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Preliminary Test Estimation in the Rayleigh Distribution Using Minimax Regret Significance Levels

Ayman Baklizi
Department of Statistics
Yarmouk University
Irbid – Jordan
E-mail: baklizi1@hotmail.com

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ABSTRAK

Memberikan ramalan awal tentang parameter yang tidak diketahui, penganggar ujian awal yang biasa berdasarkan penganggar kebolehjadian maksimum untuk parameter skala Rayleigh dibangunkan. Tahap signifikan optimum berdasarkan kriteria kesal minimaks dan nilai kritikal sepadan diperoleh secara berangka.

ABSTRACT

Given a prior guess of the unknown parameter, the usual preliminary test estimator based on the maximum likelihood estimator for the Rayleigh scale parameter is developed. The optimal significance levels based on the minimax regret criterion and the corresponding critical values are numerically obtained.

Keywords: Maximum likelihood estimator, minimax regret criterion, preliminary test estimator, optimum significance levels, Rayleigh distribution

INTRODUCTION

The Rayleigh distribution has been used in a variety of fields. It is used in life testing and reliability theory to model products with linearly increasing hazard rate. It has applications in the field of acoustics, spatial statistics and random walks (Johnson *et al.* 1994). An account of the history and properties of the distribution is given by Hirano (1986). The probability density function of the Rayliegh distribution is given by (Cohen and Whitten 1988);

$$f(x)=\frac{2^{-(k-1/2)}}{\sigma\Gamma(k/2)}\left(\frac{x}{\sigma}\right)^{(k-1)}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\quad x,\,\sigma>0.\ k \text{ is a known integer.} \tag{1}$$

where σ is a scale parameter. Various estimation methods for the parameter σ of the Rayleigh distribution have been discussed in the literature. The best linear unbiased estimator (BLUE) has been discussed by various authors (David 1981; Balakrishnan and Cohen 1991; Adatia 1994) among others. Maximum likelihood estimation is discussed in Cohen and Whitten (1988), Lee *et al.* (1980), and Tiku *et al.* (1986). Bayesian estimation is discussed by Sinha and Howlader (1983).

In some applications, the experimenter possesses some knowledge about the parameter σ. This knowledge may be obtained from past experience, or from the aquaintance with similar situations. Thus he is in a position to make an educated guess or prior estimate σ_0. This prior information may be incorporated in the estimation process using a preliminary test estimator (Ohtani and Toyoda 1978; Toyoda and Wallace 1975; Sawa and Hiromatsu 1973), thus improving the estimation process. In this paper we present a preliminary test estimator for the parameter of the Rayleigh distribution . The procedure for obtaining the optimum values of the significance levels using the minimax regret criterion of Brook (1976) is developed in Section 2. The results are given in the final section.

Preliminary Test Estimation

Consider a random sample $X_1, ..., X_n$ from the Rayleigh distribution. The maximum likelihood estimator of σ is given by (Cohen and Whitten 1988) as

$$\hat{\sigma} = \left(\sum_{i=1}^{n} x_i^2 / nk \right)^{1/2}. \tag{2}$$

It can be shown (Cohen and Whitten 1988) that $nk\,\hat{\sigma}^2/\sigma^2 \sim \chi^2_{nk}$. Assume that σ_0 is a prior guess of σ. Consider testing $H_0 : \sigma = \sigma_0$ against H_1: $\sigma \neq \sigma_0$; the likelihood ratio test rejects H_0 when $\frac{nk\hat{\sigma}^2}{\sigma_0^2} > c_1$ or $\frac{nk\hat{\sigma}^2}{\sigma_0^2} > c_2$. A preliminary test estimator $\tilde{\sigma}$ of σ may be obtained as follows

$$\tilde{\sigma} = \begin{cases} \sigma_0, c_1 < \frac{nk\hat{\sigma}^2}{\sigma_0^2} < c_2 \\ \hat{\sigma}, \quad \text{Otherwise} \end{cases} \tag{3}$$

where c_1 and c_2 are such that $p_{\sigma_0}(W < c_1) = p_{\sigma_0}(W > c_2) = \frac{\alpha}{2}$ and $W \sim \chi^2_{nk}$. Our aim is to find the optimum values of α, according to the minimax regret criterion. The mean of $\tilde{\sigma}$ is given by

$$E(\tilde{\sigma}) = \sigma_0 E\left[I\left(\delta_1 < \frac{nk\hat{\sigma}^2}{\sigma^2} \right) \right] + E\left[\hat{\sigma}\left\{ 1 - I\left(\delta_1 < \frac{nk\hat{\sigma}^2}{\sigma^2} < \delta_2 \right) \right\} \right], \tag{4}$$

where $\delta_1 = \dfrac{c_1\sigma_0^2}{\sigma^2}, \delta_2 = \dfrac{c_2\sigma_0^2}{\sigma^2}$ and $I(.)$ is the indicator function. Notice that $nk\hat{\sigma}^2/\sigma^2 \sim \chi^2_{nk}$, so that

$E\left[I\left(\delta_1 < \dfrac{nk\hat{\sigma}^2}{\sigma^2} < \delta^2\right)\right] = p\left(\delta_1 < \dfrac{nk\hat{\sigma}^2}{\sigma^2} < \delta_2\right) = \int_{\delta_1}^{\delta_2} g(w)dw$ where $g(w)$ is the pdf of a chi-squared random variable with degrees of freedom, also

$$E\left[\hat{\sigma}\left\{1 - I\left(\delta_1 < \frac{nk\hat{\sigma}^2}{\sigma^2} < \delta_2\right)\right\}\right] E(\hat{\sigma}) - E\left(\hat{\sigma}I\left(\delta_1 < \frac{nk\hat{\sigma}^2}{\sigma^2} < \delta_2\right)\right).$$

Now, $E(\hat{\sigma}) = \dfrac{\sigma}{\sqrt{nk}} E\left(\dfrac{\sqrt{nk\hat{\sigma}}}{\sigma}\right) = \dfrac{\sigma}{\sqrt{nk}} E\left(\sqrt{W}\right) = \dfrac{\sigma}{k}\sqrt{2}\dfrac{\Gamma(nk+1/2)}{\Gamma(nk/2)}$.

And so $E\left(\hat{\sigma}I\left(\delta_1 < \dfrac{nk\hat{\sigma}^2}{\sigma^2} < \delta_2\right)\right) = \dfrac{\sigma}{\sqrt{nk}} E\left(\dfrac{\sqrt{nk\hat{\sigma}}}{\sigma} I\left(\delta_1 < \dfrac{nk\hat{\sigma}^2}{\sigma^2} < \delta_2\right)\right)$

$$= \frac{\sigma}{\sqrt{np}} E\left(\sqrt{W}\, I(\delta_1 < W < \delta_2)\right) = \frac{\sigma}{\sqrt{np}} \int_{\delta_1}^{\delta_2} \sqrt{w} g(w)dw.$$

Thus

$$E(\hat{\sigma}) = \sigma_0 \int_{\delta_1}^{\delta_2} g(w)dw + \frac{\sigma}{\sqrt{nk}}\sqrt{2}\frac{\Gamma(nk+1/2)}{\Gamma(nk/2)} - \frac{\sigma}{\sqrt{nk}} \int_{\delta_1}^{\delta_2} \sqrt{w} g(w)dw. \tag{5}$$

Similarly the second moment of $\tilde{\sigma}$ is given by

$$E\left(\tilde{\sigma}^2\right) = \sigma_0^2 \int_{\delta_1}^{\delta_2} g(w)dw + \sigma^2 - \frac{\sigma^2}{nk} \int_{\delta_1}^{\delta_2} wg(w)dw \tag{6}$$

The mean squared error of $\tilde{\sigma}$ is given by

$MSE(\tilde{\sigma}) = E(\tilde{\sigma}^2) - (E(\tilde{\sigma}))^2 + (E(\tilde{\sigma}) - \sigma)^2 = E(\tilde{\sigma}^2) - 2\sigma E(\tilde{\sigma}) + \sigma^2$

Thus

$$MSE(\tilde{\sigma}) = \sigma_0^2 \int_{\delta_1}^{\delta_2} g(w)dw + \sigma^2 - \frac{\sigma^2}{nk} \int_{\delta_1}^{\delta_2} wg(w)dw -$$
$$2\sigma\left(\sigma_0 \int_{\delta_1}^{\delta_2} g(w)dw + \frac{\sigma}{\sqrt{nk}}\sqrt{2}\frac{\Gamma(nk+1/2)}{\Gamma(nk/2)} - \frac{\sigma}{\sqrt{nk}} \int_{\delta_1}^{\delta_2} \sqrt{w} g(w)dw\right) + \sigma^2 \tag{7}$$

Now, the quantity $\frac{MSE(\sigma)}{\sigma^2}$ can be considered as a risk function. Let $\psi = \frac{\sigma_0}{\sigma}$ and notice that $\delta_1 = \frac{c_1\sigma_0^2}{\sigma^2}$ and $\delta_2 = \frac{c_2\sigma_0^2}{\sigma^2}$ we get

$$RIS(\psi,\alpha) = \psi^2 \int_{c_1\psi^2}^{c_2\psi^2} g(w)dw + 1 - \frac{1}{np}\int_{c_1\psi_2}^{c^2\psi^2} wg(w)dw -$$

$$2\left(\psi \int_{c_1\psi^2}^{c_2\psi^2} g(w)dw + \frac{1}{k}\sqrt{2}\frac{\Gamma(nk+1/2)}{\Gamma nk/2} - \frac{1}{\sqrt{nk}}\int_{c_1\psi^2}^{c_2\psi^2} \sqrt{w}g(w)dw\right) + 1. \quad (8)$$

Notice that the risk function depends on through which are determined such that p_{σ_0} (W < c_1) = p_{σ_0} ($W > c_2$) = $\frac{\alpha}{2}$, where W ~ c_{nk}^2.

If $\psi \to 0$ or ∞, then $RIS(\psi, \alpha)$ tends to RIS(ψ,1) which is the risk of the maximum likelihood estimator $\hat{\sigma}$. Chiou (1988) gives us the general shapes of $RIS(\psi, \alpha)$. An optimal value of α is $\alpha = 1$ if $\psi \le \psi_1$ or $\psi \ge \psi_2$ and $\alpha = 0$ otherwise, where ψ_1 and ψ_2 are the intersections of $RIS(\psi, \alpha)$ $= (\psi - 1)^2$ with $RIS(\psi,1) = 2\left(1 - \sqrt{\frac{2}{np}}\frac{\Gamma(nk+1/2)}{\Gamma(nk/2)}\right)$.

The points of intersection are $\psi_1 = 1 - (2(1 - (2/nk)^{1/2}(\Gamma(nk+1/2)/\Gamma(nk/2))$ and $\psi_2 = 1 + (2(1 - (2/nk)^{1/2}(\Gamma(nk+1/2)/\Gamma(nk/2))$. Since ψ is unknown we seek an optimal value $\alpha = \alpha^*$ which gives a reasonable risk for all values of α. Going along the lines of Sawa and Hiromatsu (1973), the regret function is $REG(\psi,\alpha) = RIS(\psi,\alpha) - \inf RIS(\psi,\alpha)$, where

$$\inf_{\alpha} RIS(\psi,\alpha) = \begin{cases} RIS(\psi,1), & \psi \le \psi_1 \text{ or } \psi \ge \psi_2 \\ RIS(\psi,0), & \text{otherwise.} \end{cases} \quad (9)$$

For $\psi \le \psi_2$ $REG(\psi,\alpha)$ takes a maximum value at ψ_L. For $\psi > \psi_2$, takes a maximum value at ψ_U (Chiou 1988). Thus the minimax regret criterion determines α^* such that $REG(\psi_L,\alpha^*) = REG(\psi_U,\alpha^*)$. $REG(\psi_L,\alpha^*) = REG(\psi_U,\alpha^*)$ A preliminary test estimator for σ that uses the minimax regret significance levels now can be defined as

$$\tilde{\sigma} = \begin{cases} \sigma_0, & c_1 < \frac{nk\hat{\sigma}^2}{\sigma_0^2} < c_2 \\ \hat{\sigma}, & \text{otherwise.} \end{cases} \quad (10)$$

where c_1 and c_2 are such that $p_{\sigma_0}(W < c_1) = p_{\sigma_0}(W > c_2) = \dfrac{\alpha^*}{2}$ where $W \sim \chi^2_{nk}$.

RESULTS

We found numerically the optimum significance levels α^* and the corresponding critical values for n=2,3,...,15, and k = 1,2,3. The results are given in Table 1. To illustrate the use of the results in this table assume, for example, that the sp. Rayleigh distribution from which the data came has (k=2) and the sample size is n=10. Using the results in the table, the preliminary test estimator for σ (with prior estimate σ_0) is

$$\tilde{\sigma} = \begin{cases} \sigma_0, & 13.247 < \dfrac{2\sum_{i=1}^{n} x_i^2}{\sigma_0^2} < 27.065 \\ \left(\sum_{i=1}^{n} x_i^2 / 2n\right)^{1/2} & \text{Otherwise.} \end{cases}$$

TABLE 1
Optimum significance levels and the corresponding critical values

	k=1			k=2			k=3		
n	α^*	c_1	c_2	α^*	c_1	c_2	α^*	c_1	c_2
2	0.568	0.669	2.516	0.426	1.722	5.817	0.369	2.946	8.812
4	0.426	1.166	4.218	0.337	2.946	8.812	0.301	4.964	13.000
3	0.478	1.722	5.817	0.369	4.273	11.633	0.325	7.118	16.981
5	0.393	2.318	7.342	0.316	5.670	14.345	0.285	9.364	20.833
6	0.369	2.946	8.812	0.301	7.118	16.981	0.273	11.676	24.595
7	0.351	3.600	10.239	0.289	8.607	19.561	0.264	14.040	28.290
8	0.337	4.273	11.633	0.280	10.128	22.096	0.257	16.445	31.933
9	0.325	4.964	13.000	0.273	11.676	24.596	0.252	18.885	35.535
10	0.316	5.670	14.345	0.267	13.247	27.065	0.247	21.353	39.101
11	0.308	6.389	15.671	0.262	14.837	29.510	0.243	23.847	42.638
12	0.301	7.118	16.981	0.257	16.445	31.933	0.240	26.362	46.149
13	0.295	7.858	18.277	0.253	18.068	34.338	0.237	28.896	49.639
14	0.289	8.606	19.561	0.250	19.705	36.727	0.234	31.448	53.109
15	0.285	9.364	20.833	0.247	21.353	39.101	0.231	34.014	56.562

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