to be at least P.S.D. Luo and Lan 1985 recommended that the negative element of the Q matrix be set to zero to satisfy the requirement generally stated in the literature. If using the resultant feedback gain matrix, K, resulted in the eigenvalues of the closed-loop system being too different from those specified, those authors suggested choosing another set of prescribed closed-loop eigenvalues. In this paper, it was found wholly unnecessary to do this because use of the derived negative definite matrix, Q, in Eqn (18) still permits the determination of a feedback gain matrix for the optimal control law.

Using the MATLAB Control System Toolbox, the solution to the A.R.E and the linear feedback gain matrix, K, were found and are shown in Table 2. The solution to the A.R.E was found to be symmetrical and positive definite.

Using Eqn (6), the resulting closed-loop eigenvalues, ζ_i, were determined to be:

These closed-loop eigenvalues are identical to those specified. Moreover, using the negative definite Q matrix, without the modification proposed by Luo and Lan 1985, has provided a perfect match to the specified closed-loop eigenvalues.

TABLE 2
The solution to the A.R.E and the optimal feedback matrix gain for the F-15 lateral motion mode

$$P = \begin{bmatrix} 1048.9 & 8.0722\times10^{-1} & 17.308 & 11.725 & 66.152 \\ 8.0722\times10^{-1} & 1.7766\times10^{-1} & -1.8781\times10^{-1} & 6.9915\times10^{-1} & -6.1273\times10^{-2} \\ 17.308 & -1.8781\times10^{-1} & 35.767 & 1.0174 & -8.067 \\ 11.725 & 6.9915\times10^{-1} & 1.0174 & 6.8281 & -2.3277 \\ 66.152 & -6.1273\times10^{-2} & -8.067 & -2.3277 & 268.83 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.9853 & 0.1574 & 0.8136 & 0.6661 & -0.2940 \\ -3.9557 & 0.412 & -15.907 & 0.8726 & 3.5362 \end{bmatrix}$$

TABLE 3
Closed-loop eigenvalues of the S.A.S for the F-15 model

ζ_1	= -8	(Roll mode)
ζ_2	= -0.05	(Spiral mode)
$\zeta_{3,4}$	= -4.88 ± j3.66	(Dutch Roll mode)
ζ_5	= -0.7	(Integrator)

Using the software of the MATLAB Control System Toolbox to solve for the A.R.E, however, requires the state-weighting matrix to be at least P.S.D. This requirement was sometimes relaxed by MATLAB for some cases and a N.D state-weighting matrix could be used.

Whenever MATLAB failed to produce the Riccati solution, another method, based on Marshall and Nicholson (1970) but simplified by McLean, D, had to be used.

Longitudinal Motion of an Hypersonic Transport Aircraft

The second example involves the longitudinal motion of a hypersonic transport aircraft, Hyperion, flying at Mach 8 at 85000ft (7). The aircraft state-space equation can be represented by Eqn (1). The state vector, the control vector and the A and B matrices are given in Table 6. The state variables are the forward speed u, the angle of attack α, the rate of change of the pitch attitude q, the pitch attitude θ, the height h, the bending displacement η and the rate of change of bending displacement $\dot{\eta}$. The control variables are the flap δ_F, the engine duct area Ad and the engine temperature across combustor T_o. The specified eigenvalues were chosen to be as shown in Table 4.

TABLE 4
Prescribed closed-loop eigenvalues for Hyperion

$\sigma_{1,2}$	$= -5 \pm j18$	(Structural Bending)
$\sigma_{3,4}$	$= -40 \pm j12$	(Short Period)
$\sigma_{5,6}$	$= -0.04 \pm j0.012$	(Phugoid)
σ_7	$= -10.0$	(Height)

The control-weighting matrix chosen was the identity matrix:

$$G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

The software package MAPLE was again used to solve the algebraic equations for the Q matrix (in this case, seven equations) required to produce the desired eigenvalues. Using MATLAB Control System Toolbox, the solution to the A.R.E, P, and the linear feedback gain matrix, K, were obtained. The Q matrix, the solution to the A.R.E and the feedback gain matrix are shown in Table 7. Again, the state-weighting matrix obtained was N.D. The solution to the A.R.E is N.D also but symmetrical. Using Eqn (6), the closed-loop eigenvalues obtained were identical to those prescribed (see Table 5).

Dynamic Responses of Hyperion

Hyperion without any active feedback control system was found to be highly statically and dynamically unstable. The open-loop eigenvalues, λ_i, for Hyperion flying at speed Mach 8 at 85000ft are shown below.

It can be seen from Table 8 that the aircraft is highly, dynamically unstable. The first task of the automatic flight control system is to stabilise the aircraft. Once the aircraft has been stabilised, the dynamics of the aircraft should then be made to correspond to the specified flying qualities characterised by the specified eigenvalues.

By using the LQR theory, the controlled dynamics of Hyperion were made stable and by using the method of determining a particular Q matrix as discussed above, the aircraft was made to exhibit the eigenvalues specified. Shown below are the dynamic responses of Hyperion for three different situations. The rate of change of pitch attitude and bending displacement step responses are considered. Figure 1 shows the response of the basic unstable, uncontrolled aircraft. Figure 2 shows the response of the aircraft to a commanded height change of 1000ft with the optimal S.A.S displaying those closed-

TABLE 5
Hyperion's closed-loop eigenvalues

$\zeta_{1,2}$	$= -5 \pm j18$	(Structural Bending)
$\zeta_{3,4}$	$= -40 \pm j12$	(Short Period)
$\zeta_{5,6}$	$= -0.04 \pm j0.012$	(Phugoid)
ζ_7	$= -10.0$	(Height)

TABLE 6
The state and control vectors and the corresponding A and B matrices for Hyperion

The state and the control vectors for Hyperion are:

$$x' = (u \quad \alpha \quad q \quad \theta \quad h \quad \eta \quad \dot{\eta}); \quad u' = (\delta_F \quad A_d \quad T_o)$$

The coefficient matrices A and b for Hyperion flying at speed of Mach 8 at height 85000ft are:

A=

$$\begin{bmatrix} -4.1857\times10^{-3} & -3.5030\times10^{1} & 4.2686\times10^{-1} & -3.2200\times10^{1} & 7.9938\times10^{-4} & 1.8614\times10^{1} & 4.301\times10^{-1} \\ -2.3158\times10^{-6} & -5.8716\times10^{-2} & 1.0002 & 0 & 4.4227\times10^{-7} & -3.9534\times10^{-2} & 2.197\times10^{-4} \\ -9.4647\times10^{-6} & 4.3430 & -5.7885\times10^{-2} & 0 & 1.8076\times10^{-6} & 7.2990 & -5.285\times10^{-2} \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & -7.8487\times10^{3} & 0 & 7.8487\times10^{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 \\ 1.4938\times10^{-3} & 5.4953\times10^{1} & -4.1812\times10^{-1} & 0 & -2.8529\times10^{-4} & -2.6905\times10^{2} & -1.1340 \end{bmatrix}$$

B=

$$\begin{bmatrix} -1.1359\times10^{2} & -1.7159\times10^{2} & 1.3329\times10^{-2} \\ -1.4513\times10^{-2} & 4.7726\times10^{-3} & -1.6720\times10^{-7} \\ -2.3511 & -8.2859\times10^{-1} & 6.9090\times10^{-5} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -9.8249\times10^{-1} & 34421\times10^{-5} \end{bmatrix}$$

TABLE 7
The state-weighting matrix and the optimal feedback control matrix corresponding to the closed-loop eigenvalues given in Table 4

Q = diag

$$\begin{bmatrix} 3.5356\times10^{-2} & -2.6082\times10^{4} & -1.5355\times10^{2} & 6.5757\times10^{3} & 8.4045\times10^{-7} & 3.7961\times10^{6} & 2.4916\times10^{3} \end{bmatrix}$$

K=

$$\begin{bmatrix} 1.8279\times10^{-1} & -1.3002\times10^{2} & -1.6282\times10^{1} & -3.5847\times10^{1} & 7.4726\times10^{-4} & 4.6587\times10^{2} & -2.0364\times10^{1} \\ -1.9695\times10^{-2} & 1.982\times10^{2} & -9.1544\times10^{-1} & -2.4545\times10^{2} & -5.9542\times10^{-4} & -1.5847\times10^{3} & -7.5662\times10^{1} \\ 2.5061\times10^{-5} & 3.3766\times10^{-2} & -1.6882\times10^{-3} & -4.6831\times10^{-2} & -2.5246\times10^{-7} & 3.6179\times10^{-2} & 2.2694\times10^{-4} \end{bmatrix}$$

TABLE 8
Open-loop eigenvalues for Hyperion flying at Mach 8 at 85000ft

$\lambda_{1,2}$	=	$-0.55 \pm j16.44$	(Structural Bending)
$\lambda_{3,4}$	=	$-1.89 \times 10^{-3} \pm j5.78 \times 10^{-2}$	(Phugoid)
λ_5	=	-2.49	(Short Period)
λ_6	=	2.33	(Short Period)
λ_7	=	-1.56×10^{-8}	(Height)

loop eigenvalues of Table 4. To illustrate that the method is effective for any choice of closed-loop eigenvalues, another S.A.S for Hyperion was designed with the specified closed-loop eigenvalues of Table 9. The dynamic response of the new system is shown in Figure 3. The rate of change of pitch attitude and bending displacement step responses due to commanded change in height of 1000ft are again considered.

TABLE 9
The second set of prescribed eigenvalues for Hyperion

$\sigma_{1,2} = -0.9 \pm j17.9775$	(Structural Bending)
$\sigma_{3,4} = -0.60 \pm j1.91$	(Short Period)
$\sigma_{5,6} = -0.04 \pm j0.012$	(Phugoid)
$\sigma_{7} = -10.0$	(Height)

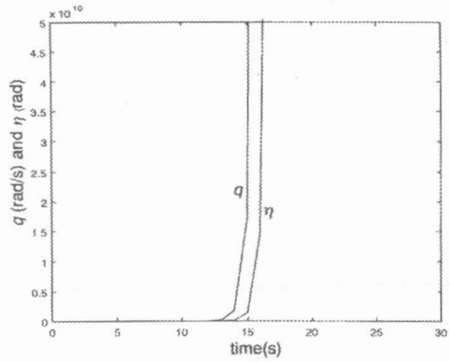

Fig. 1. Basic open-loop response of Hyperion

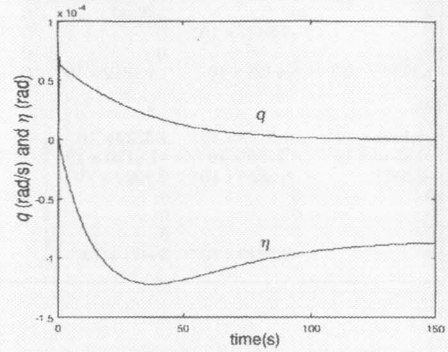

Fig. 2. Hyperion response to a step command for height change.

There is no observable effect of the short period and bending motions in either response shown in Figure 2. However, the phugoid mode oscillation can be observed in the bending displacement response, η.

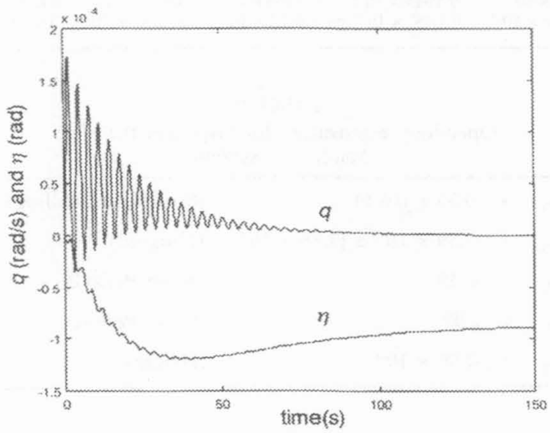

Fig. 3. Hyperion response to a step command for height change.

Figure 3 shows the effect of short period mode on the rate of change of pitch attitude and bending step responses. The short period oscillation can be seen clearly in the first 50 seconds of the simulation. It should be noted that the magnitude of oscillation is not large. Note also that the short period mode is 'superimposed' on the phugoid mode oscillation in the bending displacement step response. No effect of bending mode oscillation is visible in either case. If there is any problem with the aircraft flying qualities, then the problem can be remedied by specifying a better set of closed-loop eigenvalues for the particular mode.

CONCLUSIONS

It has been shown in this paper that the state-weighting matrix used to obtain the optimal control law using the LQR theory can be negative definite in contradiction to the conditions stated in the standard literature in the field. The advantage of using the method described in this paper is that the closed-loop eigenvalues of the controlled system can be specified before the design of the S.A.S takes place. This method of finding the required state-weighting matrix gives sufficient freedom to a designer to obtain any set of closed-loop eigenvalues specified. The choice of control-weighting matrix, G, was only restricted by the requirement that it be positive definite although the method of substituting soft for hard constraints proposed by Bryson and Ho 1975 is recommended.

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